MATHEMATICAL STUDIES IN MANAGEMENT SCIENCE

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FOREWORD

The idea of the "management sciences" is as old as science itself; it is simply the idea that man may apply his highly refined methods of gaining knowledge to the management of his own affairs, be they entrepreneurial, governmental, educational, or whatever.

This idea crops up in Western thought in Plato and Aristotle, later in the stoic philosophy, later in many medieval and renaissance writings, and in the many facets of economic and social philosophy of the nineteenth century.

In our own century there has been quite a surge forward in exploring the idea of the management sciences, and for good reasons. Everyone knows that our social systems have become much more complicated than any before; the "manager" of today is faced with problems of such magnitude and seriousness that he can no longer rely on good common sense or flashes of insight to solve them. Many managers of even a decade ago were proud of the fact that they "flew by the seat of their pants" and that their companies or government agencies "grew like Topsy." Nowadays we realize that a manager who flies by the seat of his pants is apt to have the future hit him in the same place, and that the new Topsies are monsters no Frankenstein could have imagined.

In an attempt to meet the challenges of the modern technological world, a number of professional societies and informal groups were started after World War II: information scientists, systems engineers, general systems scientists, control scientists, behavioral scientists. Of these groups, one had a very broad and specific objective: to establish a profession of scientists and engineers whose mission is to improve large, complex systems of all kinds. This group goes under the label of "operations research."

The professional interest of operations research is matched by the scientific and philosophical interest of another group with equally broad interests. In 1954 The Institute of Management Sciences was formed to identify, extend, and unify scientific knowledge that contributes to the understanding and practice of management. Thus, the idea behind the Institute was to create a union of many groups of managers, engineers, scientists, and others who have a common interest in understanding and improving man's environment as well as man himself. Whereas the cohesion of operations research lies in its development of a discipline and a profession, the unity of the management sciences lies in a common aim. In the end, this common aim of many different parties may merge with the

aim of a coherent discipline; it was with just such an idealist hope in mind that the founders of the new Institute of Management Sciences called the journal *Management Science*—i.e., used the singular "science" for the journal, the plural "sciences" for the Institute that was combining such a diversity of interests.

There are many ways to learn about the management sciences. Books and articles on the subject are appearing in increasing numbers. Unfortunately, too many of these are simply attempts to "sell" the managers on the idea of hiring professional scientists. The "unity" of interests turns out to be economic, which of course is all right, but not the whole picture. Indeed, there is no reason at all why the industrial manager shouldn't "sell" the scientist the idea of the scientist's working for the manager. Or why the government executive shouldn't "tell" the scientist about some of the realities of government life. Or why a manager shouldn't apply some of his knowledge of management to the management of science itself. In a successful Institute of Management Sciences the conversations will go in all directions with equal weight and authority.

In order to appreciate the real idea behind the management sciences, one should read the material in its original form, as people struggle to express their ideas and to establish a sound basis for their positions. This is a way that may be difficult to follow, but far more rewarding in the end.

This volume of papers selected from the first decade of *Management Science*—together with its companion volume, *Executive Readings in Management Science*—provides just such an opportunity. The editors have wisely decided to divide the papers into two volumes, one dealing with fundamental scientific work of applied mathematicians, the other with work in various scientific disciplines, as well as contributions from managers themselves. The reader should be able to acquire from these two volumes a real flavor of the current topics of conversation that make up the management sciences of today. He should also be able to sense those topics that are not being discussed but should be: the subtle problems of human values or the more complex problems of large systems like education, water, urban living, etc. He will come to realize that in this effort to develop a living conversation about the deepest problems we humans face, too many people are remaining silent, absorbed in their own little enterprises. A unity of the management sciences will occur only if everyone begins to speak.

C. West Churchman

PREFACE

In 1961 the Institute of Management Sciences, at the suggestion of The Macmillan Company, decided to publish two volumes of selected reprints of articles from *Management Science*. The purpose of this project was to make some of the significant work of contributors to the journal more readily available. One volume was intended for executives; the other was intended for mathematically oriented specialists in the management sciences.

During the latter half of 1962 Martin K. Starr and I were appointed editors of the projected anthologies. In the next year we each read those of the papers in *Management Science* that seemed respectively appropriate, and then we selected papers for inclusion in the volumes. The articles in this book were chosen from the first eight volumes of *Management Science*. The papers in Mr. Starr's book, *Executive Readings in Management Science*, were selected from the first nine volumes of *Management Science*. Our selections were reviewed by the editors of *Management Science* and approved by the editor-in-chief. This approval should not be construed as an official statement by the Institute of Management Sciences concerning the relative merits of papers included and excluded from the volumes.

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The papers in this volume are limited to those that contribute original research in mathematical aspects of the management sciences. Surveys and expository papers are specifically excluded. Even with this restriction I found that many fine papers could not be included for lack of space. However, I believe that the papers reprinted here are fairly representative and are of uniformly high quality.

An attempt has been made to correct any substantive or typographical errors that appeared in the original published papers. Where major changes were required in the original text to correct errors or to accommodate limitations of space, this fact is noted at the end of the reprinted version. Minor corrections are not noted.

The articles in this volume have been arranged according to similarity of subject matter rather than according to chronology. The papers are divided into two major parts, the first dealing with deterministic decision models and the second dealing with stochastic decision models. Each part is preceded by a

commentary in which certain relationships among the papers in this volume (and elsewhere) are described.

A few of the papers deal with topics that are not discussed by other papers in this collection. As a result I have not always found it convenient to mention such papers in the commentary. This silence should of course not be interpreted as reflecting any judgment on my part concerning the quality of these papers.

I am indebted to C. West Churchman for his guidance in the initial planning of these volumes; to R. M. Thrall, editor-in-chief of *Management Science*, for his constant support; to the editors of *Management Science*, especially William W. Cooper, Murray A. Geisler, and Morton Klein, for their comments and suggestions; to the department of industrial engineering of Stanford University for providing secretarial support; to Martin K. Starr for his excellent cooperation; and to my wife for typing and preparing the final manuscript for publication, as well as for her constant encouragement.

ARTHUR F. VEINOTT, JR.

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SECTION IV *Quadratic Programming*

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AN INTRODUCTION

This volume is a collection of reprints of some of the major research papers published in the first eight volumes of *Management Science*, the journal of the Institute of Management Sciences. Almost all of the papers represented are concerned with the problem faced by a decision maker who desires to select from among a collection of available alternatives one that is, in some sense, optimal. These papers have two general features. First, there is specified a set of variables and of empirical relations that link them. Usually some variables are under the control of the decision maker, while others are not. A combination of values of the variables satisfying the empirical relations is called a policy. Second, there is an objective function that permits the policies to be ranked according to their relative desirability. Together, the variables, empirical relations, and objective function constitute a model.

One usually formulates a model of a real decision process with either or both of the following objectives in mind: first, one desires an insight into the qualitative properties of good decision rules for the situation under study; second, one seeks an actual policy to use for some specified values of the uncontrollable variables. Sometimes these objectives may be realized by a mathematical analysis of the model without resorting to computation. However, their attainment usually requires, and is always facilitated by, computational procedures for finding an optimal policy—i.e., a policy that performs best as measured by the objective function. All of the papers in this volume present computational procedures for optimization and/or mathematical analyses directed toward qualitative characterizations of models.

Abstract models, such as those formulated in this volume, attempt to mirror the key features of certain concrete situations. The models necessarily suppress many hopefully less significant details. Because a model is, at best, approximately equivalent in a formal sense to a concrete situation, it is to be expected that a policy that is optimal for a model will not be optimal for the original concrete situation under study. Experience has shown, however, that it is often possible to construct a model whose optimal policies perform quite well in the corresponding concrete situation. It is this fact that makes model construction useful.

From a mathematical point of view, optimization of a model usually involves extremizing (maximizing or minimizing) a real valued function over a specified set. A particularly important example is the linear programming problem. It involves extremizing a linear function of finitely many real variables over a set defined by a finite number of linear restrictions.

An important feature of real decision problems is the presence of uncertainties about the future. Models that explicitly allow for uncertainties are based on probability theory. The most sophisticated of these models allow explicitly for the fact that decisions made at one point in time must be based on less information than are decisions made at later points in time.

This volume is separated into two parts. In Part One papers describing deterministic decision models are collected. Part Two includes papers discussing stochastic decision models.

PART ONE
DETERMINISTIC DECISION MODELS

I

COMMENTARY ON PART ONE: DETERMINISTIC DECISION MODELS

It is often convenient to develop models of decision problems in which uncertainties are suppressed. The models in Part One have this feature.

The models discussed in Sections II–V have the following mathematical structure. One seeks an n coordinate vector of real numbers (or, briefly, an n-vector) $x = (x_i)$ that

(1) minimizes $$c(x)$$

subject to

(2) $$\sum_{i=1}^{n} a^i x_i = b$$

(3) $$x_i \geq 0 \qquad (i = 1, 2, \cdots, n)$$

where c is a given real valued function and $a^1, \cdots, a^n$, b are given m-vectors. In Sections II and III and parts of Section V it is further assumed that (1) takes the special form

(1)′ minimizes $$cx$$

where $c = (c_i)$ is a given n-vector.[1] In this case we have a linear programming problem. In Section IV, (1) takes the form

(1)″ minimizes $$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i$$

in which case the problem is one of quadratic programming. In this event $c(x)$ is taken to be convex in x.

Linear Programming

A surprisingly large number of real decision problems can be studied fruitfully by means of linear programming. The collection of papers given in Sections II and III and parts of Sections V and VIII attests to this fact. Textbooks on linear programming and related topics are also available [5, 6, 14, 15, 18, 20].

Duality and Existence Theorems in Linear Programming

The main theorems of linear programming exploit the fact that linear programming problems come in pairs. By using the constants of the "primal"

[1] If $u = (u_i)$ and $v = (v_i)$ are n-vectors, the scalar product of the vectors is $uv = \sum_{i=1}^{n} u_i v_i$.

problem (1)′, (2), (3) we may construct a "dual" problem, which is to find an m-vector y that

(4) maximizes
$$yb$$
subject to

(5)
$$ya^i \leq c_i \qquad (i = 1, 2, \cdots, n).$$

It is important to note that the vectors $a^1, \cdots, a^n, b, c$ are the same in both the primal and dual problem.

The vectors x^* and y^* are called optimal solutions to the primal and dual problems respectively if x^* satisfies (1)′, (2), (3) and y^* satisfies (4), (5). The vectors x and y are said to be feasible for the primal and dual problems respectively if x satisfies (2), (3) and y satisfies (5). The following are the main theorems of linear programming, from Gale, Kuhn, and Tucker [21c].

Existence Theorem

There exist optimal solutions to the primal and dual problems if and only if there exist feasible solutions to the problems.

Dual Theorem

If x^* and y^* are feasible solutions to the primal and dual problems respectively, then x^* and y^* are optimal if and only if

(6)
$$cx^* = y^*b.$$

The proof of the sufficiency of (6) as a condition for the optimality of x^* and y^* is so simple that we reproduce it here. Let x and y be feasible for the primal and dual problems. Then by (2), (3), and (5)

(7)
$$cx \geq \sum_{i=1}^{n} (ya^i)x_i = y\left(\sum_{i=1}^{n} a^i x_i\right) = yb.$$

Setting $y = y^*$ in (7) and using (6) it follows that for any feasible x for the primal problem
$$cx \geq y^*b = cx^*,$$
which establishes the optimality of x^*. A similar argument shows y^* to be optimal.

The dual theorem of linear programming is of great importance. The principle computational methods for solving linear and discrete dynamic programming problems are based upon it. The minimax theorem for two person zero sum matrix games is a simple corollary of it.

The paper by Charnes and Cooper, Chapter 7, deals with certain problems in which there are multiple linear objective functions and relies heavily upon the dual theorem. The papers by Dantzig, Chapter 6, and by Ford and Fulkerson, Chapter 4, give examples of situations in which the dual of the "natural" problem has a special structure which can be exploited in computations.

The Simplex Method

The most important computational procedure for solving linear programming problems is Dantzig's famous simplex method [21a]. Actually this method simultaneously solves both the primal and dual problems. We outline briefly the essentials of a typical iteration of this finite iterative procedure as applied to the problem of finding a pair of vectors (x, y) satisfying (1)′, (2)–(5).

Let $A = (a^1, \cdots, a^n)$. An iteration begins with the following information at hand when A has rank m:

(i) the inverse B^{-1} of a basis matrix B whose columns are m linearly independent columns of A (by relabeling we may assume that $B = (a^1, \cdots, a^m)$);
(ii) a vector $x = (x^*, 0)$ that is feasible for the primal problem and is such that $x^* = B^{-1}b$ (x is called a basic feasible solution);
(iii) a vector $y = c^*B^{-1}$ where $c^* = (c_1, \cdots, c_m)$ (the elements of y are variously called simplex multipliers or prices).

Observe from (ii) and (iii) that

$$cx = (yA)x = y(Ax) = yb,$$

so that by the sufficiency of (6) as a condition for optimality, x and y are optimal if y is feasible for the dual problem. If y is not feasible for the dual problem, one then finds a new basis matrix $\bar{B}$, inverse $\bar{B}^{-1}$, basic feasible solution $\bar{x}$, and vector $\bar{y}$ of simplex multipliers. This is done as follows.

Select an integer s for which

$$ya^s > c_s. \tag{8}$$

Then compute $t^s = B^{-1}a^s$ and let

$$\bar{x} = x + \theta\left[u_s - \begin{pmatrix} t^s \\ 0 \end{pmatrix}\right] \tag{9}$$

where u_s has $+1$ in the sth position and zeroes elsewhere, and where $\theta(\geq 0)$ is the largest number[2] for which $\bar{x} \geq 0$. Observe from (ii) and (9) that

$$A\bar{x} = Ax + \theta[a^s - a^s] = b,$$

so $\bar{x}$ is feasible for the primal problem. Using (8) one finds that

$$c\bar{x} = cx + \theta[c_s - c^*B^{-1}a^s] = cx + \theta[c_s - ya^s] \leq cx \tag{10}$$

so that the new feasible solution $\bar{x}$ is an improvement over x.

In the usual non-degenerate case we may assume that $\theta > 0$ and that there is a unique integer r, $1 \leq r \leq m$, for which $\bar{x}_r = 0$. We call this the non-degeneracy hypothesis. Thus, in order for $\bar{x}$ to be a basic feasible solution for $\bar{B}$, $\bar{B}$ must be formed by replacing the column vector a^r in B by a^s.

[2] If $\bar{x} \geq 0$ for all $\theta \geq 0$, then the objective function (1)′ of the primal problem can be made arbitrarily large and there is no feasible solution for the dual problem.

One way to form $\bar{B}$ from B is to postmultiply B by E where E is the elementary matrix obtained by replacing the rth column of the $m \times m$ identity matrix I by t^s. Then $\bar{B} = BE$. Thus, if we can find E^{-1}, we may compute $\bar{B}^{-1}$ by $\bar{B}^{-1} = E^{-1}B^{-1}$.

Now interchange the rth and sth columns of A, and the rth and sth coordinates of c and of $\bar{x}$ defined in (9). We shall then have $\bar{x} = (\bar{x}^*, 0)$ with $\bar{x}^* = \bar{B}^{-1}b$ and $\bar{y} = \bar{c}^*\bar{B}^{-1}$ where $\bar{c}^*$ is formed from c^* by replacing c_r by c_s. This step completes an iteration except for constructing E^{-1}, which we now do.

It follows from the non-degeneracy hypothesis that the rth element of t^s, say t_{rs}, is positive. Now replace t_{rs} in t^s by -1 and multiply the resulting vector by $-1/t_{rs}$. Then substitute the vector so obtained for the rth column of the $m \times m$ identity matrix. The result is E^{-1}, which one checks by verifying that $EE^{-1} = I$.

If there is a sequence of iterations that starts and ends with a common basic feasible solution, then by (10) the objective function remains constant throughout that sequence. But under the non-degeneracy hypothesis, (10) is a strict inequality, so that no basic feasible solution can recur. Because there are a finite number of basic feasible solutions (no more than $n!/(n - m)!m!$, the number of combinations of n vectors taken m at a time), the simplex method therefore must terminate after a finite number of iterations. Thus, if the dual problem is feasible, the simplex method terminates with vectors x^* and y^* that are feasible for the primal and dual problems and that satisfy (6). Hence (6) is a necessary (as well as sufficient) condition for optimality. Thus, the simplex method provides a constructive proof of the dual theorem.

Large Scale Linear Programming Problems

The simplex method has turned out to be an efficient procedure for solving linear programming problems on digital computers when there are no more than a few hundred constraints. However, in many practical problems the constraints number in the thousands. Some techniques for dealing with such problems are discussed by Dantzig in Chapter 6 and in [16a].

The most exciting idea for dealing with large scale problems is the decomposition principle of Dantzig and Wolfe [8]. Their work was stimulated in part by the important paper of Ford and Fulkerson, Chapter 3.

Transportation and Network Problems

An important special structure in linear programming arises when each column of the matrix A is composed on one $+1$, one -1, and zeroes elsewhere. A problem with this structure is called a transhipment problem in Chapter 1. The special case of a transhipment problem in which no row of A contains both $+1$'s and -1's is called a transportation problem. Both names derive from the fact that certain problems of choosing optimal routes for shipping the stocks of a product from one set of locations to another can be formulated as linear programming problems with the above structure.

The importance of the transhipment problem derives from two facts. First,

special computational procedures that exist for solving the problem are significantly more efficient than direct application of the simplex method. Indeed, transhipment problems with several thousand constraints are within the range of current computing equipment. Second, many problems that on the surface seem entirely unrelated to the distribution problems indicated above have the transhipment structure in the "natural" formulation of the problem. This is the case with the caterer problem (Chapters 18, 19, 20) and the dual of the project cost curve problem (Chapter 4), as well as with a large number of other problems [12, 13, 25].

Orden shows in Chapter 1 that any transhipment problem can be reduced to an equivalent transportation problem. This reduction is of interest because it extends the applicability of the many algorithms that have been devised for solving transportation problems to transhipment problems [12, 21b].

An important property of transhipment problems is that whenever every coordinate of b is an integer, the same is true of every basic feasible solution of the primal problem. This property permits many combinatorial problems to be solved by formulating them as transhipment problems.

It is often of interest to solve transhipment problems in which there are additional constraints of a simple form—e.g., upper bounds on certain variables or sums of variables. Wagner shows in Chapter 2 that a transhipment problem with upper bounds on certain partial sums of variables can be reduced to an ordinary transportation problem. In Chapter 5 Fulkerson investigates the problem of determining an optimal way of increasing the capacity of a network subject to a budget constraint. He solves the problem by solving a sequence of transhipment problems.

Deterministic Dynamic Programming

One special type of transhipment problem may be formulated as follows. We are given n nodes labeled $1, 2, \cdots, n$ and a collection of ordered pairs (i, j) of nodes called arcs. There is associated with each arc (i, j) in the network a number c_{ij} representing the cost of shipping one unit of product from node i to node j along the arc (i, j). The problem is to ship one unit of product from node 1 to node n as cheaply as possible.

To formulate the problem we let x_{ij} be the total amount of product shipped from node i to node j along the arc (i, j). We then seek x_{ij} that

$$\text{(11)}\quad \text{minimize} \qquad \sum_{i,j} c_{ij}x_{ij}$$

subject to

$$\text{(12)}\quad \begin{aligned} \sum_j x_{ij} &= 1 \\ \sum_j x_{ij} - \sum_j x_{ji} &= 0 \qquad (i = 2, 3, \cdots, n-1) \\ -\sum_j x_{jn} &= -1 \end{aligned}$$

$$\text{(13)}\quad x_{ij} \geq 0 \qquad (i, j = 1, 2, \cdots, n).$$

The dual of this problem is to choose numbers $f_1, \cdots, f_n$ that

(14) maximize
$$f_1 - f_n$$

subject to

(15)
$$f_i - f_j \leq c_{ij} \qquad (i, j = 1, 2, \cdots, n).$$

Because the last equation in (12) may be obtained by summing the first $n - 1$ equations, the last equation is redundant and may be omitted. This omission is equivalent to deleting f_n from the dual problem, or, as we shall assume, setting $f_n = 0$.

As we have suggested earlier, every basic feasible solution to (12), (13) will involve integer x_{ij}'s. In the present case each such x_{ij} is either 0 or 1. Furthermore, a subcollection of the arcs (i, j) for which $x_{ij} = 1$ will form a "path" from node 1 to node n—i.e., the subcollection of arcs takes the form $(1, k_1)$, $(k_1, k_2), \cdots, (k_r, n)$. For an optimal basic feasible solution, the associated path from node 1 to node n will, of course, be a minimal cost path from node 1 to node n.

If the primal and dual problems are feasible, one optimal set of dual variables, f_i^* say, will satisfy the familiar recurrence relations of dynamic programming

(16)
$$f_i^* = \min_j \{c_{ij} + f_j^*\} \qquad (i = 1, \cdots, n - 1).$$

We may interpret f_i^* as the total cost of shipping one unit from node i to node n along the cheapest possible path.

The problem formulated above is actually a prototype of virtually all discrete deterministic dynamic programming problems. In the dynamic programming point of view one considers a process that may be in one of n states. One is allowed to move the process from state i to j at a cost c_{ij}. The problem is to guide the process from state 1 through a sequence of intermediate states to state n as cheaply as possible. By identifying states with nodes and decisions to go from i to j with arcs (i, j), the equivalence of discrete dynamic programming with the minimal cost path problem is immediate. Indeed we see also that discrete deterministic dynamic programming is simply a branch of linear programming in which duality plays a central role.

In many cases a discrete deterministic dynamic programming formulation leads to an acyclic network—i.e., a network in which an arc (i, j) is not admitted if $j \leq i$. For these cases one may calculate an optimal set of dual variables immediately from (16) by computing $f_{n-1}, f_{n-2}, \cdots, f_1$ in that order. By reference to the interpretations given above or, formally, by the theorem of the alternative, an optimal solution to the primal problem is then immediately at hand.

Charnes and Cooper [4] first employed the foregoing method to solve the warehouse problem. In Chapter 6, Dantzig extends the method to a broad class of problems of which the problem given above is a special case. Although Charnes, Cooper, and Dantzig recognized the computational advantages of this

solution method, it remained for d'Epenoux [9] to develop the dynamic programming interpretation offered here.

In much of the literature of dynamic programming a generalization of (16) is studied in which the number of possible states and decisions is infinite. This study amounts to considering a minimal cost path problem with infinitely many nodes and arcs. In order to actually solve such problems, however, one is forced to use a finite approximation or to exploit the structure of the problem to simplify the computations. Excellent examples of this latter approach in certain problems of production planning and inventory control are given by Bellman in Chapter 18, Dreyfus in Chapter 14, and Wagner and Whitin in Chapter 16.

Quadratic Programming

The quadratic programming problem arises as an approximation to the problem of minimizing a convex function subject to linear inequalities. The problem also arises directly in certain circumstances. This is the case, for example, where the coefficients of the objective function of the linear programming problem (1)′, (2), (3) are random variables and where x must be chosen before the values of those random variables are known. In this event it is generally impossible to choose a single x that solves (1)′, (2), (3) for all c. If instead one adopts the reasonable alternative of choosing an x that minimizes the expected value of cx, a linear programming problem is again obtained. If, however, one desires to have a small probability of obtaining very high values of the objective function while at the same time achieving a low expected value of the objective function, a different approach is called for. It is then reasonable to seek to minimize the variance $\sum_i \sum_j c_{ij}x_ix_j$ of cx (c_{ij} is the covariance of c_i and c_j) subject to a restriction that the expected value of cx not exceed a certain maximal level. The resulting problem is clearly one in quadratic programming. This application was initially suggested by Markowitz [23] in his analysis of efficient investment portfolios.

The first algorithm for solving the quadratic programming problem was given by Beale [2]. Since that time several other algorithms have been proposed. Many of these are similar to the simplex method in various respects. One proposal by Dantzig [7] is a direct generalization of the simplex method. By this statement is meant that if $c_{ij} = 0$ for all i, j in (1)″, the resulting application of Dantzig's algorithm is equivalent to the usual simplex method.

A generalization of the dual theorem of linear programming has been developed for quadratic programming as well as more general nonlinear problems. Dorn [10] surveys these results and discusses several of the proposed algorithms for solving quadratic programming problems.

Among the proposals for solving quadratic programming problems surveyed by Dorn are those presented by Theil and Van de Panne in Chapter 9 and Lemke in Chapter 11. These two algorithms have the common feature that both begin by finding the absolute minimum of the objective function without regard to the constraints. If this solution satisfies the constraints, it is optimal; if not, the algorithms proceed separately. Theil and Van de Panne give a

systematic procedure for adding constraints until an optimal solution is found. An alternative justification for this procedure using the Kuhn-Tucker conditions is given by Boot in Chapter 10. Lemke instead uses the fact that the dual problem (also a quadratic programming problem) can be simplified after the absolute minimum of the primal objective function is found. He then develops an algorithm for solving the simplified dual problem in which the dual is feasible at each step. There are corresponding solutions of the primal problem which are infeasible until the final step, at which an optimal solution is found. These features are reminiscent of Lemke's dual simplex method [22] for solving linear programming problems.

Production and Inventory Control

The most fertile field for applications in the management sciences has been production and inventory control. There are now several books [1, 17, 19, 24] available which give accounts of recent research in the field. A variety of models have been proposed of which those in Section V are representative. Each paper in Section V develops a specially designed algorithm for solving a problem more efficiently than would be possible with an appropriate, general purpose algorithm like the simplex method. We shall here attempt to point out relationships between the papers in Section V by formulating a model that includes as special cases several of the closely related models in that section.

Consider the problem of finding numbers $p_1, \cdots, p_n, s_1, \cdots, s_n, y_1, \cdots, y_n$ that minimize

$$\sum_{i=1}^{n} [c_i(p_i) + g_i(s_i) + h_i(y_i)] \tag{17}$$

subject to ($s_0 \equiv 0$)

$$y_i = y_{i-1} + p_i - s_{i-1} \qquad (i = 1, \cdots, n) \tag{18}$$

$$\underline{y}_i \leq y_i \leq \bar{y} \qquad (i = 1, \cdots, n) \tag{19}$$

$$0 \leq p_i \qquad (i = 1, \cdots, n) \tag{20}$$

$$\underline{s}_i \leq s_i \leq \bar{s}_i,\ s_i \leq y_i \qquad (i = 1, \cdots, n) \tag{21}$$

where g_i, c_i, and h_i are given continuous, real, valued functions and where $y_0, \underline{y}_1, \cdots, \underline{y}_n, \bar{y}, \underline{s}_1, \cdots, \underline{s}_n, \bar{s}_1, \ldots, \bar{s}_n$ are given constants. We may interpret this problem as one of choosing the amounts of a single product to produce and sell during each of n successive time periods $1, 2, \cdots, n$ so as to minimize the total manufacturing and selling costs over those periods while satisfying given capacity constraints. In this interpretation p_i is the amount produced at the beginning of period i, s_i is the amount sold at the end of period i, and y_i is the inventory on hand after production but before sales in period i. The production cost function for period i is c_i; g_i is the sales cost function for period i,[3] and h_i is the inventory carrying cost function for period i.

[3] Since sales revenue is a negative cost, g_i is usually negative; g_i accounts for the fact that the selling price of a product may vary with the amount sold.

Table I gives various specializations of the above problems that are discussed in Chapters 12–16 and 33.

TABLE I

Chapter	$c_i(p_i)$	$g_i(s_i)$	$h_i(y_i)$	$\bar{y}$	$\underline{y}_i$	$\bar{s}_i$	$\underline{s}_i$
12, 33	convex	0	linear	∞	0	$= \underline{s}_i$	≥ 0
13	linear stationary	linear stationary	convex	∞	≥ 0	∞	0
14	linear	linear	linear	≥ 0	0	∞	0
15*	quadratic (convex)	quadratic (convex)	quadratic (convex)	∞	$-\infty$	∞	$-\infty$
16	concave†	0	linear	∞	0	$= \underline{s}_i$	≥ 0

* In this model (19), (20), and (21) are omitted.

† The concave cost function in Chapter 16 is assumed to take the special form

$$c_i(p_i) = \begin{cases} 0 & , p_i = 0 \\ K + c_i \cdot p_i, & p_i > 0 \end{cases}$$

where $K \geq 0$.

The assumption that $\bar{s}_i = \underline{s}_i (i = 1, \cdots, n)$ in Johnson's paper (Chapter 12) means that the sales to be realized in each period are known in advance. The problem is then to plan production to meet sales at minimal cost. Johnson devises an extremely simple algorithm for solving his problem. The procedure is simply to satisfy each unit of sales in order of occurrence as cheaply as possible. An alternative procedure for solving the same problem is given by Charnes, Cooper, and Symonds in Section 7 of Chapter 33. An extension of Johnson's method is shown to be applicable by Wagner [27] where the planner is allowed to vary his volume of sales by adjusting his selling price.

In Johnson's paper, inventories are held as a means of satisfying future requirements as cheaply as possible. By contrast, in Chapter 13 Karush and Vazsonyi consider situations in which inventories are held to provide service—e.g., inventories of labor and equipment. The problem is then to plan changes in inventory levels to meet fluctuating but known requirements for service in each period as cheaply as possible. Karush and Vazsonyi's simple algorithm for solving their problem is based on an interesting property of the optimal inventory levels, viz., that the inventory levels equal the minimal required levels except during intervals over which the inventory level is held constant. Their procedure is essentially to search through schedules with the above property to find one that is optimal. Recent work on this problem has been described by Veinott and Wagner [25].

Dreyfus develops an interesting dynamic programming algorithm for solving the warehousing problem in Chapter 14. In this model selling and buying prices vary over time in a known way. The problem is to time buying and selling of stock so as to minimize total costs while assuring that the inventory levels never exceed a fixed storage limit. An alternative algorithm for solving this problem was proposed by Charnes and Cooper [4]. Their algorithm is out-

lined by Dantzig in Chapter 6. Further study of the problem has been given by Veinott and Wagner [25].

Veinott [26] has made a study encompassing all three models discussed above.

The production planning problem studied by Holt, Modigliani, and Muth in Chapter 15 is more general than indicated in Table 1. In particular, inventories of product and labor are dealt with separately and a cost of changing the size of the labor force is introduced. The interesting consequence of their assumption of quadratic cost functions and unconstrained variables is that the optimal inventory and labor force levels in one period are linear functions of those same quantities in the preceding period. An excellent book [19] gives a complete account of the use of quadratic cost functions in production planning.

The model studied by Wagner and Whitin in Chapter 16 differs from those discussed above in that the cost function is concave. This concavity means that an optimal solution occurs at an extreme point of the set of feasible solutions. Wagner and Whitin show that there is an optimal production schedule with the property that if there is a positive inventory level at the beginning of a period, then no production takes place during that period. This property was first noted by Manne in the article reprinted as Chapter 17. Wagner and Whitin find an optimal policy with the aid of the dynamic programming recurrence relation (16) where c_{ij} is the cost of ordering enough stock in period i to satisfy the requirements in periods $i, \cdots, j - 1$ plus the cost of carrying inventory over that interval. The hypotheses of the model are further weakened in an article by Wagner [27].

Manne (Chapter 17) studies a generalization of the model proposed by Wagner and Whitin in which several products are permitted and labor is rationed among the products so as to minimize costs. Manne's (approximate) formulation of the problem as one in linear programming is a good example of how a clever choice of variables can make a seemingly intractable problem solvable.

Bellman (Chapter 18) and Prager (Chapter 19) give algorithms for solving the caterer problem, a variation of a special case of the problem studied by Karush and Vazsonyi in Chapter 13. In the caterer problem, one assumes that g_i and h_i are identically zero. One also assumes that after a unit of product provides service for one period, it must undergo repair during several periods before it can be used again. This contrasts with the assumption of Chapter 13 that necessary repair time is negligible. There are two types of repair service, one fast and one slow, with the former being more expensive.

As Beale suggests in Chapter 20, Prager's algorithm in Chapter 19 exploits the following property of an optimal solution: Suppose we fix the total number of units produced over the n periods. Then the optimal way of providing service is to satisfy each unit of service requirements in order of occurrence according to the rule: First use product that has not previously been used; second, use product that can be repaired on slow service and be available for use; finally, use product that can be repaired only on fast service starting with the product that was used latest. This procedure can be interpreted as satisfying each unit of service in order of occurrence as cheaply as possible if the cost

of production is taken to be zero.[4] Notice that with this interpretation Prager's procedure is similar to that used by Johnson in Chapter 12 on a related problem discussed above.

Derman and Klein (Chapter 21) and Lieberman (Chapter 22) consider inventory problems in which there is initially a stockpile of items of different ages that are to be used as efficiently as possible. In one problem the useful service life of an item is a function of its age at issue. One then seeks to select the order of issuing the items so as to maximize the total service life from all items when used sequentially. In a second problem the future demands for the product are known and the return received from issuing an item to satisfy a demand is a function of the age of the item at the time the demand occurs. This time, one seeks to select the order of issuing items so as to maximize the total return received from all items. For both problems conditions are given which ensure the optimality of first-in-first-out (FIFO) and last-in-first-out (LIFO) issuing policies. Subsequent investigations of these problems are given in a number of papers [3, 11, 28].

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II–1

THE TRANSHIPMENT PROBLEM

ALEX ORDEN

Burroughs Corporation

1. Introduction

The "transportation problem" in linear programming refers to a class of linear programming problems whose first example was that of selection of most economical shipping routes for transfer of single commodity from a number of sources to a number of destinations. A number of linear programming models have since been developed involving manpower assignment, machine loading, and others, in which the algebraic equations are identical in form to those of the original transportation problem. Currently it appears that this special class covers the majority of the applications of linear programming which are in practical use or under active development.[1] We will refer to all linear programming problems which fall into this mathematical mold as "transportation problems". These problems take the form

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1 \cdots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1 \cdots n$$

$$x_{ij} \geqq 0$$

$$\sum_{i,j} c_{ij}\, x_{ij} = \text{Min. or Max.}$$

where a_i, b_j, and c_{ij} are given parameters.

In the course of work which has led to transportation problem type algebraic formulations for various applications it has become clear that this class of linear programming problems has important distinguishing characteristics. Perhaps the most valuable from a practical point of view is that methods of solution have been developed which make it possible to handle problems with a large number of unknowns, x_{ij}. (Problems with m and n up to about 200, i.e. involving 40,000 x_{ij}'s can be handled on high speed computing machines.) An equally significant characteristic is that the transportation problem offers an approach to some problems which appear at first to be purely combinatorial. Specifically, the so-called "assignment problem"; e.g., optimal assignment of men to jobs, can be formulated as a transportation problem.[2]

The connection of the transportation problem to combinatorial problems is a

[1] L. W. Smith, "Current Status of the Industrial Use of Linear Programming", *Management Science*, Vol. II, No. 2.

[2] D. F. Votaw and A. Orden, "The Personnel Assignment Problem", Symposium on Linear Inequalities and Programming, Hq. U. S. Air Force, (1952) pp. 155–163.

strong one. In the "assignment problem" one asks for the most efficient assignment combination. When this combinatorial problem is formulated in the algebra of the transportation problem, it appears at first that the solution will permit fractional assignments, i.e. might divide each man's time among several jobs. It turns out, however, that the linear programming solution always leads to a one man to one job solution, just as though only combinations of this type were eligible.

Thus the transportation problem has offered two mathematical facets:

(1) as a specialized type of linear programming problem,

(2) as a method of representation of some combinatorial problems.

The distinction between these two aspects of the problem may not appear important in applications. Both aspects deal with selection problems which call for an optimum pair-wise relation of one group of items to another group, as in relating shipping points to destinations in the original transportation problem. The main difference between the linear programming aspect and the combinatorial aspect is that the latter calls specifically for taking advantage of the fact that integer values of the quantities a_i and b_j always provides at least one optimum solution with integer values of the x_{ij}.

In this paper a third aspect of the mathematical properties of the transportation problem is developed. It is shown that the same mathematical framework can be extended beyond pair-wise connections, to the determination of optimum linked paths over a series of points. This extension although viewed here as a linear programming problem, takes advantage of the combinatorial aspect of the transportation problem, and applications may arise which, like the assignment problem, appear to be combinatorial problems, but which can be solved by linear programming.

The main part of the treatment which follows is in terms of extension of the original transportation problem to include the possibility of transhipment; i.e. any shipping or receiving point is also permitted to act as an intermediate point in seeking an optimum solution. A supplementary example (Section 6) isolates this extension of the transportation problem more specifically as a problem of determination of an optimum linked path. The transhipment technique is used to find the shortest route from one point in a network to another.

2. The Transhipment Procedure

The original transportation problem deals with selection of shipping routes so as to minimize the cost of shipping a uniform commodity from specified origins to specified destinations. The amounts to be sent from each origin, the amounts to be received by each destination, and the cost per unit shipped from any origin to any destination are specified. Transhipment is not considered, that is, each point acts as a shipper only or as a receiver only. The problem without transhipment will be denoted in this paper by "T_0". In extending the problem to permit transhipment the situation is the same as in "T_0" with the additional feature that shipments may go via any sequence of points rather than being restricted to direct connections from one of the origins to one of the destinations.

We will denote this problem by "T_1". The transhipment problem will be solved by converting it in a specified way to a computation problem which has the form of T_0. After conversion to a T_0 problem, Dantzig's simplex technique[3] provides a satisfactory computation technique. That technique and variants of it have been coded for solution on several electronic digital computers.

The general nature of the solution is as follows: In T_0 there are distinct shippers and receivers. The transhipment problem, T_1, is to be converted to the form of T_0 by treating each point as a pair of points, one acting as a shipper and one as a receiver. The unit cost of shipment from a point considered as a shipper to the same point considered as a receiver is set equal to zero. It is assumed (for computation purposes only) that a large amount of the material to be shipped is available at each point and acts as a stockpile which can be drawn or replenished. The solution to the transhipment problem lies in the fact that withdrawals from and compensating additions to the stockpiles are equivalent to transhipment. The stockpile sizes do not matter provided they are large enough to permit all desirable shipments which can reduce the cost (see Sec. 4). In the computation excessively large stockpiles are arbitrarily introduced. The excesses of stockpiles over amounts actually shipped drop out of the final solution (they appear as shipments from a point to itself at zero cost). The procedure is as follows:

Step 1. Problem formulation

Assume M points which are either shippers or receivers.

Let $g_i = g_1, g_2, \cdots, g_M$ be specified net amounts to be *shipped* by each point, where some g_i are positive and some negative, and $\sum (g_i \mid g_i > 0) = -\sum (g_i \mid g_i < 0)$, i.e. the total to be shipped is equal to the total to be received.

Let a_i = amount shipped by each point including transhipment and b_i = amount received by each point including transhipment; a_i and b_i are not yet specified but must satisfy:

$$g_i = a_i - b_i \qquad i = 1, 2, \cdots M \tag{1}$$

Let c_{ij} for $i \neq j$ be specified unit costs of shipment from point i to point j. All these c_{ij} are assumed > 0.

Let $c_{ii} = 0$ be the unit cost of shipping from a point to itself.

Step 2. Set up the transhipment problem in the form of a T_0 problem (without transhipment) in which there are M origins and M destinations. The amounts shipped, a_i, and the amounts received, b_i, must be $\geqq 0$. On the basis of the specified net amounts, g_i, the smallest value which can be used as a_i and b_i are a_i^0 and b_i^0 as defined by (2).

$$\begin{aligned} &\text{If } g_i > 0, \text{ set } a_i^0 = g_i, b_i^0 = 0 \\ &\text{If } g_i < 0, \text{ set } a_i^0 = 0, b_i^0 = |g_i| \end{aligned} \tag{2}$$

These a_i^0 and b_i^0 satisfy (1).

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(3) Let
$$\begin{cases} a_i' = a_i^0 + s \\ b_i' = b_i^0 + s \end{cases}$$

where s is a positive constant (a stock pile at each point).

The a_i' and b_i' defined by (3) are to be used in the computation. s is to be large enough to permit all possible transhipments, viz., by adding s to a_i^0, the total shipped, a_i', may include (a) the net amount, g_i, originally specified, (b) additional amounts received in the course of transhipment, and (c) redundant amounts which are shipped from the point to itself at zero cost.

It will be shown in Section 4 that s must be taken large enough that in the minimum cost solution all x_{ii} differ from zero. A suitable value for s, since it obviously introduces artificial stockpiles which equal any possible transhipments, is:

(4)
$$s = \sum (g_i \mid g_i > 0) = -\sum (g_i \mid g_i < 0) = \tfrac{1}{2} \sum_{i=1}^{m} |g_i|$$

Step 3. Compute the min. cost solution to the T_0 problem defined by (3) and the c_{ij}'s. Let C' be the total cost and x_{ij}' be shipments of the min. cost solution (all $x_{ii}' \neq 0$).

Step 4. The final x_{ii}' are redundant. Discard them as follows: Each x_{ii}' is contained in both the amount shipped, a_i', and the amount received, b_i', and may be deducted from both. Replace (3) by:

(5)
$$\begin{cases} a_i'' = a_i' - x_{ii}' \\ b_i'' = b_i' - x_{ii}' \end{cases}$$

Then a *feasible* solution to the T_0 problem given by (5) is:

(6)
$$\begin{cases} x_{ij}'' = x_{ij}' & \text{for } i \neq j \\ x_{ii}'' = 0 \end{cases}$$

where x_{ij}' are the values obtained by computation in Step 3.

Corresponding to any possible solution to the T_0 problem based on the a_i'' and b_i'' of (5) there is a feasible solution to (3) with the same cost, namely introduce the "diagonal" shipments x_{ii}'. Therefore, since C' is the Min. cost for (3), it must also be the Min. cost for (5).

Step 5. Convert the results in the form (5) and (6) to the solution to the transhipment problem. This conversion is straightforward since (5) gives the total amounts shipped and received at each point, and (6) gives the amounts shipped along each route. (See the example in the next section.)

The Min. cost solution involves $(M - 1)$ point-to-point paths for which $x_{ij} \neq 0$. (In degenerate cases the number of paths may be smaller.) It will be shown later that the Min. cost solution to the transhipment problem requires no more paths than would be required if only direct shipment were permitted.

3. Example of the Transhipment Problem

The direct shipment solution and the transhipment solution to a small problem will be compared. There are 5 shipping and receiving points involved. They will be denoted by Q, R, S, T, U.

TABLE IA

Direct Shipment Problem

		(Q)	(R)	(S)	a_i	
c_{ij}	→	6	4	6	5 (T)	amounts shipped
		8	6	5	6 (U)	
b_j		2	5	4		

amounts received

TABLE IB

Min. Cost Solution to the Direct Shipment Problem

		(Q)	(R)	(S)	a_i	
x_{ij}	→	0	5	0	5 (T)	Cost = 56
		2	0	4	6 (U)	
b_j		2	5	4		

The direct shipment problem, a T_0 problem, is shown by Table IA. The body of the table contains the unit shipping costs, c_{ij}, and the marginal numbers are the amounts to be shipped and received. The solution is shown in Table IB, where the body of the table contains the x_{ij} for minimum total cost. The simplex technique for T_0 was used to obtain the solution.

The transhipment problem is shown in Table IIA. The two shippers and three receivers of the direct shipment problem become five points which are both shippers and receivers. The a_i and b_j of Table I become

$$g_1 = -2 \text{ (for point } Q), g_2 = -5 \ (R), g_3 = -4 \ (S), g_4 = +5 \ (T), g_5 = +6 \ (U).$$

The a_i' and b_i' in Table II have been set up according to Eq. (3) with $s = 20$. The c_{ij}'s enclosed by bold rule are the same as in Table I. The other c_{ij}'s are additional unit cost information required when transhipment is permitted. All $c_{ii} = 0$. The minimum cost solution, obtained by Dantzig's method, is given in Table IIB.

Using (5) to eliminate the x_{ii} and to reduce the a_i' and b_i' accordingly to a_i'' and b_i'', the result is shown by Table IIC. The minimum cost of 56 in the direct shipment solution is reduced to 52 by the transhipment solution. If the problem were non-degenerate there would be four non-zero x_{ij}'s, i.e. four routes required, in both the direct shipment and the transhipment problems. In this example both forms of the problem turn out to be degenerate, requiring only three routes.

The transhipment routes are easily obtained from Table IIC. Point U ships 6 units to S. But S is to receive a net of 4, (b_3^0); therefore 2 units are available for transhipment. This provides the 2 units, (a_3''), which are shipped from S to Q. There appears to be no difficulty involved in carrying out this type of resolution of transhipment paths in larger problems.

TABLE IIA

Transhipment Problem

	Q	R	S	T	U	a_i'
$c_{ij} \rightarrow$	0	2	1	3	1	20 (Q)
	4	0	2	2	3	20 (R)
	1	5	0	3	2	20 (S)
	6	4	6	0	1	25 (T)
	8	6	5	2	0	26 (U)
b_i'	22 (Q)	25 (R)	24 (S)	20 (T)	20 (U)	

TABLE IIB

Min. Cost Solution to the Transhipment Problem

	Q	R	S	T	U	a_i'	
$x_{ij} \rightarrow$	20	0	0	0	0	20 (Q)	
	0	20	0	0	0	20 (R)	
	2	0	18	0	0	20 (S)	Cost = 52
	0	5	0	20	0	25 (T)	
	0	0	6	0	20	26 (U)	
b_i'	22 (Q)	25 (R)	24 (S)	20 (T)	20 (U)		

TABLE IIC

Final Solution to the Transhipment Problem

	Q	R	S	T	U	a_i''	a_i^0		
$x_{ij} \rightarrow$	0	0	0	0	0	0	0	(Q)	
	0	0	0	0	0	0	0	(R)	
	2	0	0	0	0	2	0	(S)	Cost = 52
	0	5	0	0	0	5	5	(T)	
	0	0	6	0	0	6	6	(U)	
b_i''	2	5	6	0	0				
b_i^0	2	5	4	0	0				
	(Q)	(R)	(S)	(T)	(U)				

4. Proof that the Procedure Reaches the Minimum

A proof follows that the procedure described above determines the minimum cost for the transhipment problem.

Step 1. The possible range of values of s for use in (3) is $0 \leqq s < +\infty$. For any s let $C(s)$ be the *minimum cost solution* to the T_0 problem specified by (3). For any $s_2 > s_1$ we have:

$$C(s_2) \leqq C(s_1) \tag{7}$$

Proof of (7): The minimum cost solution using s_1 becomes a feasible solution for s_2 by increasing all x_{ii} by $(s_2 - s_1)$. The change in the x_{ii} does not affect the cost, therefore $C(s_2)$ is certainly no larger than $C(s_1)$. $C(s_2)$ may however be less than $C(s_1)$.

Thus $C(s)$ is a monotone decreasing function. Assuming all $c_{ij} \geqq 0$, it is bounded from below, therefore it has a greatest lower bound. Let $C^* = \text{g.l.b. } C(s)$. The object of the computation is to find C^*, which is the minimum cost for the transhipment problem.

It will be proved that at some s the monotone decreasing function $C(s)$ becomes a constant; i.e. reaches C^*. The point is that $C(s)$ does not approach C^* asymptotically in any manner, e.g., as a curve or as a series of line segments which approach nearer and nearer to C^*, but actually reaches and remains at C^*. If the situation were asymptotic, the minimum cost could be approached but would not actually be reached.

Step 2. *A value of s can be found such that the minimum cost solution to (3) has all $x_{ii} \neq 0$*

Proof: It has been assumed that all $c_{ij} > 0$ for $i \neq j$. Choose a positive constant, c_0, such that $c_0 < c_{ij}$ for all these c_{ij}. Suppose for all s in $(0 \leqq s < \infty)$ that at least one $x_{ii} = 0$. Then, for any s we would have:

$$C(s) > c_0 s$$

since in some row where $x_{ii} = 0$ all costs for material shipped would be $> c_0$. Since s can be made arbitrarily large this would contradict (7), therefore for large s we have all $x_{ii} \neq 0$.

Step 3. Consider some s_1 which has the property specified by Step 2, that the Min. cost solution to (3) has all $x_{ii} \neq 0$. Then, for all $s > s_1$ we have:

$$C(s) = C(s_1) = C^* \tag{8}$$

Proof: Let $x_{ij}(s_1)$ be a Min. cost solution for s_1. A feasible solution for any $s_2 > s_1$ is:

$$\begin{cases} x_{ij}(s_2) = x_{ij}(s_1) \text{ for } i \neq j \\ x_{ii}(s_2) = x_{ii}(s_1) + (s_2 - s_1) \end{cases} \tag{9}$$

The theory of Dantzig's simplex technique for solution of T_0 can be used as follows to show that (9) is a Min. cost solution for s_2: Let u_i and v_i be the final set of u's and v's obtained in the simplex process in solving (3) using s_1; these

u_i and v_i satisfy $u_i + v_j \leqq c_{ij}$. The same set of u's and v's can be applied to the computation for s_2 and the relations $u_i + v_j \leqq c_{ij}$ are still satisfied since the c_{ij} are unchanged. Therefore the i, j locations of the Min. cost "basic solution" for s_2 are the same as for s_1. Since the x_{ij}'s of a "basic solution" are unique, (9) is the Min. cost solution for s_2.[4] By (9), $C(s_2) = C(s_1)$ since the only x_{ij} which differ are the x_{ii}, which do not affect the cost. Since s_2 is any $s > s_1$, and $C(s)$ is monotone decreasing, we have (8).

It has thus been shown that by choice of a large s the procedure for the transhipment problem gives the minimum cost. It can be seen intuitively that (4) provides a large enough value for s. Any s which leads to a solution to (3) for which all $x_{ii} \neq 0$ is satisfactory.

5. Number of Direct Routes

Consider a T_0 problem in which there are m origins and n destinations. The number of paths required in a basic Min. cost solution is $(m + n - 1)$ (in non-degenerate cases).

The corresponding transhipment problem T_1, deals with M points where $M = m + n$. The solution procedure makes use of M origins and M destinations, therefore leads initially to a solution involving $2M - 1$ paths. Of these, however, M are x_{ii} terms which are reduced to zero. Thus the final solution to T_1 makes use of $M - 1 = m + n - 1$ paths, which is the same as for the T_0 problem.

6. Optimum Linked Paths

It has been assumed in the preceding sections that unit costs are given for direct transportation from any point to any other. The c_{ij}'s might, for example, refer to the cost of non-stop airplane transportation. In general, however, many of the c_{ij}'s initially given would be for routes which involve transhipment, i.e. which pass through one or more en-route points. In the latter case the given data for i to j would be the unit cost, c_{ij}, and a route, $(i, i', i'', \cdots j)$. Whether the given c_{ij}'s are for direct shipments or involve transhipments, they are not necessarily the lowest possible costs for each of the links i to j. There is a least unit cost and some associated route from each of the M points to each of the others.

The lowest possible unit cost for each i to j, which is not known at the start of the problem, will be denoted by $\bar{c}_{ij}$. Let i_1 and j_1 be particular points for which $\bar{c}_{ij}$ is desired. To obtain $\bar{c}_{i_1j_1}$ one can solve a transhipment problem using the c_{ij}'s initially given

and
$$g_i = \begin{cases} 1 \text{ for } i = i_1 \\ -1 \text{ for } i = j_1 \\ 0 \text{ for all other } i \end{cases}$$

[4] The u_i and v_j of Dantzig's paper are a solution to the "dual scalar problem" described in the paper "Linear Programming and the Theory of Games" by Gale, Kuhn, and Tucker—Chapter XIX of "Activity Analysis of Production and Allocation", John Wiley and Co., 1951. The argument above could have been stated directly in terms of the duality properties of linear programming problems.

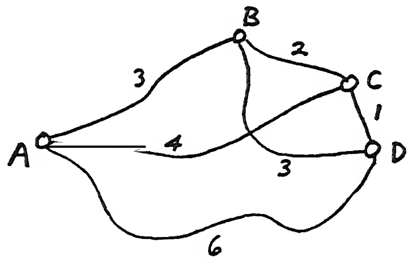

FIG. 1

TABLE IIIA

Distance Table for Point to Point Linkage

	A	B	C	D	a_i
A	0	3	4	6	11
B	3	0	2	3	10
C	4	2	0	1	10
D	6	3	1	0	10
b_i.......	10	10	10	11	

TABLE IIIB

Minimizing Solution x_{ij} Values

	A	B	C	D	a_i
A	10	0	1	0	11
B	0	10	0	0	10
C	0	0	9	1	10
D	0	0	0	10	10
b_i.......	10	10	10	11	

TABLE IIIC

Reduced Solution X_{ij} Values

	A	B	C	D	a_i
A	0	0	1	0	1
B	0	0	0	0	0
C	0	0	0	1	1
D	0	0	0	0	0
b_i........	0	0	1	1	

i.e. a transhipment problem in which the net shipment is $+1$ at i_1, -1 at j_1 and zero at all other points. The resulting $\bar{c}_{i_1j_1}$ will have associated with it a route $(i_1, i', i'', \cdots j_1)$.

Suppose all the best routes $\bar{c}_{ij}$ were found. Then if any set of net shipments, g_i, were specified, it would not be necessary to solve the transhipment problem using the c_{ij}'s initially given. One would, instead, set up a T_0 problem in which the shipments were $a_i = (g_i \mid g_i > 0)$, receipts were $b_j = (-g_i \mid g_i < 0)$, and use the least unit costs, $\bar{c}_{ij}$, already known, in place of the original c_{ij}'s. The solution would contain transhipment information in the form of the routes $(i, i', i'', \cdots j)$ associated with each $\bar{c}_{ij}$, but this would not enter into the computation. The transhipment problem computation would be reduced from M origins and M destinations to m origins and n destinations where $m + n = M$.

The situation is analogous to the relation between solution of simultaneous equations and matrix inversion. If one has a single set of simultaneous equations to solve, one does not invert the matrix. However, if there are many sets to solve, all involving the same matrix, then it pays to invert. Here, if one has a single transhipment problem to solve, one would not find all the $\bar{c}_{ij}$, but if there were many to solve, all with the same original c_{ij}'s, it would pay to find all the $\bar{c}_{ij}$'s initially.

The following is a small example of determination of an optimum linked path—the shortest path from one point in a network to another. The network is shown in Figure 1. We wish to find the shortest route from point A to point D. The direct route distances between each pair of points is shown in the diagram.

The amounts to be shipped are:

$$\begin{aligned} g_1 &= 1 \text{ for point A} \\ g_2 &= 0 \text{ for point B} \\ g_3 &= 0 \text{ for point C} \\ g_4 &= -1 \text{ for point D} \end{aligned}$$

As a fictitious stockpile to permit transhipment we can take $s = 10$. The amounts to be shipped and received in applying the transhipment technique become:

$$\begin{aligned} a_1 &= 11 & b_1 &= 10 \\ a_2 &= 10 & b_2 &= 10 \\ a_3 &= 10 & b_3 &= 10 \\ a_4 &= 10 & b_4 &= 11 \end{aligned}$$

Table IIIA shows the "cost table", using the distances from Figure 1 as the costs, and the marginal totals, a_i and b_i. Solution of a transportation problem based on Table IIIA yields the minimizing solution shown in Table IIIB. Upon deducting the diagonal values, x_{ii}, in Table IIIB from the marginal quantities, the final results appear in Table IIIC as the series of links: A to C (distance $= 4$), and C to D (distance $= 1$). The path A-C-D, for a total distance of 5, is the shortest route from A to D.

In networks with a large number of points the linear programming solution should be practical even when the number of combinatorial possibilities is immense.

II-2

ON A CLASS OF CAPACITATED TRANSPORTATION PROBLEMS*

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Transportation models (ordinary and transhipment) having certain types of capacity constraints on the flows between origins and destinations are studied from the point of view of transforming them into enlarged standardized (non-capacitated) models. Specifically constraints on the flow from disjoint and/or nested sets of origins to all destinations, and from any single origin to disjoint and/or nested sets of destinations are considered. Dual formulations are indicated for constraints on the flow to destinations from origins. In the case of a set of capacity constraints on the flow from each origin to each destination, the models proposed are easily seen to be of minimal dimension for any "standardized" version of such a capacitated problem.

I. Introduction

In this paper we consider techniques which transform transportation type problems subject to a certain class of capacity flow constraints into enlarged uncapacitated transportation problems. The simple capacitated Hitchcock problem in which each flow x_{ij} from origin O_i, $i = 1, 2, \cdots, m$, to destination D_j, $j = 1, 2, \cdots, n$, is bounded by a positive integer c_{ij} has been considered by several authors [1, 4, 6]. These previous papers have offered special and elegant algorithms for solving the problem; the methods are more involved than standard transportation simplex routines but do have the distinct advantage of utilizing a transportation tableau having a single row for each origin and a single column for each destination. Dantzig [3] has indicated how this model may also be viewed as an $(mn + m)$ row and $(mn + n)$ column ordinary transportation problem. We permit a somewhat wider class of capacity constraints. In the special case of the simple model just mentioned, our method yields an ordinary transportation problem with (mn) rows and $(m + n)$ columns.

There are two principal reasons why the approaches to be offered seem of interest. First, from the theoretical side, they provide (as does Dantzig's scheme in the simple case) the connection between the standard and several types of capacitated models, and thus demonstrate that the additional restrictions can be viewed conceptually as merely straightforward extensions of the familiar model[1]. Second, on the practical side, they allow a capacitated problem be to

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[1] The motivating idea may be contrasted to that of "secondary constraints" [2, 5, 8, 9]. The latter technique suggests the use of a subsystem of constraints which may yield an optimal feasible solution to the entire model and is computationally amenable by virtue of its abbreviated size; the technique herein suggests the use of an enlarged system which does yield an optimal solution to the original model and may be computationally amenable by virtue of its special form.

To demonstrate the legitimacy of the transformations, first one needs to show that any

solved by personnel familiar with the usual algorithm or by an automatic computer previously coded for the standard method without expending any new effort, provided of course that the dimensions of the enlarged problem do not become prohibitive. An ancillary reason for interest in the methods is that demonstration of the equivalence relationships may lead to the formulation of special algorithms utilizing the m row and n column tableau as have been offered for the solution of the simple capacitated model [1, 6].

In Sections II, III, and IV we consider the usual transportation model subject to a certain class of capacity constraints, the transformation schemes for conversion of the models, and applications to several examples. We extend our analysis to a transhipment model [7] in Sections V and VI.

II. Models of Capacitated Transportation Problems

The basic transportation model underlying our discussion in Sections II, III, and IV is

$$\text{(1a)} \qquad \text{minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}$$

subject to the constraints

$$\text{(1b)} \qquad \sum_{j=1}^{n} x_{ij} \leqq a_i \qquad i = 1, 2, \cdots, m$$

$$\text{(1c)} \qquad \sum_{i=1}^{m} x_{ij} \geqq b_j \qquad j = 1, 2, \cdots, n$$

$$\text{(1d)} \qquad x_{ij} \geqq 0$$

where the a_i and b_j are positive integers corresponding to the supply and demand at origin O_i and destination D_j, respectively, and x_{ij} is the shipment between these points at the unit cost d_{ij}.

It is convenient to put (1) into the canonical form

$$\text{(2a)} \qquad \text{minimize} \sum_{i=0}^{m} \sum_{j=0}^{n} d_{ij} x_{ij}$$

subject to the constraints

$$\text{(2b)} \qquad \sum_{j=0}^{n} x_{ij} = a_i \qquad i = 0, 1, \cdots, m$$

$$\text{(2c)} \qquad \sum_{i=0}^{m} x_{ij} = b_j \qquad j = 0, 1, \cdots, n$$

$$\text{(2d)} \qquad x_{ij} \geqq 0$$

---

feasible solution to the original capacitated problem is also a feasible solution to the enlarged model and has the same cost; consequently the value of the optimal solution of the transformed problem is a lower bound to the value of the optimal solution of the original model. Secondly one must demonstrate that given an optimal solution to the enlarged problem, there is a corresponding feasible solution to the original model having the same cost value.

$$\sum_{i=0}^{m} a_i = \sum_{j=0}^{n} b_j \tag{2e}$$

where

$$a_0 = \text{maximum}\left(0, \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i\right) \tag{2f}$$

$$b_0 = \text{maximum}\left(0, \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\right) \tag{2g}$$

$$d_{0j} = d_{i0} = 0. \tag{2h}$$

We have added a fictitious origin O_0 and destination D_0 which have the function of providing for the possible excess of demand over supply and of draining the possible excess of supply over demand, respectively. Since either demand exceeds supply, supply exceeds demand, or they are equal, no more than one of a_0 and b_0 will be strictly positive. In an uncapacitated problem, the inessential fictitious location(s) would be omitted; in the transformations imposing capacity constraints, we shall find it convenient to assume that both fictitious points exist. Once all the transformations have been applied, if either a_0 or b_0 (or both) equals zero, the corresponding location(s) may be eliminated from the model. We shall think of (2) as being arrayed in the familiar transportation tableau, each row of which corresponds to an origin and each column to a destination.

Letting I and J be a set of row and column indices, we associate with the positive integer $c_{I;J}$ the capacity constraint

$$\sum_{i \in I} \sum_{j \in J} x_{ij} \leqq c_{I;J}\,. \tag{3}$$

Throughout this paper we consider constraining relations of the forms

(i) multiple-row constraint: I contains more than one index and $J = \{1, 2, \cdots, n\}$

(ii) single-row constraint: I contains a single index and J is an arbitrary set of column indices

(iii) multiple-column constraint: J contains more than one index and $I = \{1, 2, \cdots, m\}$

(iv) single-column constraint: J contains a single index and I is an arbitrary set of row indices.

The simple capacity constraint (on a single flow x_{ij}) falls into categories (ii) and (iv). We say that constraint $c_{K;L}$ is nested within constraint $c_{I;J}$ if $I \supseteq K$ and $J \supseteq L$; we say that the two constraints are row (column) disjoint if I and K (J and L) have no row (column) indices in common.

III. Elementary Transformations

In this section we show how each of the constraining relations above in conjunction with model (2) may be transformed into the format of a standard transportation problem. We further indicate a manner for transforming certain *systems* of the above constraints, some of which may be nested within others, into an enlarged problem in canonical form.

1A. One multiple-row constraint. Assume that I contains a collection of k indices. We create a single (fictitious) new origin O^*_{m+1} and k new destinations $D^*_{n+1}, \cdots, D^*_{n+k}$, each associated with a particular $i \in I$. We describe the relation associating each $i \in I$ to one of the new destinations by the notation $j(i)$, and let

$$a_{m+1} = c_{I;1,2,\cdots,n} \tag{4a}$$

$$b_{j(i)} = a_i \qquad i \in I. \tag{4b}$$

In order that the capacity constraint be truly binding

$$\sum_{i \in I} a_i > c_{I;J}. \tag{4c}$$

If $b_0 = 0$, then total supply does not exceed total demand and (4c) would imply that a certain amount of infeasibility were being introduced by the constraint. If $b_0 > 0$, then the new constraint may or may not create infeasibility. In any case we make the revisions

$$a'_0 = a_0 - \text{minimum}\ (0, b_0 + c_{I;J} - \sum_{i \in I} a_i) \geqq 0 \tag{4d}$$

$$b'_0 = \text{maximum}\ (0, b_0 + c_{I;J} - \sum_{i \in I} a_i) \geqq 0 \tag{4e}$$

and note that if $a'_0 > a_0$, then we have formally avoided the problem of infeasibility by creating an addition to the fictitious supply at O_0.

Denoting an arbitrarily large positive number by M, we (i) revise the unit shipment costs so that

$$d_{i0} = M \qquad i \in I \tag{4f}$$

(ii) assign a zero unit cost to the entries at the intersection of O^*_{m+1} with D_0, $D^*_{n+1}, \cdots, D^*_{n+k}$, (iii) assign a zero unit cost to the entry at the intersection of the i-th row, $i \in I$, with $D_{j(i)}$, and (iv) assign the unit cost M to all other entries in the new row and columns.

The relationship between an optimal solution to the enlarged problem, say y_{ij}, and an optimal solution to the original capacitated problem is given by

$$x_{i0} = y_{ij(i)} \qquad i \in I \tag{5a}$$

$$x_{ij} = y_{ij} \qquad \begin{cases} i \in I, & j = 1, 2, \cdots, n \quad (5b) \\ i \notin I, & j = 0, 1, \cdots, n \quad (5c) \end{cases}$$

Note that

$$\sum_{j=1}^{n} x_{ij} = y_{m+1,j(i)} \qquad i \in I \tag{5d}$$

and that the correspondence in (5) is such that all shipments to D_j, $j = 1, 2, \cdots, n$, are found in column D_j of the optimal tableau of the enlarged problem.

Origins	Destinations							Availabilities
	D_0	D_1	D_2	D_3	D_4	D_5^*	D_6^*	
O_0	0	0	0	0	0	M	M	a'_0
O_1	M	d_{11}	d_{12}	d_{13}	d_{14}	0	M	a_1
O_2	M	d_{21}	d_{22}	d_{23}	d_{24}	M	0	a_2
O_3	0	d_{31}	d_{32}	d_{33}	d_{34}	M	M	a_3
O_4^*	0	M	M	M	M	0	0	$c_{1,\,2;\,1,\,2,\,3,\,4}$
Demands	b'_0	b_1	b_2	b_3	b_4	a_1	a_2	

Constraint: $c_{1,\,2;\,1,\,2,\,3,\,4}$

FIGURE 1

An example of the transformation applied to a three origin and four destination model is pictured in Figure 1.

1B. A disjoint set of multiple-row constraints. If there are several multiple-row constraints, every pair of which is row disjoint, the transformation in (1A) may be applied successively for each constraint.

1C. Nested multiple-row constraints. If one multiple-row constraint is nested within another multiple-row constraint, then the former may be considered as being transformed by (5d) into a capacity constraint on flows from the new origin O^*_{m+1} to certain of the new destinations. Consequently a method for handling nested multiple-row constraints will emerge from our method for handling capacity constraints on entries within a single row.

2A. A disjoint set of constraints on a single row[2]. If there are a set of h constraints on a single row $\bar{\imath}$, each pair of which is column disjoint, then we define h new origins $O^*_{m+1}, \cdots, O^*_{m+h}$, one corresponding to each constraint, and a single new destination D^*_{n+1}. We describe the relation associating each constraint to one of the new origins by the notation $i(J_l)$, $l = 1, 2, \cdots, h$, and let

$$a_{i(J_l)} = c_{\bar{\imath};J_l} \qquad l = 1, 2, \cdots, h \tag{6a}$$

$$b_{n+1} = \sum_{l=1}^{h} c_{\bar{\imath};J_l} \tag{6b}$$

We (*i*) revise the unit costs of shipment so that

$$d_{\bar{\imath},j} = M \qquad j \in J_l, \qquad l = 1, 2, \cdots, h \tag{6c}$$

(ii) assign a zero unit cost to the entries at the intersection of O_i, $O^*_{m+1}, \cdots, O^*_{m+h}$, with D^*_{n+1}, (iii) assign the cost

$$d_{i(J_l),j} = d_{\bar{\imath},j} \qquad j \in J_l, \qquad l = 1, 2, \cdots, h \tag{6d}$$

and (iv) assign a unit cost M to all other entries in the new rows and column.

[2] The condition of one constraint on a single row is a special case of this category.

Origins	Destinations						Availabilities
	D_0	D_1	D_2	D_3	D_4	$D_5{}^*$	
O_0	0	0	0	0	0	M	a_0
O_1	0	M	M	M	d_{14}	0	a_1
O_2	0	d_{21}	d_{22}	d_{23}	d_{24}	M	a_2
$O_3{}^*$	M	d_{11}	M	M	M	0	$c_{1;\,1}$
$O_4{}^*$	M	M	d_{12}	d_{13}	M	0	$c_{1;\,2,\,3}$
Demands	b_0	b_1	b_2	b_3	b_4	$c_{1;\,1} + c_{1;\,2,\,3}$	

Constraints: $c_{1;\,1}$ and $c_{1;\,2,\,3}$

FIGURE 2

Origins	Destinations							Availabilities
	D_0	D_1	D_2	D_3	D_4	$D_5{}^*$	$D_6{}^*$	
O_0	0	0	0	0	0	M	M	a_0
O_1	0	M	M	M	d_{14}	0	M	a_1
O_2	0	d_{21}	d_{22}	d_{23}	d_{24}	M	M	a_2
$O_3{}^*$	M	d_{11}	M	M	M	0	0	$c_{1;\,1,\,2,\,3}$
$O_4{}^*$	M	M	d_{12}	d_{13}	M	M	0	$c_{1;\,2,\,3}$
Demands	b_0	b_1	b_2	b_3	b_4	$c_{1;\,1,\,2,\,3}$	$c_{1;\,2,\,3}$	

Constraints: $c_{1;\,1,\,2,\,3}$ and $c_{1;\,2,\,3}$

FIGURE 3

The relationship between the optimal solution to the enlarged problem, y_{ij}, and the optimal solution to the original capacitated problem is given by

(7a) $$x_{\bar{i}j} = y_{i(J_l),j} \qquad j \in J_l, \qquad l = 1, 2, \cdots, h$$

(7b) $$x_{ij} = y_{ij} \qquad i \neq \bar{i}.$$

As in the multiple-row transformations, the correspondence in (7) is such that all shipments to D_j, $j = 1, 2, \cdots, n$, are found in column D_j of the optimal tableau of the enlarged problem.

An example of a single-row capacitated model with two origins and four destinations is pictured in Figure 2.

2B. A disjoint set of single-row constraints nested within another single-row constraint. The transformation (2A) is such that for each constraint $c_{\bar{i};\,J_l}$ a new row $i(J_l)$ is defined in which by (7a) there is a one-to-one correspondence between the original $x_{\bar{i}j}$, $j \in J_l$, and the $y_{i(J_l),j}$. Consequently a nested constraint $c_{\bar{i};J_{l^*}}$, where by definition $J_{l^*} \subset J_l$, may be considered as being transformed into a constraint on a single-row $i(J_l)$ and handled accordingly. Similarly a set of column-disjoint constraints nested within the constraint $c_{\bar{i};J_l}$ may be handled as a disjoint set of constraints on a single row $i(J_l)$.

Origins	Destinations					Availabilities
	D_0	D_1	D_2	$\cdots$	D_{12}	
O_1	0	d_{11}	d_{12}	$\cdots$	$d_{1,12}$	a_1
O_2	0	d_{21}	d_{22}	$\cdots$	$d_{2,12}$	a_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
O_7	0	d_{71}	d_{72}	$\cdots$	$d_{7,12}$	a_7
Demands	b_0	b_1	b_2	$\cdots$	b_{12}	

Assumption: $a_0 = 0$

Constraints

1. $c_{1;\,1,\,2}$
2. $c_{1;\,3}$
3. $c_{1;\,5,\,7}$
4. $c_{1;\,8,\,9,\,10,\,11}$
5. $c_{1;\,9,\,10,\,11}$
6. $c_{1;\,10,\,11}$
7. $c_{2,\,3,\,4,\,5;\,1,\,2,\,\ldots\,,\,12}$
8. $c_{2,\,3;\,1,\,2,\,\ldots\,,\,12}$
9. $c_{4;\,2,\,3,\,4}$
10. $c_{4;\,2}$
11. $c_{4;\,7,\,9}$
12. $c_{4;\,12}$

FIGURE 4a

Figure 3 gives a two origin and four destination model with a nested constraint.

2C. A collection of sets of single-row constraints. If more than one origin has a set of single-row capacity constraints on the shipments therefrom, the transformations (2A) and (2B) may be performed successively by taking, for example, one row at a time and by exhausting all of the constraints upon it before advancing to the next row.

3. A collection of column constraints. It is left as an "exercise" for the interested reader to verify that single and multiple column capacity constraints may be handled by transformations analogous to the ones above by interchanging the rôle played by the additional fictitious origins and destinations (including O_0 and D_0).

4. A collection of row and column constraints. As we noticed in the suggested transformations for single and multiple row constraints, the shipments to any destination D_j, $j = 1, 2, \cdots, n$, are always to be found in the enlarged problem in column D_j, and furthermore no "fictitious" shipments to D_j are introduced in the transformations[3]. Thus if a collection of row and column constraints is imposed on (2), we may proceed first to make all necessary row transformations, and subsequently to make all necessary column transformations.

IV. Examples

In Figure 4 we trace the development of a standardized transportation model stemming from a model with seven origins and twelve destinations, subject to

[3] The reader will recall that in adding new origins, either the entry at the intersection of the new origin with D_j was assigned a unit cost of M, prohibiting the use of the route in an optimal (feasible) solution, or was assigned a unit cost d_{ij} that had been "displaced" from another entry in D_j where the unit cost had been changed to M.

Origins	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}^*	Availabilities
O_1	0	M	M	M	d_{14}	M	d_{16}	M	M	M	M	M	$d_{1,12}$	0	a_1
O_2	0	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	$d_{2,10}$	$d_{2,11}$	$d_{2,12}$	M	a_2
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
O_7	0	d_{71}	d_{72}	d_{73}	d_{74}	d_{75}	d_{76}	d_{77}	d_{78}	d_{79}	$d_{7,10}$	$d_{7,11}$	$d_{7,12}$	M	a_7
O_8^*	M	d_{11}	d_{12}	M	M	M	M	M	M	M	M	M	M	0	$c_{1;1,2}$
O_9^*	M	M	M	d_{13}	M	M	M	M	M	M	M	M	M	0	$c_{1;3}$
O_{10}^*	M	M	M	M	M	d_{15}	M	d_{17}	M	M	M	M	M	0	$c_{1;5,7}$
O_{11}^*	M	M	M	M	M	M	M	M	d_{18}	d_{19}	$d_{1,10}$	$d_{1,11}$	M	0	$c_{1;8,9,10,11}$
Demands	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	

(Destinations: D_0 to D_{13}^*)

Constraints Number 1, 2, 3, and 4

$b_{13} = c_{1;1,2} + c_{1;3} + c_{1;5,7} + c_{1;8,9,10,11}$

FIGURE 4b

Origins	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}^*	D_{14}^*	D_{15}^*	Availabilities
O_1	0	M	M	M	d_{14}	M	d_{16}	M	M	M	M	M	$d_{1,12}$	0	M	M	a_1
O_2	0	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	$d_{2,10}$	$d_{2,11}$	$d_{2,12}$	M	M	M	a_2
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
O_{10}^*	M	M	M	M	M	d_{15}	M	d_{17}	M	M	M	M	M	0	M	M	$c_{1;5,7}$
O_{11}^*	M	M	M	M	M	M	M	M	d_{18}	M	M	M	M	0	0	M	$c_{1;8,9,10,11}$
O_{12}^*	M	M	M	M	M	M	M	M	M	d_{19}	M	M	M	M	0	0	$c_{1;9,10,11}$
O_{13}^*	M	M	M	M	M	M	M	M	M	M	$d_{1,10}$	$d_{1,11}$	M	M	M	0	$c_{1;10,11}$
Demands	b_0	b_1	b_3	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	

(Destinations: D_0 to D_{15}^*)

Constraints Number 5 and 6

$b_{14} = c_{1;9,10,11}$

$b_{15} = c_{1;10,11}$

FIGURE 4c

Origins	Destinations								Availabilities
	D_0	D_1	...	D^*_{15}	D^*_{16}	D^*_{17}	D^*_{18}	D^*_{19}	
O_1	0	M	...	M	M	M	M	M	a_1
O_2	M	d_{21}	...	M	0	M	M	M	a_2
O_3	M	d_{31}	...	M	M	0	M	M	a_3
O_4	M	d_{41}	...	M	M	M	0	M	a_4
O_5	M	d_{51}	...	M	M	M	M	0	a_5
...	...	...	...	...	...	...	...	...	...
$O_{13}{}^*$	M	M	...	0	M	M	M	M	$c_{1;\,10,\,11}$
$O_{14}{}^*$	0	M	...	M	0	0	0	0	$c_{2,\,3,\,4,\,5;\,1,\,2,\,\ldots,\,12}$
Demands	b'_0	b_1	...	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	

Constraint Number 7
Under Assumption:

$$b'_0 = b_0 + c_{2,\,3,\,4,\,5;\,1,\,2,\,\ldots,\,12} - (a_2 + a_3 + a_4 + a_5) \geqq 0$$
$$b_{16} = a_2$$
$$b_{17} = a_3$$
$$b_{18} = a_4$$
$$b_{19} = a_5$$

FIGURE 4d

Origins	Destinations									Availabilities
	D_0	D_1	...	D^*_{15}	D^*_{16}	D^*_{17}	D^*_{18}	D^*_{19}	D^*_{20}	
O_1	0	M	...	M	M	M	M	M	M	a_1
...	...	...	...	...	...	...	...	...	...	...
$O_{13}{}^*$	M	M	...	0	M	M	M	M	M	$c_{1;\,10,\,11}$
$O_{14}{}^*$	0	M	...	M	M	M	0	0	0	$c_{2,\,3,\,4,\,5;\,1,\,2,\,\ldots,\,12}$
$O_{15}{}^*$	M	M	...	M	0	0	M	M	0	$c_{2,\,3;\,1,\,2,\,\ldots,\,12}$
Demands	b'_0	b_1	...	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	

Constraint Number 8

$$b_{20} = c_{2,\,3;\,1,\,2,\,\ldots,\,12}$$

FIGURE 4e

twelve row constraints. In Figure 4a we give the model (2) and a statement of the twelve constraints. We first use transformation (2A) to account for constraints Number 1, 2, 3, and 4, Figure 4b. Constraints Number 5 and 6 are added by transformation (2B) in Figure 4c. The multiple-row constraint Number 7 is added by transformation (1A) in Figure 4d, and the nested multiple-row constraint Number 8 is included by transformation (1C) in Figure 4e. Transformations (2A) and (2B) are used to impose constraints Number 9, 10, 11, and 12, Figure 4f.

Origins	Destinations																Availabilities
	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	$\cdots$	D_{20}^*	D_{21}^*	D_{22}^*	
O_1	0	M	M	M	d_{14}	M	d_{16}	M	M	M	M	M	$d_{1,12}$	$\cdots$	M	M	M	a_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
O_4	M	d_{41}	M	M	M	d_{45}	d_{46}	M	d_{48}	M	$d_{4,10}$	$d_{4,11}$	M	$\cdots$	M	0	M	a_4
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
O_{15}^*	M	M	M	M	M	M	M	M	M	M	M	M	M	$\cdots$	0	M	M	$c_{2,3;1,2,\ldots,12}$
O_{16}^*	M	M	M	d_{43}	d_{44}	M	M	M	M	M	M	M	M	$\cdots$	M	0	0	$c_{4;2,3,4}$
O_{17}^*	M	M	M	M	M	M	M	d_{47}	M	d_{49}	M	M	M	$\cdots$	M	0	M	$c_{4;7,9}$
O_{18}^*	M	M	M	M	M	M	M	M	M	M	M	M	$d_{4,12}$	$\cdots$	M	0	M	$c_{4;12}$
O_{19}^*	M	M	d_{42}	M	M	M	M	M	M	M	M	M	M	$\cdots$	M	M	0	$c_{4;2}$
Demands	b'_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	$\cdots$	b_{20}	b_{21}	b_{22}	

Constraints 9, 10, 11, and 12

$b_{21} = c_{4;2,3,4} + c_{4;7,9} + c_{4;12}$

$b_{22} = c_{4;2}$

FIGURE 4f

Origins	Destinations					Availabilities
	D_0	D_1	D_2	D_3^*	D_4^*	
O_0	0	0	0	M	M	a_0
O_1	0	M	M	d_{11}	d_{12}	a_1
O_2	0	M	M	d_{21}	d_{22}	a_2
O_3	0	d_{31}	d_{32}	M	M	a_3
O_4	0	d_{41}	d_{42}	M	M	a_4
O_5^*	M	0	M	0	M	$c_{1,2;1}$
O_6^*	M	M	0	M	0	$c_{1,2;2}$
Demands	b_0	b_1	b_2	$c_{1,2;1}$	$c_{1,2;2}$	

Constraints: $c_{1,2;1}$ and $c_{1,2;2}$

FIGURE 5

A second example having four origins and two destinations is explored in Figure 5. Suppose the origins O_1 and O_2 supply a somewhat "inferior" item to that available at the remaining origins, and as a consequence, each destination D_j must not receive more than $c_{1,2;j}$ of product from O_1 and O_2. We thus have a set of single-column capacity constraints leading to the enlarged model in Figure 5.

As a third illustration, we consider the simple capacitated Hitchcock problem. Since a capacity on a single x_{ij} may be viewed as a single-row *or* a single-column constraint, we have two types of transformations from which to choose. Figure 6 shows a two origin and three destination model and the associated enlarged problems under the alternatives that the simple constraints are handled as row or column limitations. Note the condition that each x_{ij} is constrained allows our modifying the tableau resulting from transformation (2C) so as to eliminate the original set of O_i, $i = 1, 2, \cdots, m$, or D_j, $j = 1, 2, \cdots, n$. In general if the capacities are viewed as either single-row or single-column constraints, the resultant standardized model consists of a total $(m + mn + n + 2)$ rows and columns[4]. This number may be reduced by one as either a_0 or b_0 (or both) equals zero. It is noteworthy that both the row and column formulation are of minimal dimension for a *standardized* simple capacitated transportation model as the underlying mathematical model consists of $(m + mn + n + 1)$ restrictions[5].

[4] In the single-row interpretation, we are assuming

$$\sum_{j=1}^{n} c_{ij} \geqq a_i .$$

If the opposite inequality holds, we revise a_i to

$$a'_i = \sum_{j=1}^{n} c_{ij}$$

and eliminate destination D^*_{n+i}. A similar remark holds for the single-column interpretation.

[5] As we noted previously, Dantzig's transformation results in a standardized model having a total number of rows and columns of the order $(m + 2mn + n)$.

Origins	Destinations						Availabilities
	D_0	D_1	D_2	D_3	D_4^*	D_5^*	
O_0	0	0	0	0	M	M	a_0
O_3^*	0	d_{11}	M	M	0	M	c_{11}
O_4^*	0	M	d_{12}	M	0	M	c_{12}
O_5^*	0	M	M	d_{13}	0	M	c_{13}
O_6^*	0	d_{21}	M	M	M	0	c_{21}
O_7^*	0	M	d_{22}	M	M	0	c_{22}
O_8^*	0	M	M	d_{23}	M	0	c_{23}
Demands	b_0	b_1	b_2	b_3	$\sum_{j=1}^{3} c_{1j} - a_1$	$\sum_{j=1}^{3} c_{2j} - a_2$	

Simple Capacitated Hitchcock Model—Single Row Interpretation
FIGURE 6a

Origins	Destinations							Availabilities
	D_0	D_4^*	D_5^*	D_6^*	D_7^*	D_8^*	D_9^*	
O_0	0	0	0	0	0	0	0	a_0
O_1	0	d_{11}	M	d_{12}	M	d_{13}	M	a_1
O_2	0	M	d_{21}	M	d_{22}	M	d_{23}	a_2
O_3^*	M	0	0	M	M	M	M	$\sum_{i=1}^{2} c_{i1} - b_1$
O_4^*	M	M	M	0	0	M	M	$\sum_{i=1}^{2} c_{i2} - b_2$
O_5^*	M	M	M	M	M	0	0	$\sum_{i=1}^{2} c_{i3} - b_3$
Demands	b_0	c_{11}	c_{21}	c_{12}	c_{22}	c_{13}	c_{23}	

Simple Capacitated Hitchcock Model—Single Column Interpretation
FIGURE 6b

V. Model of a Capacitated Transhipment Problem

Orden [7] has considered an extension of the transportation model to allow for the transhipment of the resource through locations, i.e., a location may both receive and ship amounts of the resource. We assume that there are locations L_g, $g = 1, 2, \cdots, p$, to which are associated integral numbers r_g; a positive r_g signifies that location g has that net amount of available resource for shipment elsewhere, and a negative r_g signifies that location g has that net requirement of the resource to be shipped from elsewhere. For the sake of simplicity of exposition, we assume that a transhipment may be made through any L_g; the reader may verify that if some locations are restricted to be origins only and others to be destinations only, the transportation tableau is easily altered by removing the corresponding columns and rows, respectively.

Analogous to (2f) and (2g), we introduce an artificial location L_0 with an associated

$$r_0 = -\sum_{g=1}^{p} r_g. \tag{8a}$$

That is, if the total available resources exceed the total requirements, location L_0 has a net demand for the excess; and if total requirements exceed the total available resources, L_0 provides a fictitious supply for the deficiency. The mathematical model may be summarized as

$$\text{minimize} \sum_{g'=0}^{p} \sum_{g=0}^{p} d_{g'g} x_{g'g} \tag{8b}$$

subject to the constraints

$$\sum_{g'=0}^{p} x_{gg'} - \sum_{g'=0}^{p} x_{g'g} = r_g \qquad g = 0, 1, \cdots, p \tag{8c}$$

$$x_{g'g} \geqq 0 \tag{8d}$$

where

$$d_{g'g} = 0 \quad \text{if } g = g' \tag{8e}$$

$$d_{0g} = d_{g'0} = 0. \tag{8f}$$

Letting

$$r_g^+ = \text{maximum } (0, r_g) \geqq 0 \tag{9a}$$

$$r_g^- = \text{minimum } (0, r_g) \leqq 0 \tag{9b}$$

we define

$$s = \sum_{g=0}^{p} r_g^+. \tag{10a}$$

We may interpret s as the maximum amount of the resource which would ever be transhipped through any location.[6] Following Orden's presentation, we transform (8) into the following standardized transportation model

$$\text{minimize} \sum_{i=0}^{p} \sum_{j=0}^{p} d_{ij} x_{ij} \tag{10b}$$

subject to the constraints

$$\sum_{j=0}^{p} x_{0j} = \text{maximum } (0, r_0) = a_0 \tag{10c}$$

$$\sum_{j=0}^{p} x_{ij} = s + r_i^+ = a_i \qquad i = 1, 2, \cdots, p \tag{10d}$$

[6] More specifically, we are assuming that s represents the total amount of the resource available in the entire system. If there are additional buffer stocks which may be used in transhipments, then s should be increased by the corresponding amount.

(10e) $$\sum_{i=0}^{p} x_{i0} = -\text{minimum } (0, r_0) = b_0$$

(10f) $$\sum_{i=0}^{p} x_{ij} = s - r_j^- = b_j \qquad j = 1, 2, \cdots, p$$

(10g) $$x_{ij} \geqq 0$$

An example of a four location model is exhibited in Figure 7.

VI. Transformations for the Transhipment Model

Having put the transhipment problem into a standardized model (10), we are ready to examine to what extent the techniques given in Section III may be applied. Clearly from a formal point of view, there are no difficulties in execut-

From	To					Availabilities
	L_0	L_1	L_2	L_3	L_4	
L_0	0	0	0	0	0	max $(0, r_0)$
L_1	0	0	d_{12}	d_{13}	d_{14}	$s + r_1^+$
L_2	0	d_{21}	0	d_{23}	d_{24}	$s + r_2^+$
L_3	0	d_{31}	d_{32}	0	d_{34}	$s + r_3^+$
L_4	0	d_{41}	d_{42}	d_{43}	0	$s + r_4^+$
Demands	$-\min(0, r_0)$	$s - r_1^-$	$s - r_2^-$	$s - r_3^-$	$s - r_4^-$	

Transhipment Model

FIGURE 7

From	To					Availa-bilities
	L_0	L_1	L_2	L_3	L_4	
L_5^*	0	0	d_{12}	M	M	c_{12}
L_6^*	0	0	M	d_{13}	M	c_{13}
L_7^*	0	0	M	M	d_{14}	c_{14}
L_8^*	0	d_{21}	0	M	M	c_{21}
L_9^*	0	M	0	d_{23}	M	c_{23}
L_{10}^*	0	M	0	M	d_{24}	c_{24}
L_{11}^*	0	d_{31}	M	0	M	c_{31}
L_{12}^*	0	M	d_{32}	0	M	c_{32}
L_{13}^*	0	M	M	0	d_{34}	c_{34}
L_{14}^*	0	d_{41}	M	M	0	c_{41}
L_{15}^*	0	M	d_{42}	M	0	c_{42}
L_{16}^*	0	M	M	d_{43}	0	c_{43}
Demands	b_0	$\sum_{j\neq 1} c_{1j} - r_1$	$\sum_{j\neq 2} c_{2j} - r_2$	$\sum_{j\neq 3} c_{3j} - r_3$	$\sum_{j\neq 4} c_{4j} - r_4$	

Simple Capacitated Transhipment Model—Single Row Interpretation
Assumption: $a_0 = 0$

FIGURE 8

ing the standardizing methods. But our definition of a multiple-row or multiple-column constraint (viz., one that extends over all columns or rows, except O_0 and D_0) is probably of little practical significance in the transhipment model as the constraint would be taken to include the fictitious shipments x_{gg} from a location to itself. The single-row and single-column transformations do apply without any loss of significance in meaning.

In the special case of a simple capacitated transhipment model (in which there is a capacity constraint on each flow x_{ij}, $i \neq j$) the model may be described by a transportation tableau having a total of $(p^2 + 1)$ rows and columns,[7] which is of minimal dimension for a standardized capacitated transhipment model. Figure 8 illustrates the tableau for a four location example.

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[7] We are employing the same assumptions as in footnote 4.

II–3

A SUGGESTED COMPUTATION FOR MAXIMAL MULTI-COMMODITY NETWORK FLOWS*

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A simplex computation for an arc-chain formulation of the maximal multi-commodity network flow problem is proposed. Since the number of variables in this formulation is too large to be dealt with explicitly, the computation treats non-basic variables implicitly by replacing the usual method of determining a vector to enter the basis with several applications of a combinatorial algorithm for finding a shortest chain joining a pair of points in a network.

1. Introduction

A problem of some importance in applications of linear programming is the determination of maximal multi-commodity flows in networks. For example, some of the linear programming problems which have been proposed recently by Juncosa and Kalaba in their studies of communication networks [5] can be cast in this form. Straightforward application of the simplex method to such problems is usually not feasible, since even small networks may generate linear programs which are too large for present machine capacity. What is needed are specialized computing schemes that take advantage of the structure of such problems. For the single commodity case, various easy computations are known [1, 3, 4], but the multi-commodity problem has remained relatively unexplored.

Consideration of simple examples makes it appear that the multi-commodity flow problem is considerably more complex than the single commodity one. Certainly the nice combinatorial features of the single commodity case are lost in the generalization—simplex bases (for any formulation of the problem known to us) are not triangular, hence addition and subtraction do not suffice to solve such problems by the simplex method, the max flow min cut theorem, true for single commodity networks, is false [3], and no simple-minded modification of the labeling process [4] seems to work.

The purpose of this note is to suggest a computation which makes some use of the structure of one formulation of the multi-commodity problem within the framework of a simplex computation. For this particular formulation, the matrix of the linear program is the incidence matrix of arcs vs. all chains joining sources and sinks for the various commodities, and thus the number of variables is too large to be dealt with explicitly. The suggested computation treats non-basic variables implicitly by replacing the "pricing" operation of the simplex method (i.e. the determination of a vector to enter the basis) with several applications of a combinatorial algorithm for finding a shortest chain joining a pair of points in a network.

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2. Arc-chain Formulation

Let $A_1, \cdots, A_m$ be a list of the arcs of the network, $C_1, \cdots, C_n$ a list of all chains that join, for the various commodities, all the sources for a commodity with all sinks for the same commodity, and let $A = (a_{rs})$ be the $m \times n$ incidence matrix of arcs vs. commodity chains:

$$a_{rs} = \begin{cases} 1 \text{ if } C_s \text{ contains } A_r, \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

Thus, for example, if the network is that of Fig. 1, with sources P_1, P_2, sink P_3 for one commodity, and source P_4, sink P_1 for a second commodity, the matrix A is as shown in Fig. 2.

If we let x_s, $s = 1, \cdots, n$, denote the amount of commodity flow along C_s, and b_r the flow capacity of A_r, then the multi-commodity maximal flow problem is represented by the linear program:

$$\text{maximize} \sum_{s=1}^{n} x_s \tag{2}$$

subject to the constraints

$$\sum_{s=1}^{n} a_{rs} x_s + x_{n+r} = b_r, \qquad x_1, \cdots, x_{n+r} \geqq 0. \tag{3}$$

The assumption in (2) that commodities are valued equally is not essential to the method we propose, as will be clear from our discussion in the following section. Another thing we wish to point out is that it is immaterial whether the

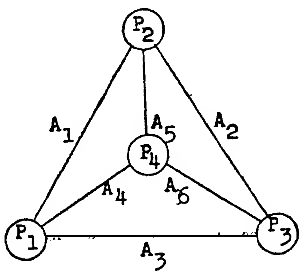

FIG. 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
A_1			1		1			1		1		1			1
A_2			1	1		1								1	1
A_3	1							1	1				1	1	
A_4		1		1					1	1	1				
A_5				1	1		1		1			1		1	
A_6		1			1		1			1			1		1

Commodity 1 (C_1–C_{10}) Commodity 2 (C_{11}–C_{15})

FIG. 2

problem involves directed or undirected arcs. Thus, for example, if there are "one-way streets," or if, in a communication network, say, it is desired to place an upper bound on the number of messages that can be transmitted from P_i to P_j, and an upper bound on the messages that can be sent from P_j to P_i, one considers two arcs, one from P_i to P_j, the other from P_j to P_i, and directed chains from sources to sinks.

Since the number of chains is usually very large in practical applications, the arc-chain formulation of the problem might seem to be impossible to deal with computationally. Indeed, the enumeration of all chains from commodity sources to sinks in a network of moderate size would be a lengthy task, to say the least. Fortunately, there is no need to write down the entire matrix A, since the selection of a variable entering the basic set at any stage of the simplex computation (or the recognition that a basis is optimal) can be accomplished without explicit knowledge of the non-basic column vectors of A. All we need is the basis $B = (b_{rj})$ (or its inverse), a square submatrix whose order is the number m of arcs in the network, to compute the simplex multipliers α_r $(r = 1, \cdots, m)$ satisfying, for $j = s_1, \cdots, s_m$,

$$\sum_{r=1}^{m} \alpha_r b_{rj} = \begin{cases} 1 & \text{if } j \leqq n \\ 0 & \text{if } j > n. \end{cases} \tag{4}$$

We can then find a vector to bring into the basis (or prove that the current basis is optimal) by the method of the next section. Once such a vector has been found, determination of the vector leaving the basis is accomplished in the usual way.

3. A Shortest Chain Algorithm

Suppose we have computed the α_r in (4) corresponding to a particular basis B. If some α_r is negative, then the variable x_{n+r} may be introduced into the basic set with possibly an increase in the form (2), that is, the unit vector having 1 in the r-th position, zeros elsewhere, can be brought into the basis. (It may be that this vector also represents a one-arc chain for some commodity; in this case, a bigger increase in (2) might result, of course, by taking the latter interpretation.)

Assume, therefore, that a stage has been reached in the computation where all α_r are non-negative. In this case, the algorithm described below, which makes no use of the full incidence matrix A, can be used either to locate a column vector of A (i.e. a commodity chain in the network), that may be brought into the basis, or to prove that the current basis is optimal.

Let us interpret the α_r as lengths of the arcs. We wish to find a chain C_s, if one exists, whose length

$$\sum_{r=1}^{m} \alpha_r a_{rs}$$

is less than one, the coefficient of x_s in (2). Thus, it suffices to locate, for each commodity, a shortest chain from the commodity sources to its sinks. If each

of the chains thus selected has length at least one, the basis is optimal. Otherwise, a column vector of A corresponding to one of these chains may be introduced into the basis.

The problem of locating a shortest chain from one set of nodes to another set of nodes in a network can be reduced to a standard transshipment problem [6], and may consequently be solved in various simple ways; see [2] and [6], for example. The algorithm we describe is that of [2]. (In [2], the problem is considered to be that of finding a shortest chain from one node to another; to reduce our problem to this one, simply join each node of the first set to a new node by an arc of length zero, and similarly for the other set. We shall give a description which does not involve this device explicitly, however.)

Let the set of sources for one commodity be S, the sinks for the commodity T, and suppose the nodes of the network are $P_1, \cdots, P_N$. Let l_{ij} denote the length of the arc joining P_i and P_j, i.e. if the arc A_r joining P_i and P_j corresponds to the simplex multiplier α_r, set $l_{ij} = \alpha_r$. (If arcs are directed, then we let l_{ij} denote the multiplier corresponding to the arc from P_i to P_j, hence in this case we may have $l_{ij} \neq l_{ji}$, whereas in the undirected case, $l_{ij} = l_{ji}$.) Initially assign to each node P_i a number π_i as follows:

$$\pi_i = \begin{cases} 0 & \text{for } P_i \in S \\ \infty & \text{otherwise.} \end{cases}$$

Now scan the network for an arc P_iP_j such that

$$\pi_i + l_{ij} < \pi_j .$$

Replace π_j by $\pi_i + l_{ij}$ if such an arc is found. Continue this process. Eventually no such arcs can be found; then the number π_i represents the length of a shortest chain from S to P_i, for all i. In particular, the smallest π_i, for $P_i \in T$, is the length of a shortest chain from S to T. Let π_k be the smallest such. To find a chain from S to T of length π_k, look for an arc P_jP_k such that $\pi_j + l_{jk} = \pi_k$, then search for an arc P_iP_j such that $\pi_i + l_{ij} = \pi_j$, and so on. Eventually a node of S is reached, and the desired chain has been traced out (in reverse).

If in the process of locating shortest chains from commodity sources to sinks, for the various commodities, one is found of length less than one, we recommend that the corresponding column vector of A be introduced into the basis immediately, rather than repeating the shortest chain algorithm a number of times in order to use the usual criterion for selection of a vector to enter the basis.

We point out that the reason for getting rid of negative multipliers α_r before using the shortest chain algorithm is that the algorithm may not work if arcs have negative lengths.

To start the simplex computation, one can of course begin with the basic variables $x_{n+1}, \cdots, x_{n+r}$, corresponding to the zero flow.

4. Concluding Remarks

Except for hand computation of a few small problems, we have no computational experience with the proposed method. Whether the method is practicable

for a problem involving, say, 50 nodes, 100 arcs, and 20 commodity source-sink sets $S_1, T_1, \cdots, S_{20}, T_{20}$, is a question which can be settled only by experimentation. It would certainly be more practicable in this case than straightforward application of the simplex method to a node-arc formulation of the problem, since in the latter formulation there would be roughly 1100 equations in 2100 variables, and hence the basis matrices would be much too large, whereas in the suggested method, the basis matrices would be 100×100, and at most 20 applications of the shortest chain algorithm would be necessary on each simplex iteration. How many simplex iterations might be required is another matter, though. The incidence matrix A for such a problem could have many thousands of columns. On the other hand, there would probably be many column vectors of A dominated by others, in the sense that, for a given commodity (or for different commodities in the equal value case), if one chain C is a subset of another chain C', then C' can be ignored. (For instance, the chain C_1 of Fig. 2 dominates C_8 and C_9.) The shortest chain method takes care of such dominances automatically.

A more serious consideration is how to handle the case of limited supplies of commodities in such a problem. For example, suppose that in the two commodity maximal flow problem corresponding to the matrix of Fig. 2, there is an amount a_1 of commodity 1 at P_1, an amount a_2 of commodity 1 at P_2, and an amount a_4 of commodity 2 at P_4. We can reduce this to a problem of the same type as before by introducing three new directed arcs and nodes as follows: A'_1 from P'_1 to P_1 with capacity a_1, A'_2 from P'_2 to P_2 with capacity a_2, and A'_4 from P'_4 to P_4 with capacity a_4. We then take P'_1, P'_2 as sources for commodity 1, and P'_4 as the source for commodity 2. However, in the hypothesized large network with 20 commodities, the number of such new arcs would be $\sum_{i=1}^{20} n_i$, where n_i is the number of nodes in S_i, and since each new arc increases the size of basis matrices by one, this might take the problem out of range of present computing machines.

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II–4

A NETWORK FLOW COMPUTATION FOR PROJECT COST CURVES*

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A network flow method is outlined for solving the linear programming problem of computing the least cost curve for a project composed of many individual jobs, where it is assumed that certain jobs must be finished before others can be started. Each job has an associated crash completion time and normal completion time, and the cost of doing the job varies linearly between these extreme times. Given that the entire project must be completed in a prescribed time interval, it is desired to find job times that minimize the total project cost. The method solves this problem for all feasible time intervals.

Introduction

A linear programming problem of some practical importance that has been formulated by Kelley and Walker [6, 7] involves computing the cost curve for a "project" composed of many individual "jobs" or "activities." Here a project is a partially ordered set of jobs, the partial ordering arising from technological restrictions that force certain jobs to be finished before others can be started. It is assumed that each job has an associated normal completion time and a crash completion time, and that the cost of doing the job varies linearly between these two extreme times. Then it would be desirable to calculate the least project cost, given that the entire project must be completed in a prescribed time interval. This would yield one point on the project cost curve. Solving the problem for all feasible time intervals produces the complete project cost curve. With this information at hand, the project planner can answer either the question posed above, or the related question: given a fixed budget, what is the earliest project completion date?

We shall show that the project cost curve can be easily computed using network flow theory[1] [1, 2, 3, 4, 5].

1. The Project Network

There are at least two rather different ways of depicting the project as a directed network or linear graph. For example, suppose the project consists of jobs 1, 2, 3, 4, 5 and that the only order relations are:

1 precedes 3, 4
2 precedes 4,
3, 4 precede 5,

* Received June 1960.

[1] After the results of this paper were obtained, the author learned that a network flow approach to the project cost problem has also been developed by Kelley. See [7], which contains a statement to this effect, and [8] for a complete exposition of Kelley's method."

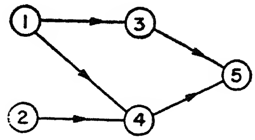

FIG. 1

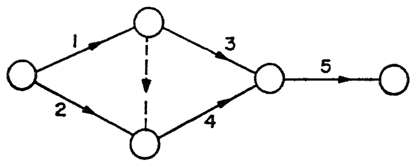

FIG. 2

and those implied by transitivity. The usual way of picturing this partially ordered set is shown in Fig. 1, where nodes correspond to jobs and directed arcs to the displayed order relations. Another way is shown in Fig. 2, where some of the arcs represent jobs, and the nodes may be thought of as events in time; the existence of a node stipulates that all inward pointing jobs at the node must be completed before any outward pointing job can be started. Notice that the second of these two representations of the project uses an arc (the dotted one of Fig. 2) not corresponding to any job. This need cause no concern, since a dummy job can be added to the project to correspond to such an arc, and the assumption made that such fictitious jobs have zero completion time and zero cost. It is not difficult to see that allowing dummy jobs permits such a network representation for any project. Indeed, one could merely take the kind of network shown in Fig. 1, replace each node i by a pair of nodes i', i'' and add the directed arcs (i', i'') (from i' to i'') to the network. An old arc (i, j) becomes the arc (i'', j') in the new network; these latter are now dummies. But this is not, in general, efficient in terms of the number of nodes and arcs.

Using either of these two network representations of the project, the problem of computing the cost curve can be shown to be a flow problem. We shall work with the second representation. Thus we take as given a directed network in which arcs correspond to jobs and nodes to events. This network contains no directed cycles. We may also assume, by adding "beginning" and "terminal" nodes (events), if necessary, together with appropriate arcs pointing out from the beginning node and into the terminal node, that each arc is contained in some directed chain from the beginning node to the terminal node. Finally, we may assume, since the network contains no directed cycles, that the nodes have been numbered $1, 2, \cdots, n$ in such a way that 1 is the beginning node, n the terminal node, and if (i, j) is an arc, then $i < j$.

2. The Project Cost Curve Program and Its Dual

Associated with each arc (i, j) of the project network are three nonnegative integers: $a(i, j)$, $b(i, j)$, $c(i, j)$, with

$$a(i, j) \leqq b(i, j). \tag{2.1}$$

Here the interpretation is that $a(i, j)$ is the crash time for job (i, j), $b(i, j)$ the normal completion time, while $c(i, j)$ is the decrease in cost of doing job (i, j) per unit increase in time from $a(i, j)$ to $b(i, j)$. In other words, the cost of doing (i, j) in $\tau(i, j)$ units of time is given by the known linear function

$$k(i, j) - c(i, j)\tau(i, j) \tag{2.2}$$

over the interval

$$a(i, j) \leqq \tau(i, j) \leqq b(i, j). \tag{2.3}$$

Then the problem is, given λ units of time in which to finish the project, to choose, for each job (i, j), a time $\tau(i, j)$ satisfying (2, 3) in such a way that the resulting project cost

$$\sum_{i,j} [k(i, j) - c(i, j)\tau(i, j)] \tag{2.4}$$

is minimized, or equivalently, the function

$$\sum_{i,j} c(i, j)\tau(i, j) \tag{2.5}$$

is maximized. Thus, letting $\tau(i)$ be the (unknown) time of occurrence of event i, we wish to maximize (2.5) subject to the inequalities

$$\tau(i, j) + \tau(i) - \tau(j) \leqq 0, \qquad \text{all } (i, j), \tag{2.6}$$

$$-\tau(1) + \tau(n) \leqq \lambda, \tag{2.7}$$

$$\tau(i, j) \leqq b(i, j), \qquad \text{all } (i, j), \tag{2.8}$$

$$-\tau(i, j) \leqq -a(i, j), \qquad \text{all } (i, j). \tag{2.9}$$

Here (2.6) expresses the condition that there must be sufficient time between the occurrences of events i and j to do job (i, j) in $\tau(i, j)$ units of time, and (2.7) says that the project duration time is at most λ.

The project cost $P(\lambda)$ corresponding to the assigned value of λ in (2.7) is given by

$$P(\lambda) = \sum_{i,j} k(i, j) - \max \sum_{i,j} c(i, j)\tau(i, j), \tag{2.10}$$

the maximum being taken over all $\tau(i, j)$, $\tau(i)$ that satisfy the constraints. Here we assume that the constraints are feasible, which will be the case for sufficiently large λ. Indeed, for given $\tau(i, j)$ satisfying (2.8) and (2.9), the constraints are feasible if and only if λ is at least equal to the "τ-length" of a longest directed chain from 1 to n, that is, λ must be at least equal to the maximum of $\sum \tau(i, j)$, the summation being along a directed chain from 1 to n, and the maximum being taken over all such chains. The proof of this relies on the fact that the project network contains no directed cycles.

Dummy jobs may be assumed to have lower bounds $a(i, j) = 0$, upper bounds $b(i, j) = 0$, and costs $c(i, j) = 0$ in this linear program. To construct the project cost curve $P(\lambda)$, we need to solve the program (2.5)–(2.9) parametrically in λ.

This formulation of the project cost curve program has been given by Kelley and Walker [6, 7].

It may be observed preliminarily that $P(\lambda)$, which is well defined for some λ-interval, is convex. For if λ_1, λ_2 are two given values of λ that make the constraints feasible, and if $\tau_1(i, j)$, $\tau_1(i)$, $\tau_2(i, j)$, $\tau_2(i)$ represent optimal solutions to the two corresponding programs, then averaging these two solutions gives a feasible solution to the constraints corresponding to the λ-value $\frac{1}{2}(\lambda_1 + \lambda_2)$. Hence, since we are minimizing $P(\lambda)$,

$$P[\tfrac{1}{2}(\lambda_1 + \lambda_2)] \leqq \tfrac{1}{2}P(\lambda_1) + \tfrac{1}{2}P(\lambda_2).$$

In addition, $P(\lambda)$ is piecewise linear, as will be apparent later on.

We may set $\tau(1) = 0$, since adding a constant to all event times does not alter the program. With this normalization, it follows from (2.6) that all $\tau(i)$ are nonnegative, since the job times are nonnegative by (2.9), and since each node is contained in some directed chain from 1 to n.

Let us examine the dual of the project cost program. If we assign nonnegative multipliers $f(i, j)$, v, $g(i, j)$, $h(i, j)$ to the constraints (2.6), (2.7), (2.8), (2.9) respectively, the dual of the program, for fixed λ and $\tau(1) = 0$, has constraints

$$f(i, j) + g(i, j) - h(i, j) = c(i, j), \qquad \text{all } (i, j), \tag{2.11}$$

$$\sum_j [f(i, j) - f(j, i)] = \begin{cases} 0, i \neq 1, n, \\ - v, i = n, \end{cases} \tag{2.12}$$

subject to which

$$\lambda v + \sum_{i,j} b(i, j)g(i, j) - \sum_{i,j} a(i, j)h(i, j) \tag{2.13}$$

is to be minimized. Here, we repeat, all variables are nonnegative. Equalities appear in the constraints since variables of the primal program are not explicitly restricted in sign.

It follows immediately that at least one of $g(i, j)$, $h(i, j)$ can be taken zero in an optimal dual solution, and hence we may assume

$$g(i, j) = \max[0, c(i, j) - f(i, j)], \tag{2.14}$$

$$h(i, j) = \max[0, f(i, j) - c(i, j)]. \tag{2.15}$$

Thus the dual problem becomes: find nonnegative numbers $f(i, j)$, one for each arc of the project network, and a nonnegative number v, that satisfy the equations (2.12) and minimize the nonlinear function

$$\begin{aligned} \lambda v &+ \sum_{i,j} b(i, j) \max [0, c(i, j) - f(i, j)] \\ &- \sum_{i,j} a(i, j) \max [0, f(i, j) - c(i, j)]. \end{aligned} \tag{2.16}$$

A key observation at this point is that a function of the form

$$b \max (0, c - f) - a \max (0, f - c) \tag{2.17}$$

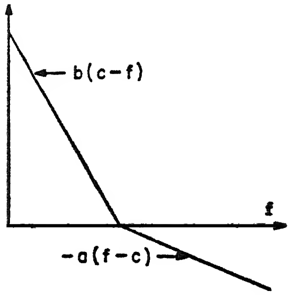

FIG. 3

(sketched in Fig. 3) is convex, and of course, piecewise linear. The convexity of (2.17) follows from the assumption $a \leqq b$. Thus, even though (2.16) is nonlinear, it is the next best thing (for minimizing), namely a sum of piecewise linear, convex functions of the individual variables $f(i, j)$. As is well known in linear programming, such a function can be dealt with by linear methods.

Here one replaces each $f(i, j)$ by a sum of two nonnegative variables, say,

$$f(i, j) = f(i, j; 1) + f(i, j; 2), \tag{2.18}$$

the new variables being subject to the upper bound or capacity constraints

$$f(i, j; 1) \leqq c(i, j), \tag{2.19}$$

$$f(i, j; 2) \leqq \infty. \tag{2.20}$$

Then $f(i, j; 1)$ has coefficient $-b(i, j)$, $f(i, j; 2)$ has coefficient $-a(i, j)$ in the new linear minimizing form. Thus, if we define

$$c(i, j; k) = \begin{cases} c(i, j), & k = 1, \\ \infty & k = 2, \end{cases} \tag{2.21}$$

$$a(i, j; k) = \begin{cases} b(i, j), & k = 1, \\ a(i, j), & k = 2, \end{cases} \tag{2.22}$$

the dual program has constraints

$$\sum_{j,k} [f(i, j; k) - f(j, i; k)] = \begin{cases} 0, & i \neq 1, n, \\ -v, & i = n, \end{cases} \tag{2.23}$$

$$0 \leqq f(i, j; k) \leqq c(i, j; k), \tag{2.24}$$

and minimizing form

$$\lambda v - \sum_{i,j,k} a(i, j; k) f(i, j; k). \tag{2.25}$$

This program has the following network flow interpretation. First enlarge the project network by doubling the number of arcs: corresponding to each arc (i, j) of the project network there are now two arcs $(i, j; 1)$ and $(i, j; 2)$ from

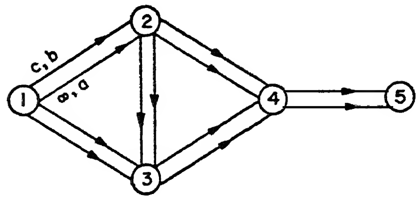

FIG. 4

i to j (see Fig. 4). Each arc $(i, j; k)$ of the new network has an assigned flow capacity $c(i, j; k)$. The problem is to construct a flow $f(i, j; k)$ from source node 1 to sink node n in the new network that minimizes (2.25).[2]

Flow problems of this kind have received a great deal of study in recent years, and computational methods far superior to general linear programming methods are known for such problems [2, 3, 4, 5]. In the next section we shall describe an efficient flow algorithm for generating the complete project cost curve. This algorithm will start with the largest λ of interest, namely λ equal to the maximal b-length of directed chains from 1 to n. The algorithm then determines sequentially a finite set of values of λ that contains all breakpoints of the convex, piecewise linear $P(\lambda)$. Corresponding to each of these λ-values, certain node numbers $\tau(i)$ are produced. We denote them by $\tau(i)$ because they do indeed have the interpretation of optimal event times in the original project program. Here we shall have $\tau(1) = 0$, $\tau(n) = \lambda$. Then optimal job times $\tau(i, j)$ for the project, corresponding to these $\tau(i)$, are given simply by defining

$$\tau(i, j) = \min [b(i, j), \tau(j) - \tau(i)]. \tag{2.26}$$

We shall discuss these assertions in more detail following the algorithm statement.

3. A Flow Algorithm for Determining $P(\lambda)$

The basic routine used in the algorithm is a labeling process in which labels are assigned to some of the nodes. In general, the labeling process is a systematic search for a path (having certain desired properties) from 1 to n. Here the word "path," as opposed to "directed chain" or "chain," means that arcs can be traversed against their orientations in going from 1 to n. Such arcs are termed reverse arcs of the path; the others, traversed with their orientation, are called forward arcs of the path.

We enter the labeling process with an integral flow $f = f(i, j; k)$ and node integers $\tau(i)$ that satisfy

$$\tau(1) = 0, \tag{3.1}$$

[2] A function f from arcs to nonnegative reals that satisfies equations (2.23) for some number v, and also satisfies the capacity constraints (2.24) on individual arcs, is called a flow from 1 to n of value v. The left-hand side of (2.23) is the net flow out of node i. Thus the net flow out of nodes 2, $\cdots$, $n - 1$ is zero, the net flow into n is v, and it follows that the net flow out of 1 is v.

$$(3.2) \qquad a(i, j; k) + \tau(i) - \tau(j) < 0 \Rightarrow f(i, j; k) = 0,$$

$$(3.3) \qquad a(i, j; k) + \tau(i) - \tau(j) > 0 \Rightarrow f(i, j; k) = c(i, j; k).$$

To shorten the notation, we shall set

$$(3.4) \qquad \bar{a}(i, j; k) = a(i, j; k) + \tau(i) - \tau(j).$$

(The properties (3.1), (3.2), (3.3) are optimality properties for the program (2.23), (2.24), (2.25) corresponding to $\lambda = \tau(n)$. That is, for $\lambda = \tau(n)$, these properties imply that the flow f minimizes (2.25).[3] The labeling process terminates in one of two ways, called "breakthrough" and "nonbreakthrough." Breakthrough means that the node n has received a label, and this in turn means that a path from 1 to n has been found having the properties that for all forward arcs of the path, $\bar{a}(i, j; k) = 0$ and $f(i, j; k) < c(i, j; k)$, whereas for all reverse arcs of the path, $\bar{a}(i, j; k) = 0$ and $f(i, j; k) > 0$. Then the old flow f is changed by adding a positive integer ϵ to the amount of flow in all forward arcs of the path, and subtracting ϵ from the flow in reverse arcs of the path. This yields a new integral flow f' from 1 to n, for which the optimality properties (3.2), (3.3) hold. Nonbreakthrough, on the other hand, means that the labeling process has terminated and node n has not been labeled. In this case, the node integers are changed by subtracting a positive integer δ from all $\tau(i)$ corresponding to unlabeled i. This doesn't change $\tau(1) = 0$ but reduces $\tau(n) = \lambda$ and thus defines a new set of optimal job times, via (2.26), corresponding to this new value of λ. The node number change δ is selected in such a way that the new node numbers $\tau'(i)$ and old flow f still satisfy the optimality properties (3.2), (3.3).

After either a breakthrough or a nonbreakthrough, the labeling process is repeated.

The project cost $P(\lambda)$ is linear between successive values of λ produced by nonbreakthroughs, as we shall see.

The labeling process described below has been divided into two parts, called first and second labelings, respectively. The first labeling seeks a directed chain from 1 to n composed of infinite capacity arcs (those corresponding to $k = 2$) such that $\bar{a}(i, j; 2) = 0$ for each arc of this chain. If such a chain is found, then the computation terminates, for the existence of such a chain means, in effect, that any further decrease in λ would make the project program infeasible. If no such chain can be found, we go on to the second labeling, in which the search is

[3] This follows from the linear programming duality theorem, or can be seen directly by observing that

$$\begin{aligned}\sum_{i,j,k} \bar{a}(i, j; k)f(i, j; k) &= \sum_{i,j,k} a(i, j; k)f(i, j; k) + \sum_{i,j,k} [\tau(i) - \tau(j)]f(i, j; k) \\ &= \sum_{i,j,k} a(i, j; k)f(i, j; k) + [\tau(1) - \tau(n)]v \\ &= \sum_{i,j,k} a(i, j; k)(f(i, j, k) - \lambda v.\end{aligned}$$

Then (3.2) and (3.3) imply that f maximizes the left-hand side of this equality, hence, the right.

extended in an attempt to find a path from 1 to n having the properties previously outlined.

The computation is initiated by the Start routine below, which generates a starting set of node numbers that, in conjunction with the zero flow, satisfy the optimality properties (3.1), (3.2), (3.3). This routine is nothing more than a way of finding a chain of maximal b-length from 1 to n, and has been used by Kelley and Walker to compute what they term "critical paths" [6]. (Their usage of the word "path" corresponds to our usage of the word "chain" or "directed chain.")

We now describe the algorithm in detail.

Start. Successively compute $\tau(1), \tau(2), \cdots, \tau(n)$ by the recursion

$$\tau(1) = 0 \tag{3.5}$$

$$\cdots$$

$$\tau(j) = \max_i [\tau(i) + a(i, j; 1)] = \max_i [\tau(i) + b(i, j)].$$

Set all $f(i, j; k) = 0$.

Iterative Procedure. Enter with an integral flow $f = f(i, j; k)$ and node integers $\tau(i)$ satisfying (3.1), (3.2), (3.3). Initially these are the ones generated by the Start routine.

A. *Labeling Process.* During this routine, a node is considered to be in one of three states: unlabeled, labeled and unscanned, or labeled and scanned. Initially all nodes are unlabeled.

1. *First Labeling.* Assign node 1 the label $[-, -, -, \epsilon(1) = \infty]$. (This node is now labeled and unscanned; all other nodes are unlabeled.) In general, select any labeled, unscanned node, say node i, and search for all unlabeled nodes j such that $(i, j; 2)$ is an arc with

$$\bar{a}(i, j; 2) = 0. \tag{3.6}$$

Label such nodes j with $[i, 2, +, \epsilon(j) = \infty]$. (Such nodes j are now labeled and unscanned and node i is labeled and scanned.) Repeat the general step until either the node n is labeled and unscanned, or no more nodes can be labeled and node n is unlabeled. In the former case, terminate. In the latter case, go on to the Second Labeling.

2. *Second Labeling.* Nodes labeled above retain their labels, and the labeling process continues as follows. All nodes revert to the unscanned state. The general step: Select any labeled, unscanned node, say node i, and scan it for all unlabeled nodes j such that either

$$(i, j; k) \text{ is an arc with } \bar{a}(i, j; k) = 0, f(i, j; k) < c(i, j; k), \tag{3.7}$$

or

$$(j, i; k) \text{ is an arc with } \bar{a}(j, i; k) = 0, f(j, i; k) > 0. \tag{3.8}$$

If (3.7), assign j the label $[i, k, +, \epsilon(j)]$, where

$$\epsilon(j) = \min [\epsilon(i), c(i, j; k) - f(i, j; k)]. \tag{3.9}$$

If (3.8), assign j the label $[i, k, -, \epsilon(j)]$, where

$$\epsilon(j) = \min [\epsilon(i), f(j, i; k)]. \tag{3.10}$$

(Such nodes j are now labeled and unscanned and node i is labeled and scanned.) Repeat the general step until either node n is labeled and unscanned (breakthrough), or until no more labels can be assigned and node n is unlabeled (nonbreakthrough). In case of breakthrough, go on to routine B. If nonbreakthrough, go to routine C.

B. *Flow Change.* The labeling process has resulted in breakthrough. Change the flow f as follows. Node n will be labeled $[j, k, +, \epsilon(n)]$. Add $\epsilon(n)$ to $f(j, n; k)$; then go on to node j and its label. The general step: if node p is labeled $[i, k, +, \epsilon(p)]$, add $\epsilon(n)$ to $f(i, p; k)$; if node p is labeled $[i, k, -, \epsilon(p)]$, subtract $\epsilon(n)$ from $f(p, i; k)$; in either case, go on to node i and its label. Repeat the general step until node 1 is reached, that is, $\epsilon(n)$ has been added to some $f(1, j; k)$. Then discard the labels and go back to A.

C. *Node Number Change.* The labeling process has resulted in nonbreakthrough. Single out the following subsets of arcs:

$$A_1 = \{(i, j; k) \mid i \text{ labeled}, j \text{ unlabeled}, \bar{a}(i, j; k) < 0\}, \tag{3.11}$$

$$A_2 = \{(i, j; k) \mid i \text{ unlabeled}, j \text{ labeled}, \bar{a}(i, j; k) > 0\}. \tag{3.12}$$

Define

$$\delta_1 = \min_{A_1} [-\bar{a}(i, j; k)], \tag{3.13}$$

$$\delta_2 = \min_{A_2} [\bar{a}(i, j; k)], \tag{3.14}$$

$$\delta = \min (\delta_1, \delta_2). \tag{3.15}$$

Change the node integers $\tau(i)$ by subtracting δ from all $\tau(i)$ corresponding to unlabeled i. Discard the labels and go back to A.

4. Discussion of the Computation

The starting set of node integers $\tau(i)$ and the zero flow f satisfy the optimality properties (3.1), (3.2), (3.3), since $\bar{a}(i, j; k) \leqq 0$ for all arcs. Moreover, if we enter the iterative procedure with node numbers $\tau(i)$ and a flow f that satisfy these properties, and if breakthrough occurs, the new function f' obtained from f by adding the positive integer $\epsilon(n)$ to all $f(i, j; k)$ corresponding to forward arcs of the path from 1 to n, and subtracting $\epsilon(n)$ from all $f(i, j; k)$ corresponding to reverse arcs of the path, is again an integral flow (of value $v + \epsilon(n)$) from 1 to n. Moreover, the old node integers $\tau(i)$ and new flow $f'(i, j; k)$ again satisfy (3.2), (3.3), simply because the flow changes made occur in arcs for which $\bar{a}(i, j; k) = 0$.

If, on the other hand, nonbreakthrough occurs, then the node number change δ of routine C is a positive integer and the resulting node integers $\tau'(i)$ and flow f again satisfy the optimality properties. We first check that δ is well defined, i.e. that at least one of the sets of arcs A_1, A_2 defined by (3.11), (3.12) is nonempty.

Indeed A_1 cannot be empty. For suppose A_1 were empty. Since there is a chain from 1 to n in the project network, and since 1 is labeled, n unlabeled, there must be a pair of arcs $(i, j; k)$, $k = 1, 2$, in the enlarged network with i labeled, j unlabeled. Then $\bar{a}(i, j; k) \geqq 0$ for this pair of arcs. It follows from (3.3) and labeling rule (3.7) that $f(i, j; k) = c(i, j; k)$, hence $f(i, j; 2) = \infty$. But this is absurd. Consequently δ is a positive integer. We show next that for any δ' satisfying

$$0 \leqq \delta' \leqq \delta, \tag{4.1}$$

the node numbers

$$\tau'(i) = \begin{cases} \tau(i) & \text{for } i \text{ labeled,} \\ \tau(i) - \delta' & \text{for } i \text{ unlabeled,} \end{cases} \tag{4.2}$$

and the flow f again satisfy the optimality properties. Thus suppose

$$\bar{a}'(i, j; k) = a(i, j; k) + \tau'(i) - \tau'(j) < 0. \tag{4.3}$$

We need to verify that $f(i, j; k) = 0$. If $\bar{a}(i, j; k) < 0$, this is immediate. If $\bar{a}(i, j; k) = 0$, it follows from (4.2) and (4.3) that i is unlabeled, j labeled at the conclusion of the labeling process. Hence, by labeling rule (3.8), $f(i, j; k) = 0$, as otherwise i would be labeled from j. Finally, suppose $\bar{a}(i, j; k) > 0$. Then, since $\bar{a}'(i, j; k) < \bar{a}(i, j; k)$, we again have i unlabeled, j labeled. But then the arc $(i, j; k)$ is in A_2 defined by (3.12), and hence $\bar{a}'(i, j; k) \geqq 0$, contradicting the assumption $\bar{a}'(i, j; k) < 0$. Thus this case cannot occur.

A similar proof shows that if $\bar{a}'(i, j; k) > 0$, then $f(i, j; k) = c(i, j; k)$. (Hence, in particular, we cannot have $\bar{a}'(i, j; 2) > 0$.)

This completes the proof that the outputs of the iterative procedure again satisfy the optimality properties if the inputs do.

That the algorithm terminates after finitely many applications of the labeling procedure can be seen in various ways. One way is as follows. Suppose that the algorithm fails to terminate, so that an infinite sequence of breakthroughs and nonbreakthroughs occurs. The number of breakthroughs in this sequence is finite. For otherwise, since the flow change following breakthrough is a positive integer, flows having arbitrarily large values v would be produced. But if the algorithm produces a flow f having sufficiently large value v, there must be at this stage a chain from 1 to n of arcs corresponding to $k = 2$ such that $f(i, j; 2) > 0$ on arcs of this chain. Hence, since $\bar{a}(i, j; 2) \leqq 0$ throughout the computation, we have $\bar{a}(i, j; 2) = 0$ for arcs of this chain. But then the first labeling results in termination. This leaves only the possibility that infinitely many successive nonbreakthroughs occur. This possibility is eliminated by observing that, following nonbreakthrough, all nodes that were previously labeled can again be labeled, and, in addition, at least one more node can be labeled. The first part of this statement follows from the fact that for labeled i and j, the new $\bar{a}'(i, j; k)$ are equal to the old $\bar{a}(i, j; k)$; the second by looking at an arc in A_1 or A_2 that determines δ.

To sum up, the algorithm produces successive integral flows and node integers

that satisfy the optimality properties (3.1), (3.2), (3.3), and eventually terminates. It is important to be a little more precise about the first part of this statement, in the following sense. Suppose that between two occurrences of nonbreakthrough in the computation, a number (possibly zero) of breakthroughs occur. Let τ_1 and τ_2 denote the node integers produced following the two nonbreakthroughs and let f be the last flow produced by the intervening breakthroughs. Then f minimizes (2.25) for all λ between $\tau_1(n)$ and $\tau_2(n) = \tau_1(n) - \delta$. We shall use this fact later on.

We next verify that (2.26) defines optimal job times corresponding to $\lambda = \tau(n)$. To this end, one can go back to the original pair of dual programs (2.5)–(2.9) and (2.11)–(2.13), using also (2.14), (2.15) to define g and h, and (2.18) to define f. It suffices to show that

$$\tau(i, j) + \tau(i) - \tau(j) < 0 \Rightarrow f(i, j) = 0, \tag{4.4}$$

$$\tau(i, j) < b(i, j) \Rightarrow g(i, j) = 0, \tag{4.5}$$

$$\tau(i, j) > a(i, j) \Rightarrow h(i, j) = 0, \tag{4.6}$$

since (with $\tau(1) = 0$, $\tau(n) = \lambda$) these are optimality properties for primal and dual. These implications follow in a straightforward manner from (3.2) and (3.3). For example, suppose the hypothesis of (4.4) holds. Then $\tau(i, j) = b(i, j)$, hence $b(i, j) + \tau(i) - \tau(j) < 0$. Consequently $a(i, j) + \tau(i) - \tau(j) < 0$ also. It then follows from (3.2) that $f(i, j; k) = 0$, $k = 1$ and 2, hence $f(i, j) = 0$, verifying (4.4). The others may be proved similarly.

Thus each new set of event times $\tau(i)$ yields a new point on the project cost curve by defining job times $\tau(i, j)$ as in (2.26) and calculating the project cost

$$P(\lambda) = P[\tau(n)] = \sum_{i,j} [k(i, j) - c(i, j)\tau(i, j)]. \tag{4.7}$$

The project cost $P(\lambda)$ is linear between successive values of $\lambda = \tau(n)$ generated in the computation. For let $\lambda_1 > \lambda_2$ be two successive λ's and suppose

$$\lambda_1 \geqq \lambda \geqq \lambda_2 . \tag{4.8}$$

Let f be the flow that produced the node number change yielding λ_2 from λ_1 and suppose f has value v. We have earlier pointed out that f minimizes (2.25) for all λ satisfying (4.8). Hence for such λ

$$P(\lambda) = K - [\lambda v - \sum_{i,j,k} a(i, j; k)f(i, j; k)]. \tag{4.9}$$

Here K is the constant

$$K = \sum_{i,j} [k(i, j) - b(i, j)c(i, j)]. \tag{4.10}$$

Thus

$$P(\lambda) - P(\lambda_1) = (\lambda_1 - \lambda)v, \lambda_1 \geqq \lambda \geqq \lambda_2 , \tag{4.11}$$

so that $P(\lambda)$ is linear in the interval (4.8).

The equation (4.11) also shows how to pick out all breakpoints of the convex,

piecewise linear $P(\lambda)$. For suppose $\lambda_1 > \lambda_2 > \lambda_3$ are three successive values of $\tau(n)$ generated in the computation, and let v be as defined above. Let v' be the value of the flow that produced the nonbreakthrough yielding λ_3 from λ_2. Then

$$P(\lambda_2) - P(\lambda_1) = (\lambda_1 - \lambda_2)v,$$
$$P(\lambda_3) - P(\lambda_2) = (\lambda_2 - \lambda_3)v'.$$

Consequently λ_2 is a breakpoint of $P(\lambda)$ if and only if $v < v'$, that is, if and only if there is an intervening breakthrough between the two nonbreakthroughs that yield λ_2 and λ_3.

For example, if a problem computation yields the sequence of breakthroughs and nonbreakthroughs (indicated by B and N)

B Ⓝ B B N Ⓝ B N N Ⓝ

then the circled N suffice to define $P(\lambda)$.

At the conclusion of the computation, a chain of arcs corresponding to $k = 2$ has been located along which the equalities

$$a(i, j) + \tau(i) - \tau(j) = 0 \tag{4.12}$$

hold. Summing (4.12) along this chain shows that $\lambda = \tau(n)$ is equal to the a-length of the chain. Consequently the project cannot be completed in any shorter time interval.

A consequence of the algorithm is that, given integral data in the problem, all the numbers produced are integers. Hence, in particular, breakpoints of $P(\lambda)$ are integers, and so are the corresponding optimal job times.

The method of this paper can also be used to compute project cost curves in case the given job costs are piecewise linear and convex between crash and normal completion times. This merely introduces more arcs into the network, in fact, one more arc from i to j for each additional breakpoint of the function giving the cost of job (i, j).

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II–5

INCREASING THE CAPACITY OF A NETWORK: THE PARAMETRIC BUDGET PROBLEM*

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The problem considered in this paper is that of allocating a budget of resources among the links of a network for the purpose of increasing its flow capacity relative to given sources and sinks.

On the assumption that the cost of increasing each link capacity is linear, a labeling algorithm is described that permits rapid calculation of optimal allocations for all budgets.

1. Introduction. Suppose that a fixed budget can be allocated among the links of a network for the purpose of increasing its flow capacity relative to a given source and sink. How should the money be spent in order to maximize the resulting network capacity?

In this note we assume that the cost of increasing the capacity of a link is linear and homogeneous, which permits direct formulation of the problem described above as a linear program, and then describe an algorithm that produces solutions to the problem, not only for a fixed budget, but for all budgets, i.e., we solve the problem parametrically. The algorithm uses a variant of the labeling procedure previously developed to solve maximal network flow problems and minimal cost transportation problems [1–4].

It is interesting that, although the budget problem does not fall within the class of transportation-type programming problems, it can still be solved by a labeling procedure. Roughly speaking, the underlying reason for this is that, for a given budget problem, one can find a pair of transportation-type linear programs such that an optimal solution to the budget problem is given by a convex combination of certain optimal solutions to the two auxiliary problems. Indeed, our algorithm is designed to solve, efficiently, a sequence of such related transportation-type problems, the sequence having the property that adjacent pairs of solutions produced by the algorithm can be used to generate a solution of the parametric budget problem.

Section 2, below, contains a formulation of the budget problem as a linear program and a statement of the dual program. In Section 3 we set up the sequence of associated programs and include some heuristic discussion. Section 4 provides a statement of the algorithm. A numerical example illustrating the computation is given in Section 5. Section 6 concludes with proofs that the algorithm produces solutions to the associated programs, and to the budget problem.

2. The Budget Problem. We suppose given a network consisting of nodes $P_0, P_1, \cdots, P_n$ and oriented links P_iP_j leading from P_i to P_j. Each link P_iP_j

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has associated with it two integers: c_{ij}, the existing flow capacity of the link, assumed nonnegative, and a_{ij}, the cost per unit of additional capacity, assumed positive. We take P_0 to be the source for flow, P_n the sink.[1]

Letting x_{ij} denote the flow from P_i to P_j along P_iP_j, y_{ij} the amount of capacity added to P_iP_j, b the total budget to be allocated for increased capacity, and v the net flow through the network from P_0 to P_n, the problem is to determine nonnegative values of x_{ij}, y_{ij}, v that

$$\text{maximize } v \tag{1}$$

subject to the constraints

$$\begin{cases} \sum_j (x_{0j} - x_{j0}) - v = 0 & \\ \sum_j (x_{ij} - x_{ji}) = 0 & (i = 1, \cdots, n - 1) \\ \sum_j (x_{nj} - x_{jn}) + v = 0 & \end{cases} \tag{2a}$$

$$x_{ij} - y_{ij} \leq c_{ij} \tag{2b}$$

$$\sum_{i,j} a_{ij}y_{ij} = b. \tag{2c}$$

Here, of course, b is assumed nonnegative.

Clearly this problem will not, in general, have integral solutions, because of the presence of constraint (2c). Nonetheless, almost all of the computation can be carried out in integers, as will be shown.

For future reference, we note that if we assign constraints (2a) the multipliers π_i $(i = 0, \cdots, n)$, constraints (2b) the multipliers γ_{ij}, and constraint (2c) the multiplier σ, one finds the dual of program (1) and (2) to be

$$\text{minimize } \sum_{i,j} c_{ij}\gamma_{ij} + b\sigma \tag{3}$$

subject to

$$-\pi_0 + \pi_n \geq 1 \tag{4a}$$

$$\pi_i - \pi_j + \gamma_{ij} \geq 0 \tag{4b}$$

$$\sigma a_{ij} - \gamma_{ij} \geq 0 \tag{4c}$$

$$\gamma_{ij} \geq 0. \tag{4d}$$

If the nonnegative numbers x_{ij}, v satisfy equations (2a), we shall call x_{ij} a *flow* (from P_0 to P_n) and v the *flow value*.

3. The Related Problems. Consider the sequence of problems

$$\text{maximize } tv - \sum_{i,j} a_{ij}y_{ij} \qquad (t = 1, 2, \cdots), \tag{5}$$

each subject to constraints (2a) and (2b) in nonnegative variables.

[1] We might equally well assume that there are several sources and sinks, provided we are interested in flows from any source to any sink. However, this situation can always be reduced to a single source and sink simply by joining all old sources to a new fictitious source by links of large capacity, and similarly for the sinks.

Notice that for t sufficiently large, e.g., if t is greater than the cost of adding a unit of capacity to each link of a chain from P_0 to P_n, the form (5) is unbounded on the convex set defined by (2a) and (2b). Thus the sequence of related problems we will need to consider is finite. We let T denote the largest value of t for which the form (5) is bounded.

Now suppose x_{ij}^t, y_{ij}^t, v^t solve the t-th one of these problems, and define

$$b^t = \sum_{i,j} a_{ij} y_{ij}^t, \qquad (t = 1, \cdots, T).$$

Then it is easy to see that x_{ij}^t, y_{ij}^t, v^t solve the budget problem for $b = b^t$. Moreover, the numbers b^t will be monotone non-decreasing in t. It might therefore seem plausible that if we are given b such that $b^t \leq b \leq b^{t+1}$, then a solution to such an intermediate budget problem could be generated by expressing b as a convex combination of b^t and b^{t+1}, and taking the same convex combination of the solutions x_{ij}^t, y_{ij}^t, v^t and x_{ij}^{t+1}, y_{ij}^{t+1}, v^{t+1}. This turns out to be almost right—that is, it is false that any two such solutions can be used in this way to solve an intermediate budget problem, but it is true that there exist solutions to the t-th and $(t + 1)$-th related problems that do generate solutions for all b lying in the interval (b^t, b^{t+1}) associated with these particular solutions.

The algorithm of the next section will, in fact, be shown to produce integral solutions x_{ij}^t, y_{ij}^t, v^t $(t = 1, \cdots, T)$ and hence a set of integers $0 = b^1 \leq b^2 \leq \cdots \leq b^T$, such that

(a) if $b^t \leq b \leq b^{t+1}$, then a solution to the budget problem corresponding to b is given by a convex combination of x_{ij}^t, y_{ij}^t, v^t and x_{ij}^{t+1}, y_{ij}^{t+1}, v^{t+1};

(b) if $b > b^T$, a solution can be obtained from x_{ij}^T, y_{ij}^T, v^T.

Moreover, the computation for the related problem t begins with the solution previously generated for problem $t - 1$, and thus the entire set of "spanning" solutions for the budget problems can be obtained efficiently.

4. The Algorithm. Before stating the algorithm for solving the sequence of related problems, we note that the dual of problem t is to find numbers π_i^t, one for each node P_i, and γ_{ij}^t, one for each arc P_iP_j, that

(6) $$\text{minimize} \sum_{i,j} c_{ij} \gamma_{ij}^t$$

subject to the constraints

(7a) $$-\pi_0^t + \pi_n^t \geq t$$

(7b) $$\pi_i^t - \pi_j^t + \gamma_{ij}^t \geq 0$$

(7c) $$0 \leq \gamma_{ij}^t \leq a_{ij}.$$

It follows that feasible solutions x_{ij}^t, y_{ij}^t, v^t and π_i^t, γ_{ij}^t to the primal and dual problems, respectively, which satisfy the conditions

(8a) $$\pi_0^t = 0, \qquad \pi_n^t = t$$

(8b) $$\pi_i^t - \pi_j^t + \gamma_{ij}^t > 0 \Rightarrow x_{ij}^t = 0$$

(8c) $$\gamma_{ij}^t > 0 \Rightarrow x_{ij}^t - y_{ij}^t = c_{ij}$$

(8d) $$\gamma_{ij}^t < a_{ij} \Rightarrow y_{ij}^t = 0,$$

are optimal solutions.

The dual variables γ_{ij}^t and primal variables y_{ij}^t need not be mentioned explicitly in describing the computation. Instead, we shall deal only with node numbers π_i^t and flows x_{ij}^t, and will construct these to satisfy

(9a) $$\pi_0^t = 0, \qquad \pi_n^t = t$$

(9b) $$\pi_j^t - \pi_i^t \leq a_{ij}$$

(9c) $$\pi_j^t - \pi_i^t = 0 \Rightarrow x_{ij}^t \leq c_{ij}$$

(9d) $$\pi_j^t - \pi_i^t = a_{ij} \Rightarrow x_{ij}^t \geq c_{ij}$$

(9e) $$\pi_j^t - \pi_i^t < 0 \Rightarrow x_{ij}^t = 0$$

(9f) $$0 < \pi_j^t - \pi_i^t < a_{ij} \Rightarrow x_{ij}^t = c_{ij}.$$

In addition, all variables will have integral values.

It is easy to check that if there are node numbers π_i^t and a flow x_{ij}^t such that (9a)–(9f) hold, then by defining

(10) $$\gamma_{ij}^t = \max(0, \pi_j^t - \pi_i^t)$$

(11) $$y_{ij}^t = \max(0, x_{ij}^t - c_{ij}),$$

one has feasible solutions to both primal and dual problems that satisfy (8a)–(8d), and hence are optimal.

To start the computation, take $\pi_i^0 = 0$ and $x_{ij}^0 = 0$. These clearly satisfy conditions (9) for $t = 0$. The computation now progresses by a sequence of "labelings" (Step A below), each of which can terminate in one of three ways: "finite breakthrough," in which case the flow is changed (Step B), "nonbreakthrough," in which case the node integers are changed (Step C), or "infinite breakthrough," in which case the computation ends, and T has been discovered.

The inputs for the t-th application of the routine composed of Steps A, B, C are π_i^{t-1}, x_{ij}^{t-1}. The node numbers π_i^{t-1} are used to divide the links P_iP_j of the network into three classes as follows. A link P_iP_j is 0-*admissible*, a-*admissible*, or *inadmissible* according as the value of $\pi_j^{t-1} - \pi_i^{t-1}$ is 0, a_{ij}, or neither of these.[2]

Step A. (Labeling process).

(1) Assign P_0 the label (P_{n+1}^+, ∞); consider P_0 as unscanned.

(2) Take any labeled, unscanned node P_i; suppose it is labeled (P_k^+, ∞). (Initially P_0 will be the only such.) To all nodes P_j that are unlabeled and such that P_iP_j is a-admissible, assign the label (P_i^+, ∞). Consider P_i as scanned

[2] Thus initially all links are 0-admissible. Steps A, B, C, then reduce to the algorithm of ref. [1] for constructing a flow of maximal value in a network with capacity limitations c_{ij} on links.

and the newly labeled P_j, if any, as unscanned. Repeat until either the sink P_n has been labeled (infinite breakthrough), or until no new labels are possible and this is not the case. In the former case, terminate; in the latter case, proceed to (3) below.

(3) (At this stage we have a labeled set of nodes including P_0 but not P_n, and each has a label of the form (P_k^+, ∞).) All nodes now revert to the unscanned state, and the labeling process continues as follows. Take any labeled, unscanned node P_i; suppose it is labeled $(P_k^\pm, h)$. (Initially we have only labels of the form (P_k^+, ∞).) To all nodes P_j that are unlabeled, such that P_iP_j is 0-admissible, and $x_{ij}^{t-1} < c_{ij}$, assign the label $(P_i^+, \min(h, c_{ij} - x_{ij}^{t-1}))$. To all nodes P_j that are now unlabeled, such that P_jP_i is 0-admissible, and

$$x_{ji}^{t-1} > 0,$$

assign the label $(P_i^-, \min(h, x_{ji}^{t-1}))$. Next, if P_j is unlabeled and P_iP_j is a-admissible, label P_j with (P_i^+, h). (Initially, when we are labeling from a node of the starting set, this case cannot occur.) Finally, if P_j is unlabeled, P_jP_i is a-admissible, and $x_{ji}^{t-1} > c_{ji}$, label P_j with $(P_i^-, \min(h, x_{ji}^{t-1} - c_{ji}))$. Consider P_i as scanned and the newly labeled P_j, if any, as unscanned. Repeat until either the sink P_n has been labeled with, say, (P_k^+, h),[3] or until no new labels are possible and this is not the case. In the former case (finite breakthrough), go to Step B. In the latter case (nonbreakthrough), go to Step C.

Step B. (Flow change).

(Here the sink P_n has been labeled with (P_k^+, h).) Replace x_{kn}^{t-1} by $x_{kn}^{t-1} + h$, and go on to P_k and its label. In general, if P_k is labeled (P_j^+, l), replace x_{jk}^{t-1} by $x_{jk}^{t-1} + h$, and if labeled (P_j^-, l), replace x_{kj}^{t-1} by $x_{kj}^{t-1} - h$, in either case turning attention then to P_j and its label. Stop the flow change when P_0 has been reached. Now discard the labels generated in (3) of Step A and repeat A3 with the new flow in place of x_{ij}^{t-1}.

Step C. (Node number change).

(The labeling process has resulted in nonbreakthrough.) Give the present flow (which may or may not be x_{ij}^{t-1}) the name x_{ij}^t and define node numbers π_i^t by

$$\pi_i^t = \begin{cases} \pi_i^{t-1} & \text{if } P_i \text{ is labeled} \\ \pi_i^{t-1} + 1 & \text{if } P_i \text{ is unlabeled.} \end{cases}$$

The entire routine is then repeated using π_i^t and x_{ij}^t as inputs.

In the concluding section we shall sketch proofs that the flows x_{ij}^t generated in the computation have the properties discussed in Section 3, but perhaps some preliminary explanatory comments are in order.

The labeling process A1–A2 is a search for a chain from P_0 to P_n of a-admissible links. If none such exists, we proceed to enlarge the search (A3) in an attempt to find a path from P_0 to P_n of admissible links (where the word "path," as

[3] The sink P_n will never receive a label of the form (P_k^-, h), since every flow generated by the algorithm will have $x_{nj} = 0$.

Similarly each flow will have $x_{j0} = 0$, so that any node P_j labeled from P_0 will have a label of the form (P_0^+, h).

opposed to "chain," means that a link may be traversed opposite its orientation in going from P_0 to P_n) having the property that the (integral) flow change h made along the path (Step B) is positive and yields a flow again satisfying (9c) and (9d). Inadmissible links correspond to (9e) and (9f), and in these we keep the flow fixed, so that these conditions are also maintained. Thus, if we enter the routine with node numbers π_i^{t-1} and a flow x_{ij}^{t-1} satisfying (9c)–(9f), the same node numbers π_i^{t-1} and the output flow x_{ij}^t still satisfy (9c)–(9f), and consequently the output flow will again be a solution to related problem $t - 1$. In addition, it is a solution to related problem t (as can be shown using the transformation of node numbers given in Step C), and hence we can repeat the process. It is this fact—that x_{ij}^t solves both problems $t - 1$ and t—which enables one to prove that the sequence of flows $x_{ij}^1, \cdots, x_{ij}^T$ produced by the algorithm are spanning solutions for the budget problem.

5. An Example. Let the network be that of Fig. 1, the capacity c_{ij} of link P_iP_j being the number in the upper left of the box, and the cost a_{ij} of adding one unit of capacity being the number in the upper right.[4] Assume that we have the node numbers π_i^3 shown in the figure, and the flow x_{ij}^3 indicated by the numbers in the lower left of the boxes, and wish to compute x_{ij}^4 and π_i^4. Using the numbers π_i^3, we divide the links into the three classes: 0-admissible (indicated in the figure by a zero in the lower right of the box), a-admissible (indicated by an

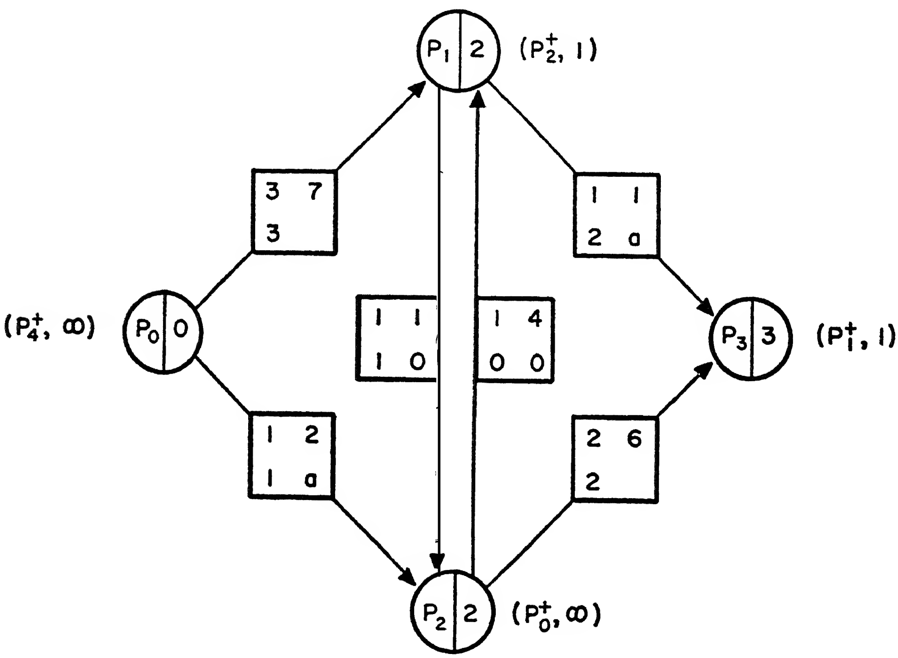
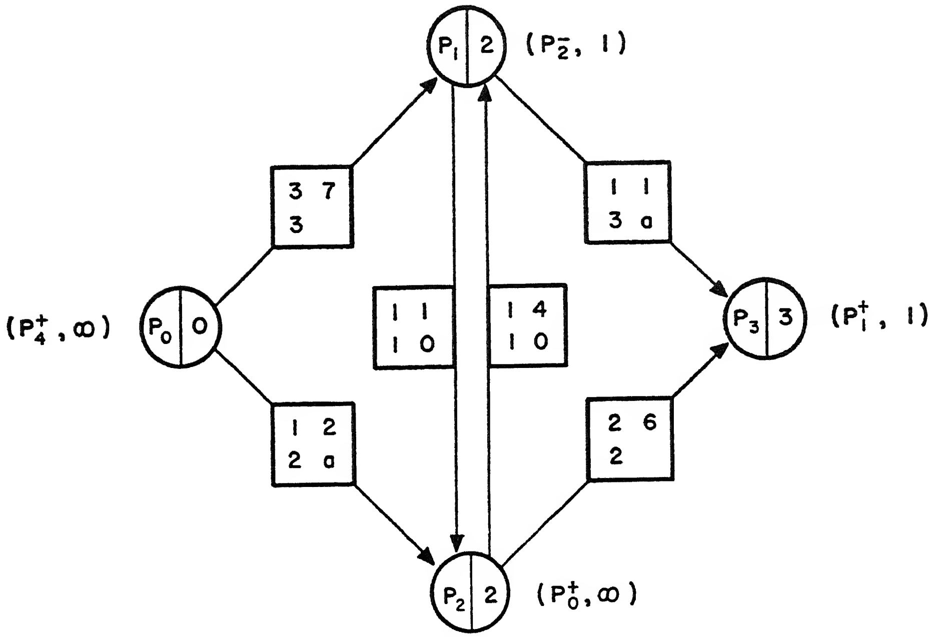


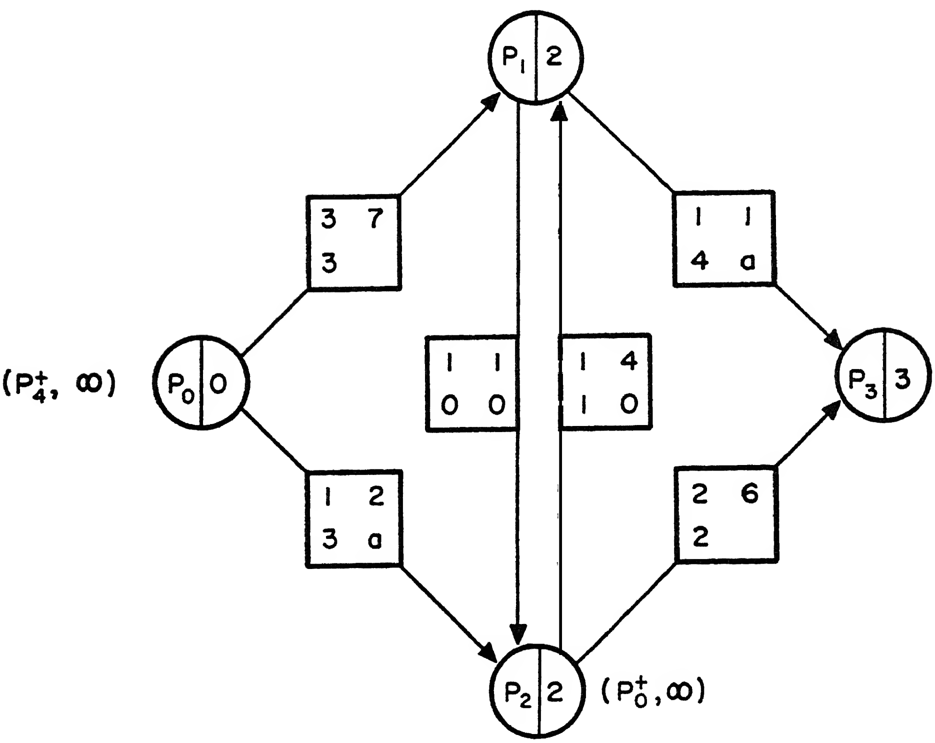
FIG. 1

[4] Links not shown in Fig. 1 are assumed to have zero capacity and large cost for additional capacity.



(P₂⁻, 1)
(P₄⁺, ∞)
(P₁⁺, 1)
(P₀⁺, ∞)
P₀ 0
P₁ 2
P₂ 2
P₃ 3
3 7 3
1 1 3 a
1 1 1 0
1 4 1 0
1 2 2 a
2 6 2

Fig. 2



(P₄⁺, ∞)
(P₀⁺, ∞)
P₀ 0
P₁ 2
P₂ 2
P₃ 3
3 7 3
1 1 4 a
1 1 0 0
1 4 1 0
1 2 3 a
2 6 2

Fig. 3

a in the lower right of the box), and inadmissible (indicated by no entry in the lower right of the box).

The labeling process A1–A2 yields the labels (P_4^+, ∞) on P_0 and (P_0^+, ∞) on P_2. We then go on to A3. Scanning P_0 gives no more labels, but from P_2 we can label P_1 with $(P_2^+, \min(\infty, 1))$, and this completes the scanning of P_2. (Notice that P_1 could also have been labeled with $(P_2^-, \min(\infty, 1))$, since the order in which the labeling rules of A3 are applied is immaterial.) Finally, from P_1 we break through to P_3 with the label $(P_1^+, 1)$, and have thus located a chain, found by tracing the labels backward from P_3, along which we can increase the flow by an additional unit.

After changing the flow, discarding the old labels, and relabeling, we obtain the labels shown in Fig. 2. Again we have a finite breakthrough, and therefore change the flow along the path indicated by the labels: add 1 to x_{13}, subtract 1 from x_{12}, and add 1 to x_{02}. We then relabel, obtaining the labels shown in Fig. 3. This time we have a nonbreakthrough, and thus go to Step C, the node-number change. The flow shown in Fig. 3 is therefore x_{ij}^4, and the new node numbers π_i^4 are given by adding 1 to the numbers on unlabeled nodes P_1 and P_3: $\pi_0^4 = 0$, $\pi_1^4 = 3$, $\pi_2^4 = 2$, $\pi_3^4 = 4$.

Observe that

$$\sum_{i,j} a_{ij} y_{ij}^4 = 2y_{02}^4 + 1y_{13}^4 = 7,$$

and thus if we are given a budget $b = 7$, we should boost the capacity of P_0P_2 by 2 units, that of P_1P_3 by 3 units, thereby achieving a total flow of 6 units from P_0 to P_3. On the other hand, we see from Fig. 1 that

$$\sum_{i,j} a_{ij}\, y_{ij}^3 = 1y_{13}^3 = 1,$$

so that with $b = 1$, the capacity of P_1P_3 should be increased by 1 unit, permitting a total flow of 4 units through the network. Notice also that

$$3v^3 - \sum_{i,j} a_{ij}\, y_{ij}^3 = 11 = 3v^4 - \sum_{i,j} a_{ij}\, y_{ij}^4$$

and hence x_{ij}^4 solves related problem 3 provided x_{ij}^3 does.

6. Theorems and Proofs. It is not difficult to see that if we enter Step A with a flow x_{ij} and obtain new numbers x_{ij}' via Step B, then x_{ij}' is a flow also, since it is obtained from x_{ij} by adding a positive amount h to the flow in links of a path from P_0 to P_n that are traversed with their orientation (in going from P_0 to P_n), and subtracting h from the flows in links traversed against their orientation. Moreover, h is no greater than the minimum of the link flows in the reverse oriented links of the path, so that nonnegativity is maintained.

The routine composed of Steps A, B, C terminates. For if A1 and A2 do not locate a chain of a-admissible links from P_0 to P_n, let L be the set of indices of nodes that are labeled in A1 and A2. Thus $0 \in L$, $n \notin L$. Now any flow x_{ij} produced via A3 and B satisfies $x_{ij} \leq c_{ij}$ for all links P_iP_j that are not a-admissible. Hence, summing equations (2a) over $i \in L$ yields

$$v = \sum_{\substack{i \in L \\ j \notin L}} (x_{ij} - x_{ji}) \leq \sum_{\substack{i \in L \\ j \notin L}} x_{ij}$$

and thus, since links P_iP_j for $i \varepsilon L, j \epsilon L$ are not a-admissible, we have

$$v \leq \sum_{\substack{i \epsilon L \\ j \notin L}} c_{ij} .$$

Consequently, since v increases by $h \geq 1$ with each occurrence of a flow change, there can be only finitely many of these.

Thus, starting with the flow $x_{ij}^0 = 0$, the algorithm successively produces flows x_{ij}^t for $t > 0$.

Theorem 1. The flows x_{ij}^t *produced by the algorithm and the corresponding* $y_{ij}^t = \max(0, x_{ij}^t - c_{ij}), v^t = \sum_j (x_{0j}^t - x_{j0}^t)$, *maximize the form* $tv - \sum_{i,j} a_{ij}\, y_{ij}$ *subject to constraints* (2a), (2b) *in nonnegative variables*, i.e. x_{ij}^t, y_{ij}^t, *and* v^t *solve related problem t.*

It suffices to show that π_i^t, x_{ij}^t satisfy (9a)–(9f).

Since it is clear that

$$\pi_i^0 = 0, \qquad x_{ij}^0 = 0, \qquad y_{ij}^0 = 0, \qquad v^0 = 0$$

satisfy (9a)–(9f) with $t = 0$, we may proceed by induction on t.

Property (9a) is clear from the induction assumption $\pi_0^{t-1} = 0$, $\pi_n^{t-1} = t - 1$, the node number change of Step C, and the fact that P_0 is labeled and P_n unlabeled in case of nonbreakthrough.

Consider (9b). Since $\pi_j^{t-1} - \pi_i^{t-1} \leq a_{ij}$, then $\pi_j^t - \pi_i^t$ could exceed a_{ij} only if $\pi_j^{t-1} - \pi_i^{t-1} = a_{ij}$ and $\pi_j^t = \pi_j^{t-1} + 1$, $\pi_i^t = \pi_i^{t-1}$. But then P_iP_j is a-admissible, P_i is labeled and P_j unlabeled at the conclusion of labeling, a contradiction.

For (9c), suppose $\pi_j^t - \pi_i^t = 0$, and consider cases. If $\pi_j^{t-1} - \pi_i^{t-1} < 0$, so that $x_{ij}^{t-1} = 0$, then, since P_iP_j is inadmissible, we also have $x_{ij}^t = x_{ij}^{t-1} = 0 \leq c_{ij}$. If $\pi_j^{t-1} - \pi_i^{t-1} = 0$, so that $x_{ij}^{t-1} \leq c_{ij}$, again we have $x_{ij}^t \leq c_{ij}$, since x_{ij}^{t-1} can be increased by at most $c_{ij} - x_{ij}^{t-1}$ in a sequence of flow changes. If

$$0 < \pi_j^{t-1} - \pi_i^{t-1} < a_{ij},$$

then P_iP_j is inadmissible and consequently $x_{ij}^t = x_{ij}^{t-1} = c_{ij}$. Finally, if

$$\pi_j^{t-1} - \pi_i^{t-1} = a_{ij},$$

then

$$\pi_i^t = \pi_i^{t-1} + 1, \qquad \pi_j^t = \pi_j^{t-1},$$

and hence P_i is unlabeled, P_j labeled at the conclusion of labeling. But if

$$x_{ij}^t > c_{ij},$$

this is a contradiction, since P_iP_j is a-admissible. Hence $x_{ij}^t \leq c_{ij}$. This completes the proof of (9c).

Proofs of the remaining properties can be given along similar lines, and so we omit them.

Corollary. The flow x_{ij}^t *and its corresponding* y_{ij}^t, v^t *solve related problem* $t - 1$.

This follows from the fact that x_{ij}^{t-1}, y_{ij}^{t-1}, v^{t-1} solve related problem $t - 1$ and the remarks at the end of Sec. 4.

Suppose that the algorithm terminates after the T-th application of the routine composed of steps A, B, C, i.e., we enter step A with π_i^T, x_{ij}^T and infinite breakthrough occurs. Thus a chain of a-admissible links from P_0 to P_n, say

$$(12) \qquad P_{i_0}P_{i_1}, P_{i_1}P_{i_2}, \cdots, P_{i_{k-1}}P_{i_k} \qquad (i_0 = 0, i_k = n),$$

has been located, and hence from (9a) and the definition of a-admissibility, it follows that

$$(13) \qquad T = \pi_n^T - \pi_0^T = \sum_{l=0}^{k}(\pi_{i_{l+1}}^T - \pi_{i_l}^T) = \sum_{l=0}^{k} a_{i_l i_{l+1}}.$$

Consequently this chain, of "a-length" T, has minimal a-length over all chains from P_0 to P_n, since if T were greater than the a-length of some chain, the form $Tv - \sum_{i,j} a_{ij}\, y_{ij}$ would obviously be unbounded, contradicting the maximality of $Tv^T - \sum_{i,j} a_{ij}\, y_{ij}^T$.

Let $b^t = \sum_{i,j} a_{ij}\, y_{ij}^t$ $(t = 1, \cdots, T)$ be the successive values of $\sum a_{ij}\, y_{ij}$ produced by the algorithm. Then

$$0 = b^1 \leq b^2 \leq \cdots \leq b^T.$$

For on the first application of the algorithm, all links are 0-admissible, hence $x_{ij}^1 \leq c_{ij}$, or $y_{ij}^1 = 0$. To establish the monotoneity, assume that $b^t < b^{t-1}$. Since y_{ij}^{t-1}, v^{t-1} and y_{ij}^t, v^t are respectively maximal in problems $t - 1$ and t, we have

$$(t - 1)\, v^{t-1} - b^{t-1} \geq (t - 1)v^t - b^t$$

$$tv^t - b^t \geq tv^{t-1} - b^{t-1},$$

whence adding gives

$$v^{t-1} \leq v^t,$$

an inequality that is also clear directly from the algorithm. Thus, if $b^t < b^{t-1}$, we get

$$(t - 1)v^t - b^t > (t - 1)v^{t-1} - b^{t-1},$$

a contradiction.

Theorem 2. Let $b = \alpha b^t + (1 - \alpha)b^{t+1}$, $0 \leq \alpha \leq 1$. *Then*

$$x_{ij} = \alpha x_{ij}^t + (1 - \alpha)x_{ij}^{t-1}$$

$$y_{ij} = \alpha y_{ij}^t + (1 - \alpha)y_{ij}^{t+1}$$

$$v \;\;= \alpha v^t + (1 - \alpha)v^{t+1}$$

solve (1) *and* (2). *If, on the other hand, we have* $b > b^T$, *then the flow* x_{ij}' *and its corresponding* y_{ij}', v' *obtained from* x_{ij}^T, y_{ij}^T, v^T *by adding* $(1/T)\,(b - b^T)$ *units of flow along the a-admissible chain* (12), *solve* (1) *and* (2).

While Theorem 2 can be proved directly, we choose to give a proof using the dual problem (3) and (4) in order to point out how to obtain solutions to the

dual of the budget problem from the node numbers generated in the algorithm.

Inasmuch as π_i^t and the associated γ_{ij}^t given by (10) satisfy the constraints (7), it follows that

$$\pi_i = \frac{\pi_i^t}{t}, \qquad \gamma_{ij} = \frac{\gamma_{ij}^t}{t}, \qquad \sigma = \frac{1}{t} \tag{14}$$

satisfy the constraints (4). Moreover, we have

$$\sum_{i,j} c_{ij}\gamma_{ij}^t = tv^t - b^t,$$

since π_i^t, γ_{ij}^t are optimal for (6) and (7). Thus

$$\begin{aligned}\sum_{i,j} c_{ij}\gamma_{ij} + b\sigma &= \frac{1}{t}\Big(\sum_{i,j} c_{ij}\gamma_{ij}^t + b\Big)\\ &= \frac{1}{t}(tv^t - b^t + b)\\ &= v^t + \frac{1}{t}(b - b^t).\end{aligned}$$

Now since x_{ij}^{t+1}, y_{ij}^{t+1}, v^{t+1} and x_{ij}^t, y_{ij}^t, v^t both solve problem t, we have

$$tv^{t+1} - b^{t+1} = tv^t - b^t.$$

Thus if $b^{t+1} = b^t = b$, then $v^{t+1} = v^t = v$, and hence $\sum c_{ij}\gamma_{ij} + b\sigma = v$. If, on the other hand, $b^t < b^{t+1}$, we have

$$\frac{1}{t} = \frac{v^{t+1} - v^t}{b^{t+1} - b^t},$$

so that

$$\begin{aligned}v &= v^t + (1 - \alpha)(v^{t+1} - v^t)\\ &= v^t + \frac{(b - b^t)}{b^{t+1} - b^t}(v^{t+1} - v^t)\\ &= v^t + \frac{1}{t}(b - b^t).\end{aligned}$$

Thus in either case, we see that

$$\sum_{i,j} c_{ij}\gamma_{ij} + b\sigma = v. \tag{15}$$

Hence, since x_{ij}, y_{ij}, v satisfy (2), and π_i, γ_{ij}, σ satisfy (4), it follows from (15) that they constitute optimal dual solutions.

Suppose, finally, that $b > b^T$. It follows from (9d) and the existence of the a-admissible chain (12) that

$$\sum a_{ij}y'_{ij} = b^T + \frac{1}{T}(b - b^T)\sum_{l=0}^{k} a_{i_l i_{l+1}},$$

and hence from (13) we have

$$\sum a_{ij}\, y'_{ij} = b.$$

Thus x'_{ij}, y'_{ij}, v' satisfy (2). Defining

$$\pi'_i = \frac{\pi_i^T}{T}, \qquad \gamma'_{ij} = \frac{\gamma_{ij}^T}{T}, \qquad \sigma' = \frac{1}{T} \tag{16}$$

again gives a pair of optimal dual solutions to the budget problem.

References

1. FORD, L. R., JR., AND D. R. FULKERSON, "A simple algorithm for finding maximal network flows and an application to the Hitchcock problem," *Can. J. Math.*, Vol. 9, 1957 pp. 210–218.
2. ——, "Solving the transportation problem," *Man. Sci.*, Vol. 3, No. 1, 1956, pp. 24–32.
3. ——, "A primal dual algorithm for the capacitated Hitchcock problem," *N. R. L. Q.* Vol. 4, No. 1, 1957, pp. 47–54.
4. ——, "Constructing maximal dynamic flows from static flows," to appear in *J. O. R. S. A.*, June, 1958.

III–6

ON THE STATUS OF MULTISTAGE LINEAR PROGRAMMING PROBLEMS*†

GEORGE B. DANTZIG

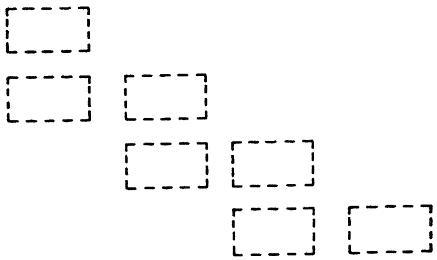
The RAND Corporation, Santa Monica, California

PART I—SPECIAL CASES

Introduction

Typical of the multistage problems are those encountered in dynamic problems. If the time span is divided into periods, the initial inventory provides the input for activities that occur in the first period or first stage. The output from the first stage provides the inventory input for activities in the second period or stage, etc. [9]. The matrix of coefficients in the linear programming model then takes on the special form (1).

(1)



To illustrate, consider the well-known warehousing problem [2, 5, 6, 17] of how much to purchase and sell of a commodity each month, given a fixed warehouse capacity, storage costs and expected prices from month to month; let α = the fixed warehouse capacity, s_0 = the initial stock in warehouse.

Consider a seasonal product to be bought and sold for each of $i = 1, 2, \cdots, n$ periods. For the ith period

c_i = cost per unit,
p_i = selling price,
w_i = warehouse cost per unit,
x_i = amount sold,
y_i = amount purchased,
t_i = amount in stock after sale of old stock,
s_i = amount in stock after purchase of new stock,
u_i = unused warehouse capacity.

* This paper was presented before the 1957 meeting in Stockholm of the International Statistical Institute and published in their I.S.I. Bulletin Vol. 36, Part 3.

† Received December 1958.

and hence from (13) we have

$$\sum a_{ij} y'_{ij} = b.$$

Thus x'_{ij}, y'_{ij}, v' satisfy (2). Defining

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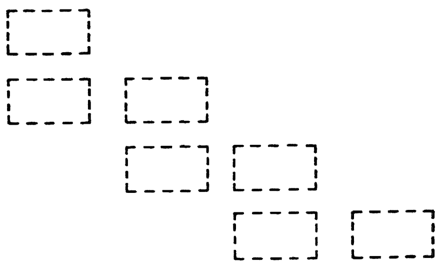

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All of these quantities will be assumed to be nonnegative. *In this problem we can distinguish two types of stages.* If the activities that take place within a period are considered as forming a stage, then within a period there are two *sub*stages: the *selling stage* which takes place before the *purchase stage.* The relations within and between periods are as follows:

$$
\begin{array}{lllllll}
s_{i-1} - x_i - t_i & & & & & & = 0 \\
 & t_i + y_i & - s_i & & & & = 0 \\
 & & + u_i + s_i & & & & = \alpha \\
 & & s_i - x_{i+1} - t_{i+1} & & & & = 0 \\
 & & & t_{i+1} + y_{i+1} & - s_{i+1} & & = 0 \\
 & & & & + u_{i+1} + s_{i+1} & & = \alpha
\end{array} \tag{2}
$$

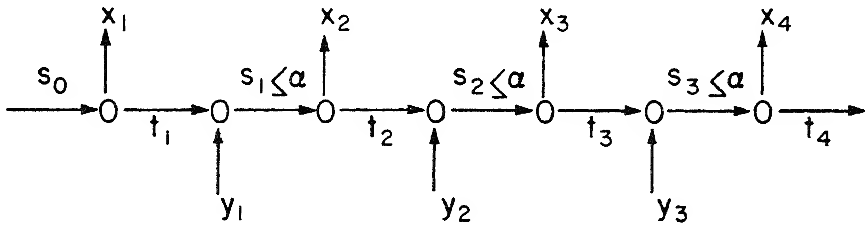
Setting aside the conditions $u_i + s_i = \alpha$ for the moment and omitting the objective function, the system for several periods takes the form:

$$
\begin{array}{lllllllr}
-x_1 - t_1 & & & & & & & = -s_0 \\
 & t_1 + y_1 - s_1 & & & & & & = 0 \\
 & & s_1 - x_2 - t_2 & & & & & = 0 \\
 & & & t_2 + y_2 - s_2 & & & & = 0 \\
 & & & & s_2 - x_3 - t_3 & & & = 0 \\
 & & & & & t_3 + y_3 - s_3 & & = 0 \\
 & & & & & & s_3 - x_4 - t_4 & = 0
\end{array} \tag{3}
$$

It will be noted that each variable appears in either one equation or it appears in two equations with opposite signs. This, however, is the condition that a linear programming problem be a *Hitchcock-Koopmans transportation (distribution) problem* [7, 21, 23]. With the conditions $s_j \leqq \alpha$ the problem clearly belongs to the class of so-called "capacitated" transportation problems and suggests that the problem can be viewed as a "network flow" problem with capacity restraints on arcs of the network [11, 16, 18, 20, 29, 30].

The network has an exceptionally simple form because of the stagewise character of the problem. In (4) each node i in the network corresponds to the

ith equation; each arc joining two nodes corresponds to a "shipment" from i to j; i.e., a variable that has (-1) in equation i and $(+1)$ in equation j; an arc with one node corresponds to an exogenous shipment to or from i; i.e., a variable with a single non-zero coefficient $+1$ or -1, respectively. The equations state that the sum of flows into or out of a node must balance to zero.

(4) 

Techniques have been worked out for solving network flow problems rapidly by hand even when a great number of periods is involved. Moreover, it is just as computationally tractable *if upper bound restraints on the amount that can be purchased or sold in any one period are imposed on the variables*; *or more generally if there are incrementally increasing costs per unit purchased or decreasing prices per unit sold* [8, 10, 11].

Dynamic Leontief Models with Substitution

The warehouse problem can be reduced to another important class of problems by the following steps. We first substitute $s_i = t_i + y_i$ in the remaining equations (and the objective form), yielding

$$
(5)\qquad
\begin{array}{rllll}
\text{I:} & -x_1 \quad - t_1 & & & = -s_0 \\
\text{I*:} & \quad u_1 + t_1 + y_1 & & & = \alpha \\
\text{II:} & \quad t_1 + y_1 - x_2 \quad - t_2 & & & = 0 \\
\text{II*:} & & \quad u_2 + t_2 + y_2 & & = \alpha \\
\text{III:} & & \quad t_2 + y_2 - x_3 \quad - t_3 & & = 0 \\
\text{III*:} & & & \quad u_3 + t_3 + y_3 & = \alpha \\
\text{IV:} & & & \quad t_3 + y_3 - x_4 - t_4 & = 0 \\
& \multicolumn{4}{l}{-p_1x_1+w_1t_1+\bar{c}_1y_1 - p_2x_2+w_2t_2+\bar{c}_2y_2 - p_3x_3+w_3t_3+\bar{c}_3y_3 - p_4x_4+w_4t_4} & = z\ (\min)
\end{array}
$$

where $\bar{c}_i = c_i + w_i$.

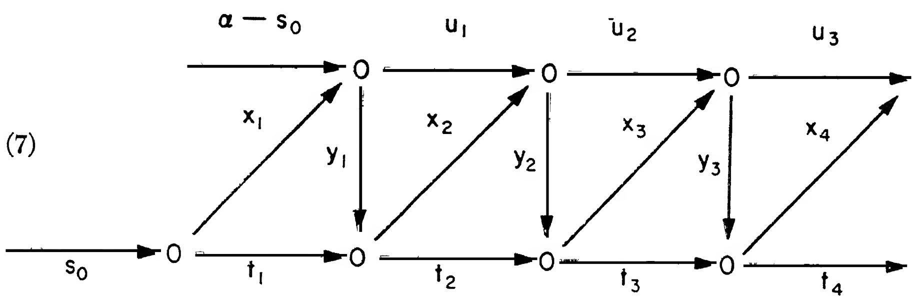
An optimal solution to (5) will be the same if (5) is augmented by the equations $s_i = t_i + y_i$ for $s_i \geqq 0$ if $t_i \geqq 0$ and $y_i \geqq 0$. System (13) which is formed from (5) by linear combinations shown in the right margin is clearly equivalent to it.

$$
(6)\qquad
\begin{array}{cccccccccccccccc|cc}
 & & & & & & & & & & & & & & & \text{Price} & \text{Operation} \\
x_1 & +t_1 & & & & & & & & & & & & & = s_0 & \pi_1 & -\mathrm{I} \\
-x_1 & & +u_1 & +y_1 & & & & & & & & & & & = \alpha - s_0 & \bar{\pi}_1 & \mathrm{I}+\mathrm{I}^* \\
 & -t_1 & & -y_1 & +x_2 & +t_2 & & & & & & & & & = 0 & \pi_2 & -\mathrm{II} \\
 & & -u_1 & & -x_2 & & +u_2 & +y_2 & & & & & & & = 0 & \bar{\pi}_2 & -\mathrm{I}^*+\mathrm{II}+\mathrm{II}^* \\
 & & & & & -t_2 & & -y_2 & +x_3 & +t_3 & & & & & = 0 & \pi_3 & -\mathrm{III} \\
 & & & & & & -u_2 & & -x_3 & & +u_3 & +y_3 & & & = 0 & \bar{\pi}_3 & -\mathrm{II}^*+\mathrm{III}+\mathrm{III}^* \\
 & & & & & & & & & -t_3 & & -y_3 & +x_4 & +t_4 & = 0 & \pi_4 & -\mathrm{IV}
\end{array}
$$

$$
-p_1x_1+w_1t_1+\bar{c}_1y_1 - p_2x_2+w_2t_2+\bar{c}_2y_2 - p_3x_3+w_3t_3+\bar{c}_3y_3 - p_4x_4+w_4t_4 = z
$$

In this form the system displays some remarkable properties. In the first place, it is still in transportation format, i.e., each column has one $+1$ and one -1 or a single $+1$ entry. The corresponding network takes the simple but quite different form from (4), and suggests that it may be worth while to make a study of equivalent networks obtained by manipulations of the equations.

(7)



Of greater importance, however, is that each column in this system has one positive coefficient and the right-hand side is nonnegative. Hence this is a *Leontief-type Model with Substitute Activities* [12]. Accordingly, the *Samuelson Substitution Theorem* [12, 24] may be applied. It states that there exists an optimal solution in which either x_1 and/or t_1 appears as a basic variable, similarly either u_1 and/or y_1, x_2 and/or t_2, $\cdots$, x_4 and/or t_4. This fact is evident if the right-hand side is perturbated to be positive. Since the number of basic variables equals the number of equations, only one of each pair can occur. This shows that *in an optimal solution, at the selling stage, either the entire inventory will be sold* ($t_i = 0$) *or there will be no sales* ($x_i = 0$); *at the buying stage, either the inventory will be at capacity* ($u_i = 0$) *or there will be no purchases* ($y_i = 0$).

The most remarkable property about such models concerns the fact that the optimal choice of basic activities *depends only on the objective form and not upon the right-hand side*. This is always true under mild conditions (e.g., there exist feasible solutions to the system for at least one positive right-hand side and z has a finite lower bound). Suppose in general we have a *"block triangular"* linear programming problem [11] of the form

$$
\begin{array}{lll}
(8) & \begin{array}{l}
A_{11}X_1 \\
A_{21}X_1 + A_{22}X_2 \\
A_{31}X_1 + A_{32}X_2 + A_{33}X_3 \\
A_{41}X_1 + A_{42}X_2 + A_{43}X_3 + A_{44}X_4 \\
C^{(1)}X_1 + C^{(2)}X_2 + C^{(3)}X_3 + C^{(4)}X_4
\end{array} & \begin{array}{l}
= b^{(1)} \\
= b^{(2)} \\
= b^{(3)} \\
= b^{(4)} \\
= z
\end{array}
\end{array}
$$

where A_{ij} are matrices, X_1 is a vector of activity levels for the first stage, X_2 a vector of activity levels for the second stage, etc., $\cdots$, $C^{(i)}$ is a row of costs, and $b^{(i)}$ a column of constraints. We assume further that (8) is also a Leontief Substitution Model so that each column of coefficients has *one and only one positive coefficient* and this occurs in the diagonal matrix A_{ii}. Also we assume all components of $b^{(i)}$ nonnegative.

Let B be any starting feasible basis. For example, in the warehouse model, the submatrix of coefficients resulting from arbitrarily selecting one of each of the paired variables can be used as a starting basis. Let B be partitioned, see (9),

$$
(9) \qquad B = \begin{bmatrix} B_{11} & & & \\ B_{21} & B_{22} & & \\ B_{31} & B_{32} & B_{33} & \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}
$$

$$
\gamma = [\gamma^{(1)} \quad \gamma^{(2)} \quad \gamma^{(3)} \quad \gamma^{(4)}]
$$

to correspond to (8) and let $\gamma^{(i)}$ be the coefficients of $C^{(i)}$ corresponding to the columns in the basis. It is easy to see that *the diagonal arrays B_{ii} must be square Leontief matrices, each possessing an inverse.* In solving a general model of this type by the revised simplex method, it is only necessary to maintain the inverse of these diagonal submatrices rather than the inverse for all of B from iteration to iteration.

Optimization in this model consists of solving a number of smaller Leontief-type models—one for each stage. What is determined is the optimum choice of activities for the last stage (but not their activity levels). This is followed by the optimum choice of activities for the next to last stage, etc., until an optimum choice is known for the first stage. Once all the columns in the basis are known, the activity levels for the first period can be computed, then for the second period, etc. To illustrate the procedure just described in a little greater detail, the first step is to compute the prices associated with equations for the last period by

$$
(10) \qquad \pi^{(4)} = \gamma^{(4)} B_{44}^{-1}
$$

If all components of the vector

$$
(11) \qquad C^{(4)} - \pi^{(4)} A_{44}
$$

are nonnegative, the activities in the basis associated with the last period are the optimal choice. If not, the column corresponding to the most negative component is next substituted for the column in the basis with a positive coefficient in the same row (the "substitute" activity) to form a new basis. It should be

noted that the activity levels are not computed. The process is repeated until an optimum selection of columns for the basis for the last period has been determined. Once obtained, it is never changed.

The process is now repeated for the third period by using the final $\pi^{(4)}$ pricing vector to replace $C^{(3)}$ by

$$\bar{C}^{(3)} = C^{(3)} - \pi_4 A_{43}. \tag{12}$$

Then prices are next computed by

$$\pi^{(3)} = \bar{\gamma}^{(3)} B_{33}^{-1} \tag{13}$$

where $\bar{\gamma}^{(3)}$ are the components of $\bar{C}^{(3)}$ corresponding to columns in the basis. The choice of third stage columns is optimal if all components of the vector

$$\bar{C}^{(3)} - \pi^{(3)} A_{33} \tag{14}$$

are nonnegative, etc. *It is seen that solution of such a dynamic system reduces to a sequence of single period problems.*

For our four-stage warehouse example this is particularly simple. Let π_i denote the price associated with the ith selling stage and $\bar{\pi}_i$ the price associated with the ith purchasing phase. Then from (10)–(14), the optimal program is found by following seven easy steps:

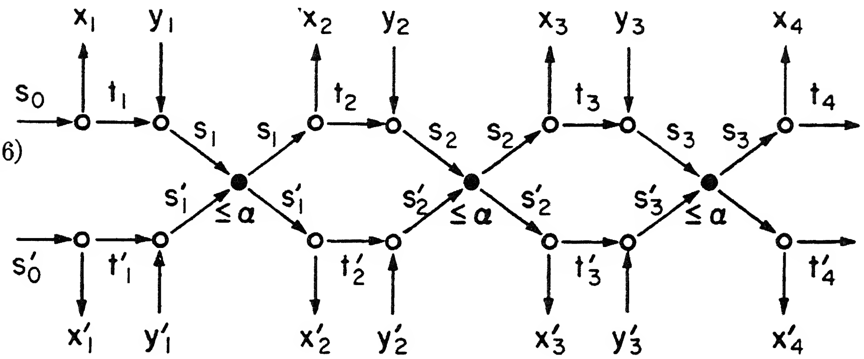
(15)

	Computation	*Decision*
1.	$\pi_4 = \text{Min}\,(-p_4, w_4)$	Sell if first term[1] is minimum
2.	$\bar{\pi}_3 = \text{Min}\,(0, \bar{c}_3 + \pi_4)$	Buy if second term is minimum
3.	$\pi_3 = \text{Min}\,(-p_3 + \bar{\pi}_3, w_3 + \pi_4)$	Sell if first term is minimum
4.	$\bar{\pi}_2 = \text{Min}\,(\bar{\pi}_3, \bar{c}_2 + \pi_3)$	Buy if second term is minimum
5.	$\pi_2 = \text{Min}\,(-p_2 + \bar{\pi}_2, w_2 + \pi_3)$	Sell if first term is minimum
6.	$\bar{\pi}_1 = \text{Min}\,(\bar{\pi}_2, \bar{c}_1 + \pi_2)$	Buy if second term is minimum
7.	$\pi_1 = \text{Min}\,(-p_1 + \bar{\pi}_1, w_1 + \pi_2)$	Sell if first term is minimum

The Generalized Warehouse Model: Charnes and Cooper [6] have shown a similar result for the case of several commodities sharing the same warehouse. This is interesting, for as a rule the theory of multicommodity transportation-type problems lacks the elegance of the single commodity case. We illustrate this result for three periods and two commodities; the procedure is general. If the variables and constants associated with the second commodity are denoted by primes, then, analogous to (4), we have the network two-commodity flow problem

[1] I.e., if $-p_4 = \text{Min}\,(-p_4, w_4)$, then *sell* in fourth selling stage; if $w_4 = \text{Min}\,(-p_4, w_4)$, *do not sell.*

(16)



where the heavy dot • represents the warehouse capacity equation in various time periods. The formal equations are given in (17). This system is equivalent to (18) where the steps required to derive the equations from (17) are shown on the right. It is clear that (18) satisfies the conditions for both a transportation problem and also a Dynamic Leontief System with Substitution.

From the latter it follows that *at each selling stage either the entire product is sold or none, at each buying stage at most only one product is purchased or none.* It may be solved iteratively backward from the last period for determination of the optimal choice of activities independent of the right-hand side. The decision rules analogous to the one-commodity case (15) are just as easy to set down.

Solving Dynamic Problems from Steady State Problems

Perhaps one of the most exciting ideas to date is a possibility that the solution to a dynamic problem might be obtained as a by-product of an iterative procedure for solving a steady state problem.

Ford and Fulkerson in [20] first used the Primal Dual Algorithm to find the maximal steady state flow in a network with fixed capacity on arcs. Later they tried out their methods on a dynamic network problem where the objective was to maximize the total flow in T time periods. For this model, in addition to the fixed capacity, there was a time to *traverse* each arc. They discovered that by slightly altering their algorithm for the solving of a steady state problem the successive cycles were producing optimal solutions to first a $T = 1$, then a $T = 2$, then a $T = 3$, etc., dynamic network flow problem.

The interesting open question is whether this idea can be generalized.

The Functional Equation Approach

This approach has been developed with special reference to multistage processes [3]. At the beginning of each stage there is a *status vector* which typically represents the inventories available to perform activities within the stage and subsequent stages. The structure of the model being such that *the status vector*

$$
(17)\quad
\begin{array}{llllll}
-x_1-t_1 & & & & =-s_0 & \mathrm{I}\\
 & & -x_1'-t_1' & & =-s_0' & \mathrm{I}'\\
t_1+y_1 & & t_1'+y_1' & +u_1 & =\alpha & \mathrm{I}^*\\
t_1+y_1-x_2-t_2 & & & & =0 & \mathrm{II}\\
 & & t_1'+y_1'-x_2'-t_2' & & =0 & \mathrm{II}'\\
t_2+y_2 & & t_2'+y_2' & +u_2 & =\alpha & \mathrm{II}^*\\
t_2+y_2-x_3-t_3 & & & & =0 & \mathrm{III}\\
 & & t_2'+y_2'-x_3'-t_3' & & =0 & \mathrm{III}'
\end{array}
$$

$$
-p_1x_1+w_1t_1+\bar{c}_1y_1-p_2x_2+w_2t_2+\bar{c}_2y_2-p_3x_3+w_3t_3-p_1'x_1'+w_1't_1'+\bar{c}_1'y_1'-p_2'x_2'+w_2't_2'+\bar{c}_2'y_2'-p_3'x_3'+w_3't_3=z(\min)
$$

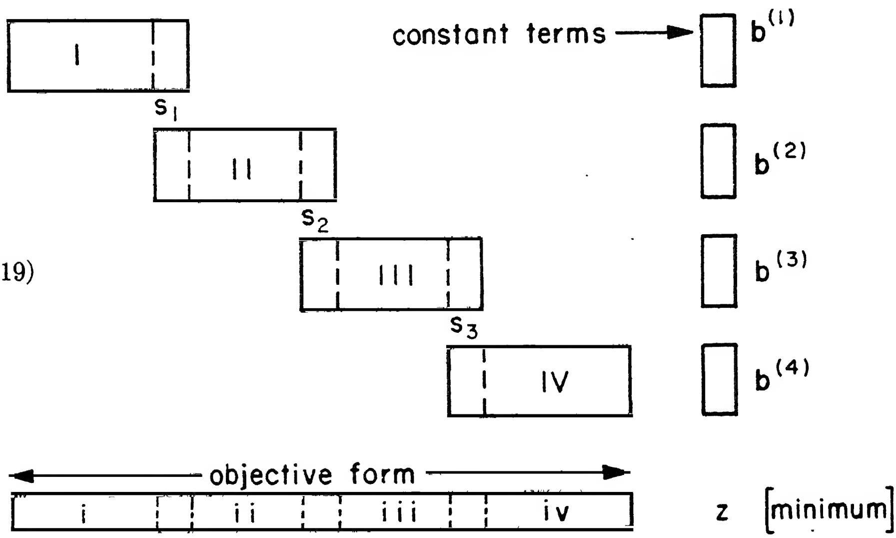
where $\bar{c}_i=c_i+w_i$, $\bar{c}_i'=c_i'+w_i'$.

$$
(18)\quad
\begin{array}{lllll}
+x_1+t_1 & & & =s_0 & -\mathrm{I}\\
 & +x_1'+t_1' & & =s_0' & -\mathrm{I}'\\
-x_1\quad +y_1 & -x_1'\quad +y_1' & +u_1 & =\alpha-s_0-s_0' & \mathrm{I}^*+\mathrm{I}+\mathrm{I}'\\
-t_1-y_1+x_2+t_2 & & & =0 & -\mathrm{II}\\
 & -t_1'-y_1'+x_2'+t_2' & & =0 & -\mathrm{II}'\\
-x_2\quad +y_2 & -x_2'\quad +y_2' & -u_1+u_2 & =0 & \mathrm{II}^*-\mathrm{I}^*+\mathrm{II}+\mathrm{III}'\\
-t_2-y_2+x_3+t_3 & & & =0 & -\mathrm{III}\\
 & -t_2'-y_2'+x_3'+t_3 & & =0 & -\mathrm{III}'
\end{array}
$$

represents the only connection between these activities and those that occur prior to the stage. The activities within a stage transform the status vector into a new status vector for initiation of the next stage. In our warehouse example the status vector has *only one component*, either s_i at the beginning of a selling stage or t_i at the beginning of a buying stage.

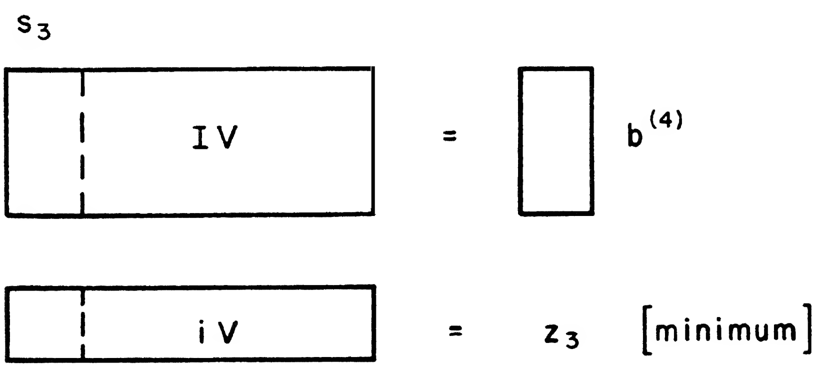
The following is an elaboration of a proposal by R. Bellman [4]. Let us suppose, for a general multistage linear programming model (1), that there is only one variable s_i shared in common between stages which we will call the status.

(19)



In order to solve the problem we begin by finding the optimal program for the last stage, see (20), where s_3 is treated as a parameter. This is done by solving the linear programming

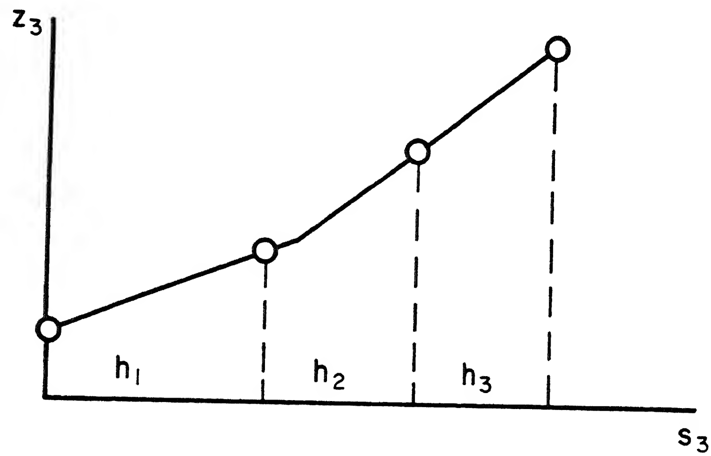
(20)



problem (20) for a particular value of s_3 (say, $s_3 = 0$). If the simplex method is used for the minimization, the solution to the dual is obtained as well. This

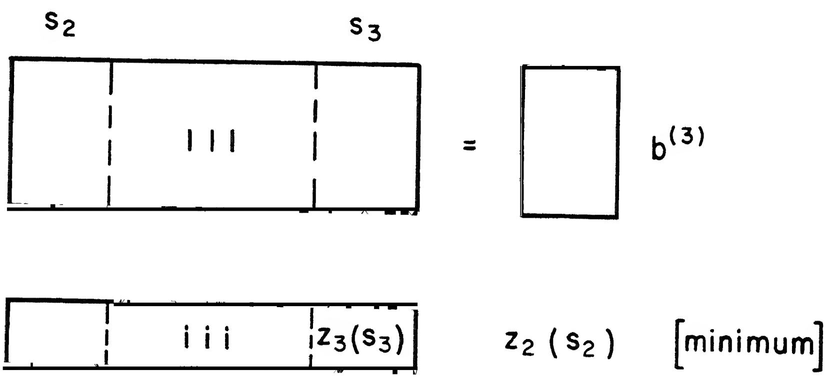
permits use of a technique known as parametric linear programming—a variant of the dual simplex algorithm—by means of which s_3 can be varied over any specified range of values, yielding the contribution z_3 of the activities in the fourth stage to the objective form as a function of s_3. Now $z_3 = z_3(s_3)$ is a broken line convex function of s_3 and only the values at the breakpoints are recorded; those between are available by linear interpolation; see (21).

(21)



The next step is to find the optimal program for the 3rd stage where s_2 is treated as a parameter; see (22):

(22)



This is not a standard linear programming problem because we have a *broken line* function $z_3(s_3)$ instead of the usual *linear* function of s_3 in the objective form. Since $z_3(s_3)$ is convex, it is possible [8, 10, 11] to substitute for s_3:

$$
\begin{aligned}
s_3 &= \lambda_1 + \lambda_2 + \cdots + \lambda_k, \qquad (0 \leqq \lambda_i \leqq h_i), \\
z_3(s_3) &= \epsilon_1\lambda_1 + \epsilon_2\lambda_2 + \cdots + \epsilon_k\lambda_k,
\end{aligned}
\tag{23}
$$

where $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_k$ are the slopes of the broken line segments and $h_1, h_2, \cdots, h_k$ are the widths of the intervals between dotted lines in (21).

After the substitution the problem is again a standard linear programming problem; hence we may repeat the procedure of stage IV for stage III, generating a convex broken line function $z_2(s_2)$.

Where upper bound methods are available it will not be necessary to express the conditions $0 \leqq \lambda_i \leqq h_i$ explicitly [11]. Under this approach no extra equations are required and very little extra work. If an upper bound technique is not available one can substitute instead

(24.0)
$$\begin{aligned} s_3 &= a_1\lambda_1 + a_2\lambda_2 + \cdots + a_{k+1}\lambda_{k+1}, \qquad \lambda_i \geqq 0, \\ z_3(s_3) &= c_1\lambda_1 + c_2\lambda_2 + \cdots + c_{k+1}\lambda_{k+1} \end{aligned}$$

and add the condition

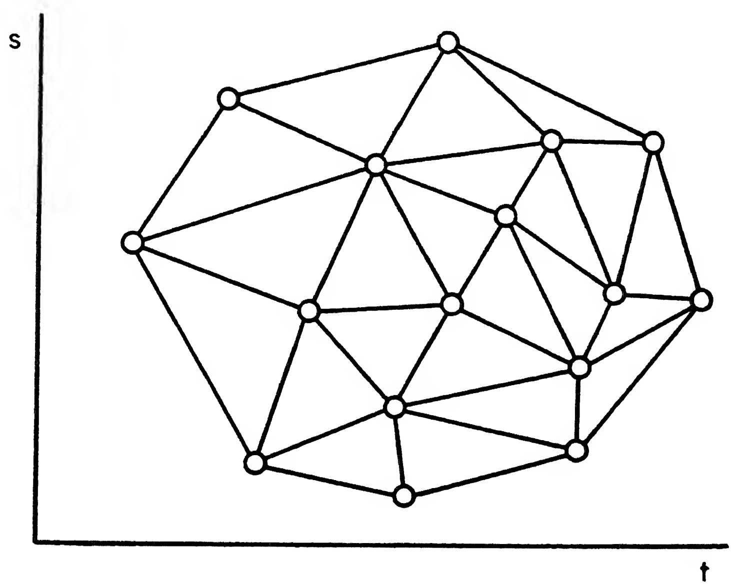
(24.1)
$$1 = \lambda_1 + \lambda_2 + \cdots + \lambda_{k+1},$$

where (a_i, c_i) are the values of s_3 and z_3 at the ith breakpoint.

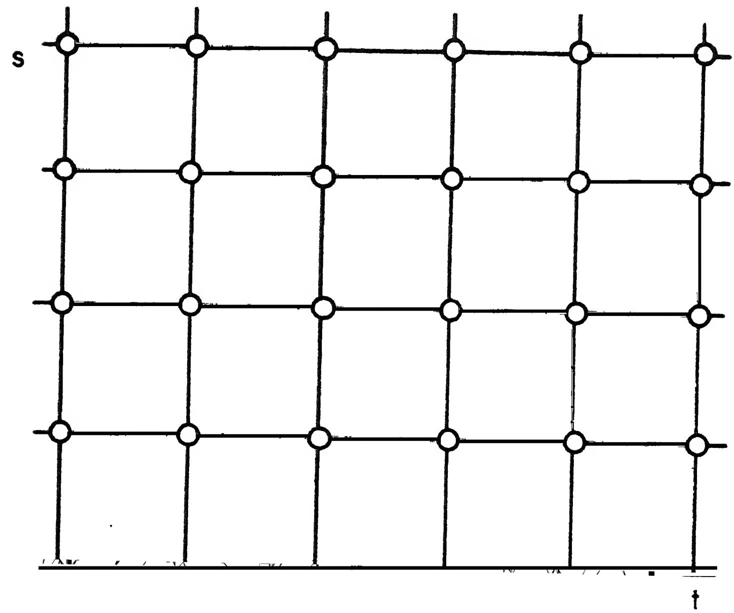
Continuing in this manner, we can compute successively the functions $z_3(s_3)$, $z_2(s_2)$, $z_1(s_1)$. Since the initial status, a_0, is known we are now in a position to solve for the optimal program for the first stage activities. The value of s_1 thus determined may be used to determine the optimal program for second stage activities, including the value of s_2, which in turn permits the solution for the third stage, etc.

In theory, the functional equation approach could be extended to the case where there are two components in the status vector, say (s, t), which are the shared variables between successive stages, by working out the convex of possible values of s, t and the value of $z(s, t)$. The resulting convex surface in three dimensions could be represented by its vertices circled points in (25.0). As an alternative, the

(25.0)



(25.1)



value of z at a number of grid points (25.1) could be determined and these points used instead of vertex points to approximate the surface. It is clear, however, that when stages are tied together by more than one variable, the functional equation approach becomes increasingly tedious to apply.

PART II—THE GENERAL CASE

The Need to Solve Large Scale Systems[2]

At the present time it is possible to solve linear programming problems of the order of two hundred equations and almost any number of variables with reasonable accuracy and costs on electronic computers. Codes are available for even larger systems, but the increased time for solution and the increased accuracy requirements place a practical limit on the size of systems that can be solved by general linear programming techniques.

For linear programming problems involving matrices which exhibit a special structure, it seems possible to develop special techniques that can extend the size of systems many times.

Thus typical of the large scale problems encountered in practice are those concerned with distributing a homogeneous product from several sources to multiple destinations. For example, the optimal shipping program for a milk company with twelve sources (milk shed areas) distributing canned milk to two thousand warehouses requires the solution of a system of more than 2000 equations in 24,000 unknowns. Fortunately there is a highly developed theory for Koopmans' Transportation-type Problem, of which this is an example, which makes it possible to solve systems of this size [7, 14, 19, 24, 29].

[2] The remarks in this section are similar to those found in 10.

Air transport problems [26] and communication problems [22] have structures similar to but unfortunately more complicated than the classical transportation problems. Because they are "multi-index" problems, even the simplest of such systems, while very special in structure, can be enormous in size.

Consider, for example, the problem of routing cargo aircraft. Let the variable x_{ijk} represent the number of aircraft of type k routed between city i and j. Let us distinguish between six types of aircraft and twenty cities. In addition, consider a second set of variables y_{ijl} which is the tons of cargo shipped between city i and j on the way to l. Our equations become

$$
\begin{aligned}
&\text{Aircraft in} = \text{Aircraft out: } \sum_j x_{cjk} = \sum_i x_{ick} \qquad (k = 1, \cdots, 6)(c = 1, \cdots, 20)\\
&\text{Cargo in} = \text{Cargo out: } a_{cl} + \sum_j y_{cjl} = \sum_i y_{icl} + b_{cl}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l = 1, \cdots, 20)(c = 1, \cdots, 20)\\
&\text{Tonnage Cap.} \geq \text{Tonnage Req.: } \sum_k \lambda_{ijk} x_{ijk} = \sum_l y_{ijl}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \cdots, 20)(j = 1, \cdots, 20)\\
&\text{Plane Months Available: } \sum_i \sum_j \mu_{ijk} x_{ijk} = p_k
\end{aligned}
\tag{26}
$$

where a_{cl} is tons of cargo arising at c for l, $b_{cl} = 0$ for $c \neq l$, and b_{ll} is total requirements at l. As we see again such a system involving only a few cities, type aircraft, and cargo destinations can generate easily a system in 1000 equations in 10,000 unknowns.

(27)

	Activities (1st Period)										Activities (2nd Period)									
	Motors		Steel		Tool Prod.			Exess Cap.			Motors		Steel		Tool Prod.			Excess Cap.		
	Prod.	Storage	Prod.	Storage	Steel Cap.	Tool Cap.	Motor Cap.	Steel	Tools	Motors	Prod.	Storage	Prod.	Storage	Steel Cap.	Tool Cap.	Motor Cap.	Steel	Tools	Motors
Variables	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$	$x_{1,9}$	$x_{1,10}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$
Initial inventories																				
Steel cap=			1					+1												
Tool cap = .					λ	μ	ν		+1											
Motor cap= ...	+1									+1										
Steel stocks=	α		−1	1	λ'	μ'	ν'													
Motor stocks=	−1	+1																		
Inventory balance (2nd period)																				
0=			−1		−1			−1					1					+1		
0=						−1			−1						λ	μ	ν		+1	
0=	−1						−1			−1	+1									+1
0=				−1							α		−1	1	λ'	μ'	ν'			
$-d_2$=		−1									−1	+1								

There have been some efforts to develop a theory for these generalized transportation-type problems with only meager results. On the other hand, many numerical examples have been solved by hand methods, suggesting that many difficulties that could arise in theory are not common in practice.

As a second example, consider a hypothetical but typical problem encountered in programming an industrial complex—in this case in the expansion of "motor" production—let us say a special type motor that requires a special type of steel and must use tools fabricated from this steel and the tools which fabricate these tools also use this steel. The tools that fabricate steel we will call below steel capacity, those that fabricate tools—tool capacity, and those that fabricate motors—motor capacity. The initial inventories must satisfy the first 5 equations in detached coefficient form given in the tableau, while the outputs from the activities in the first time period must balance, in the next 5 equations, the corresponding inputs for activities in the second time period, where d_2 is a given demand for motor stocks.

If a planner is interested in developing a program over two years by quarters that meets a specified schedule of known sales and creates the largest stockpile of motors for any future sales that may develop, then the pattern of coefficients in the tableau must be repeated for eight time periods. If we denote the upper and lower blocks by A and B, respectively, the model has the form

$$(28)\qquad \underbrace{\begin{bmatrix} A & & & & & & & \\ B & A & & & & & & \\ & B & A & & & & & \\ & & B & A & & & & \\ & & & B & A & & & \\ & & & & B & A & & \\ & & & & & B & A & \\ & & & & & & B & A \end{bmatrix}}_{10 \times 8} \; 5 \times 8$$

The resulting system of 40 equations in 80 variables with the objective to maximize a stockpile of motors, can be solved in less than a *half hour* on a modern electronic computer. Let this planner now decide that his model is entire *too coarse* and that he must plan by *months*, distinguish *two* types of motors and *two* types of steel and our resultant system becomes 7×24, 14×24 or 164×336. At this size the computation would require now a few hours. However, should the planner again decide to refine the model—either with smaller time periods, a finer breakdown of various commodities, or geographical location, he would discover that general linear programming facilities were inadequate for his problem. Yet techniques such as described in the next section have been applied to the 40×80 system, resulting in an optimal solution in *a few hours by hand*.

Solving General Block Triangular Systems

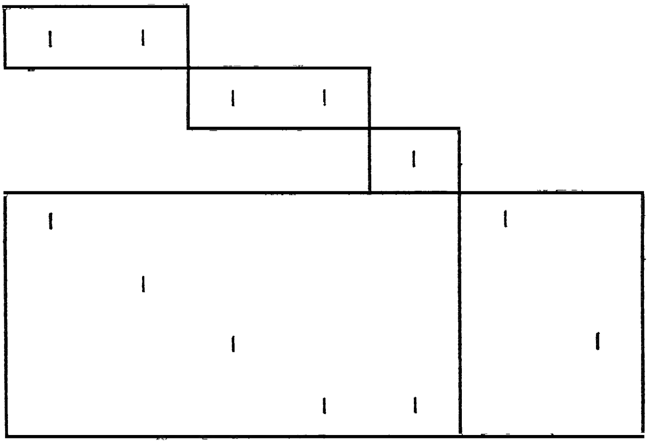
It has been observed that the vast majority of practical problems falls into the block triangular class. The successful computation of the special case of the Dynamic Leontief Substitution Model suggests that a similar approach might be tried for the more general case. Consider a simple three-by-four transportation problem. Its array of coefficients takes the form

(29)

$$
\begin{array}{cccccccccccccccc|l}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & & & & & = a_1 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & & & & & = a_2 \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & & & & & = a_3 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & x_{41} & & & & = b_1 \\
 & +x_{12} & & & & +x_{22} & & & & +x_{32} & & & & x_{42} & & & = b_2 \\
 & & +x_{13} & & & & +x_{23} & & & & +x_{33} & & & & x_{43} & & = b_3 \\
 & & & +x_{14} & & & & +x_{24} & & & & +x_{34} & & & & x_{44} & = b_4
\end{array}
$$

where we have assumed "slack" in the "column" equations to bring out more sharply the structure.[3] A similar structure can be observed for air transport models [26] and communication models [22]. However unlike the substitution model, a basis, see (9), drawn from a structure such as (29) does *not* in general have the property that the diagonal submatrices B_{ii} are square and nonsingular. For example, the basis associated with the variables x_{11}, x_{12}, x_{23}, x_{24}, x_{34}, x_{41}, x_{43} takes the form

(30)



[3] The *General Block Triangular Case* consists of submatrices (blocks) on and below the diagonal as in (8). When these blocks are vacuous except along the diagonal and bottom strip as in (29) it is called *Angular*.

Here the first and second partitions have an *excess* of columns over rows while the last partition has a shortage. This situation is quite typical and greatly complicates the computations. Research work has been concentrated on trying to reduce the computation time of such systems in two main ways:

(*a*) decreasing the number of iterations;

(*b*) finding a compact form for the inverse of the basis.

Experience with larger systems of the order of 20 equations in 500 to 1000 unknowns indicates that they tend to go many hundreds of iterations (using the simplex method) before an optimum is reached. This is most unfortunate as the number of operations required to use the inverse of a basis per iteration goes up roughly as the square of the number of equations and more decimal places have to be carried to maintain accuracy.

To cut down the number of iterations, a number of variants of the simplex method have been proposed whose general purpose is to replace the usual phase I of the simplex method (which seeks a basic feasible solution) by a procedure that produces either optimal or near optimal basic feasible solutions. Of these I mention a few proposals:

(*a*) Beale—*Method of Leading Variables.* A variant of the dual-simplex method [25] that optimizes considering first one, then two, three, etc., constraint equations [1, 13].

(*b*) Orchard-Hays—*Composite Simplex Algorithm.* Uses artificial variables but instead of minimizing their sum as is usual in phase I, starts by minimizing the objective form. Results in a feasible solution to the dual with some infeasibilities remaining in the primal. The dual-simplex method is then applied [28].

(*c*) Dantzig, Ford, Fulkerson—*Primal-Dual Algorithm.* This is a generalization of the Ford-Fulkerson proposal for transportation problems [19]. Using a feasible solution to the dual (or a pseudo-solution to the dual), the infeasibility of the primal problem is minimized over a restricted set of variables whose corresponding dual variables are zero. The dual variables are then adjusted and the process repeated until no infeasibility remains in the primal [15], at which point the solution is optimal.

(*d*) Markowitz—*Maximum decrease of objective form, z, per decrease of infeasibility form, w.* The proposal (unpublished) is to replace the usual criterion for phase I, which introduces into the basic set a variable x_j such that $\partial w/\partial x_j < 0$ is minimal; instead, x_j is chosen such that $\partial w/\partial x_j < 0$ and the ratio $(\partial z/\partial x_j)/(\partial w/\partial x_j)$ is maximal.

The above proposals are applicable in general for any linear programming problem; however, it is believed their use can at best cut down somewhat the number of iterations and perhaps make the difference between success and failure in the solution of a large multistage system. An intuitive suggestion peculiar to multistage systems will now be discussed: Consider a system of type (28) where the submatrices are repeated from period to period. The idea is to try to obtain inductively an optimum solution for a $T = 1, 2, \cdots$ period model. To solve a $T + 1$ period model that maximizes some output (motors, in the example) one could first maximize a T period model. Next, *translate the entire solution forward*

one time period so that the activity levels of period t become those for $t + 1$. For the starting period $t = 0$ start with a set of storage activities. The hope is that the assumed future activities will generate a set of prices that are representative of future actions and therefore a good guide for selection of activities for the first period. It is also hoped that there will be very little in the way of substitution of earlier type activities for later ones. To the extent that this is true, the adjustments are like solving a one-period model. This approach was used successfully on some tests with the motor-steel-tool model referred to earlier (27).

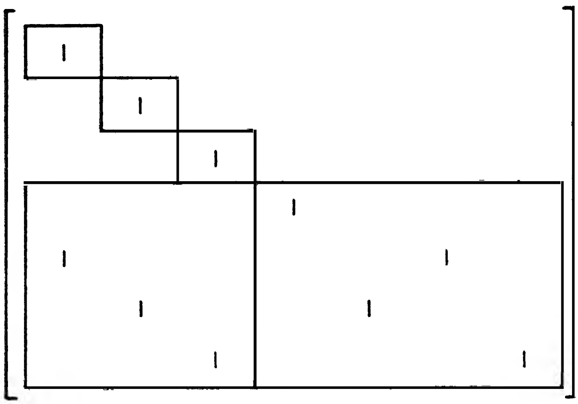
We now turn to proposals for finding a compact form for the inverse of the basis.

(a) The first proposal, due to Markowitz [27], is particularly applicable *whenever the basis* $B = [b_{ij}]$ *is composed largely of zeros.* Consider the linear system

$$\Sigma b_{ij} x_j = y_i , \qquad i = 1, 2, \cdots, m. \tag{31}$$

Markowitz essentially mechanized a hand elimination procedure for solving such a system for x in terms of y: for the pivot element choose a *column with as many zeros as possible.* The selection of the next variable to be eliminated can be made the same way on the reduced system. [It is also possible to seek out rows with many zero entries and carry out certain transpose operations on the matrix.] The information recorded is (a) the operations performed and (b) the back solution. This results in the inverse of the matrix being represented as a product of elementary matrices (i.e., matrices that are the same as identity matrices except for either one row or one column). These columns or rows, as a rule, have a large per cent of zeros. This technique has worked out well in practice. The inverse of the basis from iteration to iteration is maintained by multiplying it by additional elementary matrices. After a number of iterations, however, the compact representation is lost and it is necessary to reinvert the basis from "scratch" to make it compact again.

(b) The second proposal, due to the author, is designed for block triangular structures. It consists in taking a basis such as (30) that does not have square submatrices down the diagonal, and modifying a number of the columns. For

(32) $\bar{B} =$ 

example, suppose the second and fourth columns of (30) are replaced by unit vectors with unity in the third from last and in the last component, respectively. Then upon rearrangement of the columns, the original basis B has been replaced by a pseudo basis $\bar{B}$ as shown in (32).

It is now possible to represent B^{-1} as the product of $\bar{B}^{-1}$ and a matrix which is an identity matrix except for two columns (in this case). The inverse of $\bar{B}$ is never developed explicitly; instead, only the inverses of submatrices down the diagonal are recorded and modified from one iteration to the next. This work is described in [11]. Alan Manne, W. Orchard-Hays, Ted Robacker, and the author have made extensive studies of other ways to transform the basis B into the form $\bar{B}$ for the Angular case. The above proposal is representative. It should be remarked that *this approach is efficient only if the number of excess columns is relatively small.* It is conjectured that this will be so for most block-triangular systems encountered in practice and that it is worth while to build up a computing procedure along these lines.

(*c*) Recently the author, jointly with Philip Wolfe, developed a new procedure that is particularly applicable to angular systems and multistage systems of the staircase type (1). This is reported in preliminary form in RAND P-1544 (November 10, 1958) under the title, "A Decomposition Principle for Linear Programs." The system consists of certain goods shared in common among several parts and certain goods (including facilities, raw materials) peculiar to each part. In short the system is angular in structure.

Although the entire procedure is one intended to be carried out internally in an electronic computer it may also be viewed as a *decentralized decision making process*. Each independent part initially offers a possible bill of goods (a vector of the *common* outputs and supporting inputs including outside costs) to a central coordinating agency. As a set these are mutually feasible with each other and the given common resources and demands from outside the system. The coordinator works out a system of "prices" for paying for each component of the vector plus a special subsidy for each part that just balances the cost.

The management of each part then offers, based on these prices, a new feasible program for his part with lower cost *without regard to whether it is feasible for the system as a whole*. The coordinator, however, combines these new offers with the set of earlier offers so as to preserve mutual feasibility and consistency with exogeneous demand and supply and to minimize cost. Using the improved over-all solution he generates a revised set of prices, subsidies, and receives new offers. The essential idea is that old offers are never forgotten by the central agency (unless using "current" prices they are unprofitable); the former are mixed with the new offers to form new prices.

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III–7

MANAGEMENT MODELS AND INDUSTRIAL APPLICATIONS OF LINEAR PROGRAMMING*

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Little progress in "activity analysis models," at least as far as industrial applications are concerned, appears to have been made since Koopmans' original research [50d]. Perhaps this lack of progress has resulted from the rather extreme example used by Koopmans, a completely delegated, almost egalitarian model of an organization. Perhaps the orientation toward the general problems of an economic system, in the classical tradition, or Koopmans' lack of detailed attention to expedient computation devices account for the fact that inadequate attention has been devoted to the possible value of further developments for industrial applications. This situation should be remedied by devoting attention to the adaptations, modifications and extensions required to make these models suitable for industrial applications.

It is true that Koopmans' formulation needs to be interpreted if it is to be brought within the framework of the more usual forms of linear programming. It is not one, but a series of linear programming problems. (See appendix.) The crux of Koopmans' formulation rests on the concept of efficiency prices. These prices, or their "accounting" counterparts, are intended as internal guides for a decentralized organization—analogous to, say, the so-called internal profit-and-loss control systems employed by many large commerical organizations.[1] The objective is to supply price guides, including prices of fixed facilities, which can be used for bidding by the various departments both for services supplied within the firm itself and from outside sources.

In order to see what is involved consider the system of inequalities in Table I. Each column of the Table represents an "activity" and each row a "commodity." The variables x indicate the levels at which the activities are to be run and the

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[1] *Vide*, Peter Drucker, *The Concept of the Corporation* (New York: John Day and Co., 1946) for a discussion of the system employed by the General Motors Corporation. Gregory Brenstock, Aaron Yuzow and Solomon Schwartz, *Management in Russian Industry and Agriculture* (London: Oxford University Press, 1944) describe a system used by the Soviet Government.

TABLE I
Activity Analysis Model

Activities				Net Outputs and Inputs	Stipulations
X_1	X_2	X_3	X_4		
1				Y_1	$\geqq 0$
	-1			Y_2	$\geqq 0$
-1	-1	1		Y_3	$= 0$
		-1	1	Y_4	$= 0$
-3	-2			Y_5	$\geqq -12$
-5				Y_6	$\geqq -10$
-1	-2			Y_7	$\geqq -9$

values y represent the corresponding amounts of each commodity. The commodities (goods or services) are divided into final, y_1 and y_2; intermediate, y_3 and y_4; and primary products, y_5, y_6, and y_7. These divisions correspond to the relations the variables, y, bear to the stipulations and to sign conventions used to distinguish between inputs and outputs. Final products (y_1 and y_2) are constrained to be non-negative, primary ones (y_5, y_6, and y_7) are constrained to be non-positive and to conform to stipulated limits while intermediate products (y_3 and y_4) are zero. Within any column (in the body of the table) a negative sign attached to any coefficient designates an input to the activity and a positive sign an output.

Koopmans' organization model may be summarized as follows: Each commodity (row) is placed in charge of a custodian and each activity (column) in charge of a manager.[2] Custodians and managers are each to maximize their own "profits." The issue is whether it is possible for a central office committee—a "helmsman" in Koopmans' terminology—to devise a system of prices, or price rules, which will guarantee certain results (not necessarily optimal) to the overall entity. As has already been indicated, only limited guarantees can be offered unless further intervention is allowed. Under certain circumstances efficiency can be achieved. Moreover, as Koopmans shows, by following specified rules of pricing both "inside" and "outside" transactions may be comprehended by these efficiency conditions.

It is important to emphasize both the differences and similarities that exist between Koopmans and the linear programming approaches that have previously been presented. In one interpretation an efficient program is only one that is not obviously wasteful. Thus a point y with coordinates y_i, $i = 1, 2, \cdots$, is said to be efficient if and only if there does not exist a point $\bar{y}$, with coordinates $\bar{y}_i$, $i = 1, 2, \cdots$, which is better. The term "better" is used in the sense of a partial ordering: No coordinate of $\bar{y}$ is less than the corresponding coordinate

[2] This terminology is borrowed from Koopmans [50d], p. 93.

of y, and at least one coordinate is greater. Formally, $\bar{y}$ is better than y if

$$\bar{y} \geq y, \tag{1}$$

where "$\geq$" means $\bar{y} \geqq y$ and $\bar{y} \neq y$. If such a point $\bar{y}$ is available, then y is not efficient.

There are, to be sure, many problems which need to be considered before introducing this concept into an industrial organization. It does, however, have two virtues. First, it focuses attention on the fact (not always recognized in currently employed internal pricing systems) that (a) the maximizing objectives of any particular supervisor[3] and of the overall entity may conflict and (b) the objectives of the various supervisors may also fail to coincide. In short, the improvement secured by one may worsen the position of others. Second, there are regions in which the improvement secured by any supervisor redounds not only to his own benefit but also to the benefit of the entity and, perhaps, other supervisors as well. Thus, if y is not efficient then it is possible for at least one custodian or manager (under properly conceived price rules) to improve his own position without worsening the lot of any other custodian or manager. The prices designed to produce the "correct" behavior under these circumstances are the efficiency prices.[4] Although dynamics of these price arrangements have not been fully worked out, it is possible to ensure, under certain circumstances, that custodians and managers dealing with each other will not be foregoing benefits which they might otherwise obtain by dealing with sources outside the entity.

Koopmans has made one start on problems which are important in cost allocations as well as in organization theory. In dealing with the question of multiple objectives it was necessary for him to alter features of the usual linear programming model. The usual objective, scalar optimization of a single quantity (e.g. total profits or costs) is replaced by a problem in vector optimization. As will be shown in the appendix the activity analysis approach can be reconciled with linear programming. It replaces one linear programming problem by a series of such problems and their duals.

It is conceivable that incorporation of hierarchical[5] and hierarchoid arrangements into the models of activity analysis may provide a start toward adjusting them for industrial applications. But the activity analysis approach is important in its own right. The relations between linear programming and zero-sum two-person games are well known. It is possible that the activity analysis approach may provide a similar bridge for other types of games as well. The so-called Pareto-Nash[6] equilibrium points in non-cooperative game solutions

[3] The term supervisor is here used to refer to both custodians and managers.

[4] A more general formulation would specify other kinds of information, or "misinformation," to be supplied as a means of correcting potential misbehavior by supervisors. This kind of extension is being studied by the authors in collaboration with Martin Shubik of the General Electric Co.

[5] *Vide* [60] for a simplified example and further references.

[6] Cf. [16], [64] and [68].

suggest an affinity and common origin with Koopmans' approach. Extended versions of delegation models may also offer a means of dealing with some of the difficult problems of sub-optimization[7] that are often faced in applied work. In particular, it should be possible to evolve methods for imputing prices (initially or finally) to omitted elements of the system. These possibilities alone would seem to warrant the further research required for industrial applications.

Appendix

Table I of the text was used to illustrate some of the constructs of activity analysis. A major purpose of this approach to programming concerns the analysis of rules which might be employed to guide the activities of an entity under a decentralized management regime. The objectives of each official in such an entity may assume a variety of forms relative to the objectives of other officials, and the entity itself. They may conflict or complement one another, or they may be entirely neutral. (The economic model of a free price economy illustrates the possibilities.) The purpose of the rules (e.g., efficiency pricing) in activity analysis is to ensure that certain levels of attainment (e.g., efficient points) will be secured when each official is allowed to pursue his own objectives.

How can multiple objectives, such as these, be restated in order to make this problem amenable to the methods of linear programming? The purpose of this appendix is to reformulate the models of activity analysis so that this can be done and then to develop computational procedures for locating all efficient points and efficiency prices.[8]

Recall that the matrix A of coefficients[9] in any activity analysis model may be partitioned into three major sectors—*viz.*,

A_F, for final commodities

A_I, for intermediate commodities

A_P, for primary commodities.

A vector $y' = (y_F', y_I', y_P')$ of "commodities" can then be similarly partitioned and associated with each such array of coefficients, by a vector x, of activity levels, defined so that

$$\begin{aligned} &Ax = y \\ (3)\qquad &x \geqq 0, \quad y_F \geqq 0, \quad y_I = 0, \quad -y_P \leqq -\eta_P \quad \eta_P \leqq 0. \end{aligned}$$

Among the possible vectors y are subsets called efficient points. These points are distinguished by the property that it is not possible to improve any component of y_F (within the limits allowed by the restriction) without worsening at least one (and possibly more) of the others.

[7] *Vide* [38].

[8] Also called "shadow prices" and "accounting prices". See Koopmans [50d], p. 65.

[9] E.g., Table I in the text. Such matrices are called "technology matrices." *Vide, loc. cit.*, p. 37.

The following necessary and sufficient conditions of efficiency are established by Koopmans:[10] A vector y is efficient if and only if there exists a vector p[11] with

$$
\begin{aligned}
&p'\,y = 0\\
(4)\qquad &p'\,A \leqq 0\\
&p_F > 0, \qquad p_{P=} \geqq 0, \qquad p_{P>} = 0,
\end{aligned}
$$

where p_F and p_P are prices of final and primary commodities, respectively. $p_{P=}$ indicates those primary commodities (e.g., factors of production) which are utilized to capacity and $p_{P>}$ those which are not.

In order to construct a class of special linear programming problems for locating such y's and p's, it is useful to introduce the concept of "antecedents" of efficient points and prices. These antecedents can be interpreted as activity vectors with special properties. They are here characterized as optimal solutions to dual linear programming problems so constructed that they can be associated with solutions to the problem in vector optimization stated by Koopmans. Means which are available for determining all optimum extreme point solutions[12] to any linear programming problem can then be used to determine all efficient points. An easy extension provides solutions to the corresponding duals. Thus all efficient points and the corresponding efficiency prices can be readily ascertained by linear programming techniques.

The sets of all antecedents of efficient points are unions of convex polyhedral sets. In general, such unions do not form convex sets. Also, since the linear image of a convex set is convex, the sets of all efficient points are unions of polyhedral convex sets and the same holds for the efficiency prices.

The linear programming problem to be considered may now be stated in matrix form as

$$\max.\ v^{0\prime}\,A_F\,x$$

subject to

$$
\begin{aligned}
-A_P\,x &\leqq -\eta_P\\
(5)\qquad -A_I\,x &= 0\\
x &\geqq 0\\
v^0 &> 0
\end{aligned}
$$

with its dual

$$\min\ w_P'(-\eta_P)$$

[10] Theorem 5.4.1, p. 82, *loc. cit.*

[11] This will be called a price vector, following Koopmans, although (as he shows) it may also be related to the concept of marginal rates of substitution. Cf. *loc. cit.*, pp. 66 ff.

[12] E.g., by the use of "Tarry data" for the labyrinth problem. See [9]. There is no loss of generality in confining attention to extreme point optima since all others can be secured from them. See [16].

subject to

$$v^{0\prime}A_F \leqq -w_P{}'A_P - w_I{}'A_I$$

(6) or

$$v^{0\prime}A_F + w_I{}'A_I + w_P{}'A_P \leqq 0,$$

with

$$w_P{}' \geqq 0 \text{ and } w_I{}' \text{ unrestricted.}$$

THEOREM: If

(7) $$(y_F{}', y_I{}', y_P{}') = (x^{*\prime}A_F{}', x^{*\prime}A_I{}', x^{*\prime}A_P{}') \text{ and } x^*$$

is an optimum solution to (5) then y is necessarily an efficient point, and

(8) $$p' = (p_F{}', p_I{}', p_P{}') \equiv (v^{0\prime}, w_I{}^{*\prime}, w_P{}^{*\prime})$$

the corresponding efficiency prices where, of course, $w_I{}^*$ and $w_P{}^*$ are parts of an optimal solution to the dual, (6).

PROOF: It is necessary only to show that these optimum solutions conform to the conditions (4). Constructive procedures for locating all such optima will then be supplied. These solutions provide all efficient points and prices. The condition $p'A \leqq 0$ is, of course, equivalent to (6) since

$$v^{0\prime}A_F + w_I{}'A_I + w_P{}'A_P = p'A.$$

Also

(9) $$p'y = v^{0\prime}A_Fx^* + w_I{}^{*\prime}A_Ix^* + w_P{}^{*\prime}A_Px^* = v^{0\prime}A_Fx^* + w_P{}^{*\prime}\eta_P$$

since $A_Ix^* = 0$. Moreover, by the theorem of the alternative[13] $(w_P{}^*)_r = 0$ whenever $-(A_Px^*)_r < (-\eta_P)_r$. Hence $(A_Px^*)_k = (\eta_P)_k$, $k \neq r$, so that

$$w_P{}^{*\prime}A_Px^* = w_P{}^{*\prime}\eta_P.$$

It therefore follows that

(10) $$p'y = v^{0\prime}A_Fx^* + w_P{}^{*\prime}\eta_P = w_P{}^{*\prime}(-\eta_P) + w_P{}^{*\prime}\eta_P = 0$$

since, by the dual theorem, $v^{0\prime}A_Fx^* = w_P{}^{*\prime}(-\eta_P)$.

The first two conditions in (4) are thus established. The remaining properties, $p_F > 0$, $p_{P=} \geqq 0$, and $p_{P>} = 0$, on the price vector, are also obtained. The condition on p_F is true by the assumptions on v^0. The properties of $w_P{}^*$ used in establishing (9) and (10) are precisely those exhibited in (4)—*viz.*, $(w_P{}^*)_r = 0$ whenever $(A_Px^*)_r > (\eta_P)_r$ and $(w_P{}^*)_k \geqq 0$ for $k \neq r$.

The proof is therefore complete. Any y which has x^* as its antecedent is efficient and $p' = (v^{0\prime}, w_I{}^{*\prime}, w_P{}^{*\prime})$ is the corresponding vector of efficiency prices.

To determine all efficient points and prices it is sufficient to program parametrically[14] over the set of prices

[13] See [80].
[14] Cf. [35].

(11) $$\sum_{r=1}^{n} (v^0)_r = 1$$

with

$$(v^0)_r \geqq \epsilon > 0,$$

for ϵ arbitrarily small. The x^* and w^* obtained by tracing out the labyrinthine path over all such extreme point optima provide the required efficiency prices and efficient points.[15] The procedure is as follows: Start with an optimal solution to $(v^0)_r = \epsilon$. Next, parametrically vary v^0, obtaining new optimal tableaus until such tableaus have been obtained for all[16]

(12) $$v^0 \text{ in } \left\{ v^0 \middle| \sum_r (v^0)_r = 1, \quad (v^0)_r \geqq \epsilon > 0 \right\}.$$

Then for each such optimal tableau develop the alternate basic optima (e.g., by the labyrinth traversal method)[17] noting the dual basic optima as well. When this procedure is completed the efficiency prices will be available along with the antecedents (extreme points) of the efficient points.

Reverting to Table I of the text for an illustration, let $v^{0\prime} = (1, \alpha)$, $\alpha > 0$, then the linear programming model to be used for this example of activity analysis is

$$\max.\ v^{0\prime} A_F x = x_1 + \alpha x_2 + 0x_3 + 0x_4$$

[15] The perturbation procedure provided in [9] resolves all ambiguity with respect to degeneracy or alternate optima that may be encountered in these "wanderings."

[16] In general a tableau will remain optimal for a complete convex subset of the v^0. For, consider a basic solution x^* associated with expression

$$P_j = \sum\nolimits_{i \epsilon I} P_i y_{ij}.$$

Since the solution is optimal,

$$z_j = \sum\nolimits_{i \epsilon I} c_i\, y_{ij} \geqq c_j.$$

For present purposes $c_j > 0$. Fixing the y_{ij} and allowing the c_k's to be variable, then the inequalities

$$\sum\nolimits_{i \epsilon I} c_i\, y_{ij} - c_j \geqq 0$$

define the intersection of n halfspaces. This intersection is non-empty and convex. Its further intersection with

$$\sum\nolimits_j c_j = 1,\ c_j \geqq \epsilon > 0$$

is a bounded, closed convex set and, thus, a polyhedral convex set.

It follows that the set

$$\left\{ v^0 \middle| \sum_{r=1}^{n} (v^0)_r = 1, v_r \geqq \epsilon > 0 \right\}$$

will be swept out in a finite number of parameter variations (each sufficient to induce a change from the previous basic solution) because:

(1) Every optimal set corresponding to a v^0 vector contains at least one basic solution.
(2) There are only a finite number of basic solutions.

[17] See [9].

subject to:

$$
(13)\qquad
\begin{aligned}
-A_I x = 0&: \begin{cases} -x_1 \quad -x_2 + x_3 \qquad = 0 \\ \qquad\qquad - x_3 + x_4 = 0 \end{cases} \\
-A_P x \leqq -\eta_P&: \begin{cases} 3x_1 + 2x_2 \qquad \leqq 12 \\ 5x_1 \qquad\qquad \leqq 10 \\ x_1 + 2x_2 \qquad \leqq 9 \end{cases}
\end{aligned}
$$

with $x \geqq 0$. For simplex solutions, or variants thereof,[18] this is all that is really needed. The solutions $x \geqq 0$ provide the extreme point antecedents of the corresponding efficient y's. Solutions to the dual are read from the $z_j - c_j$ row immediately under the slack vectors or their artifical counterparts.[19] The values $w_I{}^*$, $w_P{}^*$ and $v^{0\prime}$ are the corresponding efficiency prices. By varying $v^{0\prime}$ parametrically, in the parameter α, all such solutions may be obtained.

Inserting artificial and slack vectors, as required in (13), the arrangement shown in Table II-A is secured.[20] For $0 < \alpha < \frac{2}{3}$ the tableau of $II - B$ is a unique optimum with the activity levels $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 5$ appearing under P_0 and the unit value appearing opposite S_5 (in the stub) indicating that one unit of slack is programmed on the receiving facility. These are the antecedents of the efficient point y with final commodities $y_1 = 2$, $y_2 = 3$ and primary commodities $y_5 = -12$, $y_6 = -10$ and $y_7 = -8$.[21] The corresponding efficiency prices may be obtained from $v^{0\prime}$ and from the values for $z_j - c_j$ shown under I_1, P_4, S_3, S_4 and S_5 in Tableau $II - B$. Therefore, for any "prices"[22] $1, 0 < \alpha < \frac{2}{3}$, established for the final products the corresponding prices on the intermediate ones are zero[23] while those for the primary commodities are $\alpha/2$, $\frac{2}{10} - \frac{3}{10}\alpha$ and 0, respectively.

Tableau $II - B$ remains uniquely optimal until $\alpha = \frac{2}{3}$. At this point an alternate optimum is apparent with S_4 in place of S_5. The resulting substitution (indicated by the arrows in $II - B$) yields the alternate shown in $II - C$. For $\frac{2}{3} < \alpha < 2$ the latter is uniquely optimal with activity levels $x_1 - \frac{3}{2}$, $x_2 = \frac{15}{4}$, $x_3 = x_4 = \frac{21}{4}$ and $\frac{5}{2}$ units of slack on M_2. The corresponding program is,

[18] See [12].

[19] See [16].

[20] The artificial vector is I_1, a unit vector with non-zero component in the first row. The slack vectors are $P_4 = S_2$ and S_3, S_4, S_5 where the subscript indicates the row in which the non-zero component appears in these unit vectors.

[21] These signs may be altered, if desired, to denote input-output relations as defined by Koopmans [50d]. The intermediate commodities y_3 and y_4 need not be written down since they must always equal zero.

[22] "Prices", "net profits", or other measures of relative desirability. If $1, \alpha$, are profits then any prices which yield these net results may be used. When "outside trading" is allowed, however, the prices must be established with these possibilities in mind. *Vide* Koopmans, *loc. cit.*, pp. 91ff.

[23] I.e., the values for $z_j - c_j$ shown under I_1 and P_4 in Table II-B. For this purpose the penalty rate, M, associated with the artificial vector, I_1, is ignored.

A: Initial Tableau

			$-M$					1	α	
		P_0	I_1	P_4	S_3	S_4	S_5	P_1	P_2	P_3
	I_1	0	1					-1	-1	1
	P_4	0		1						-1
	S_3	12			1			3	2	
	S_4	10				1		5		
	S_5	9					1	1	2	
Z_j-c_j								-1	$-\alpha$	

B: Optimum Tableau for $0<\alpha<\frac{2}{3}$

1	P_1	2				$\frac{1}{5}$		1		
α	P_2	3			$\frac{1}{2}$	$-\frac{3}{10}$			1	
	P_3	5	1		$\frac{1}{2}$	$-\frac{1}{10}$				1
	P_4	5	1	1	$\frac{1}{2}$	$-\frac{1}{10}$				
←	S_5	1			-1	$\frac{3}{5}$	1			
Z_j-c_j		$2+3\alpha$	M	0	$\alpha/2$	$\frac{2}{10}-\frac{3}{10}\alpha$	0	0	0	0

C: Optimum Tableau for $\frac{2}{3}<\alpha<2$

← 1	P_1	$\frac{3}{2}$			$\frac{1}{2}$		$-\frac{1}{2}$	1		
α	P_2	$\frac{15}{4}$			$-\frac{1}{4}$		$\frac{3}{4}$		1	
	P_3	$\frac{21}{4}$	1		$\frac{1}{4}$		$\frac{1}{4}$			1
	P_4	$\frac{21}{4}$	1	1	$\frac{1}{4}$		$\frac{1}{4}$			
	S_4	$\frac{5}{2}$			$-\frac{5}{2}$	1	$\frac{5}{2}$			
Z_j-c_j		$\frac{3}{2}+\frac{15}{4}\alpha$	M	0	$\frac{1}{2}-\frac{1}{4}\cdot\alpha$	0	$-\frac{1}{2}+\frac{3}{4}\alpha$	0	0	0

D: Optimum Tableau for $\alpha>2$

	S_3	3			1		-1	2		
α	P_2	$\frac{18}{4}$					$\frac{1}{2}$	$\frac{1}{2}$	1	
	P_3	$\frac{18}{4}$	1				$\frac{1}{2}$	$-\frac{1}{2}$		1
	P_4	$\frac{18}{4}$	1	1			$\frac{1}{2}$	$-\frac{1}{2}$		
	S_4	10				1	0	5		
Z_j-c_j		$\frac{18}{4}\alpha$	M	0	0	0	$\alpha/2$	$\alpha/2-1$	0	0

TABLE II A–D

Efficient Point Calculations

final commodities: $y_1 = \frac{3}{2}$, $y_2 = \frac{15}{4}$, intermediate ones zero and primary ones $y_5 = -12$, $y_6 = -7\frac{1}{2}$ and $y_7 = -9$. This new program is also efficient with final "prices" 1, $\frac{2}{3} < \alpha < 2$, intermediate ones at zero and primary prices of $\frac{1}{2} - \alpha/4$, 0, and $-\frac{1}{2} + \frac{3}{4}\alpha$.[24]

When $\alpha = 2$ an alternate optimum is again made available with S_3 in place of P_1, as shown in Tableau $II - D$. This Tableau completes the possibilities since it remains uniquely optimal for all $\alpha > 2$. For all such cases $x_1 = 0$, $x_2 = x_3 = x_4 = 4\frac{1}{2}$. Thus, $y_1 = 0$, $y_2 = 4\frac{1}{2}$, $y_3 = 0$, $y_4 = 0$, $y_5 = -9$, $y_6 = 0$, $y_7 = -9$. Hence the efficiency prices, for final commodities are 1, α on y_1 and y_2, respectively; all intermediate and primary products, with the exception of y_7, receive imputed values of zero and y_7 a price of $\alpha/2$ for all $\alpha > 2$.

[24] At $\alpha = \frac{2}{3}$, or $\alpha = 2$, trouble is caused by the presence of alternate optima so that the price information may not be sufficient for guidance.

This completes the exposition. Further understanding of what is involved and some useful additional information can be secured from the above tableaus. As has already been noted, these efficiency prices can be related to marginal productivities and the marginal rates of substitution of economic theory. They can therefore also be brought to bear in indicating the levels at which the relevant substitutions will be made and thus used to establish sensitivity limits.

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III-8

APPLICATIONS OF LINEAR PROGRAMMING IN THE OIL INDUSTRY*[1]

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This paper is the result of a survey made during the summer of 1956. It is a progress report on applications of linear programming by a number of oil companies. Examples are presented of applications to a variety of problems arising in the areas of Drilling and Production, Manufacturing, and Marketing and Distribution. The examples were selected to illustrate both the power and the limitations of present linear programming methods when applied to actual problems.

1. Introduction

Plans were made during early 1956 for a symposium on industrial application of linear programming to be presented at the Fall Meeting of the Institute. As the theme of that meeting was "A Progress Report" and some of the earliest applications of linear programming were made in the oil industry, it seemed fitting to include in the program a progress report on what the oil industry had been able to accomplish thus far in this field.

We were requested by George Dantzig to present such a review and to include in it not only some of our applications but, if possible, applications by other oil companies as well. With this in mind, about a dozen major oil companies were contacted by us and were invited to contribute linear programming applications or studies they had made which were of general interest and of nonconfidential nature. The response was most encouraging. Because of limited time available arrangements were made to visit personnel of six oil companies for the purpose of discussing their work and ours in the linear programming field.

The oil industry became aware of linear programming through the pioneering work of Charnes, Cooper, and Mellon (1952, 1954) and the work of Gifford Symonds (1953). We owe a great deal to these gentlemen and to Alan Manne (1956) for pointing out to us that linear programming has a place in our business. A few years ago, there were few people indeed in the oil industry who had ever heard of such things as "basic solution" or "convex set". Today, these terms are much more familiar and as a result much less frightening to some. What is involved here is an educational process and educational processes are notoriously slow. It is amazing, therefore, to see how much has been done in such a comparatively short time.

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As technology advances and improves, problems become more interwoven and complex. The problems of the oil industry are no exception. They can logically be grouped into categories according to the different phases of our business as shown in Figure 1. An integrated oil company must first of all carry out exploration activities to determine the spots where oil is most likely to be found. The land must then be acquired or leased and an exploratory well or "wildcat" as it is called is drilled. If luck is with us, we hit oil. Additional wells are drilled to develop the field and production gets underway. The oil is transported by various means to the refinery where a variety of products are manufactured from it. The products in turn leave the refinery, enter the distribution system and are marketed.

Needless to say, each of the areas shown in Figure 1 is full of unanswered questions and problems. Different methods exist for exploring the oil potentialities of a region. How should they be combined for maximum effectiveness? An oil field can be produced in many different ways. Which is best? The complexity of a modern refinery is staggering. What is the best operating plan? And what precisely do we mean by "best"? Of course, not all the problems in these areas lend themselves to linear programming but some of them do. What we would like to do is to pick out a few representative *LP* type problems from each area, show how they were formulated and in some cases, discuss the results that were obtained.

We had hoped to find applications in all four of the areas shown in Figure 1. Unfortunately, we were successful only in three. We did not find any nonconfidential applications in the field of exploration. Exploration is one of the most confidential phases of our business and it is for that reason that oil companies are not very explicit about their studies in this field. We can state, however, from personal experience, that a number of applications to exploration are under investigation.

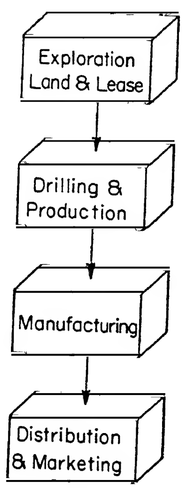

FIG. 1

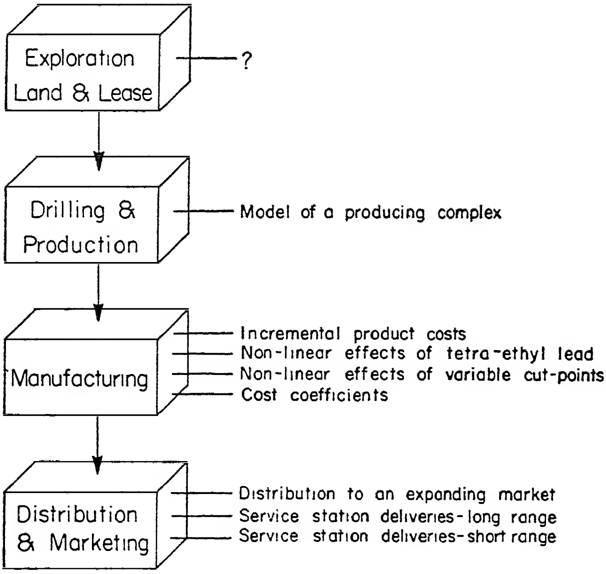

FIG. 2

Let us therefore turn our attention to the remaining three areas of Drilling and Production, Manufacturing, and Distribution and Marketing. Figure 2 shows an outline of the applications that will be discussed. Out of the Drilling and Production area the problem of devising a model for a producing complex was selected. In the case of Manufacturing, the selection was difficult because historically this was the first area of application and much work has been done in this field. The problems shown were selected because they either illustrate an important concept or because they illustrate a peculiar twist in mathematical formulation. The problem of incremental product costs illustrates the technique of parametric programming and also shows what can happen if too many simplifications are introduced. The methods developed for handling tetra-ethyl lead and variable cut points illustrate how, under certain conditions, nonlinearities can be introduced into the system. The problem of cost coefficients will illustrate the need for realistic refinery costs. Finally, three problems out of the area of Distribution and Marketing were selected—a bulk plant distribution problem having to do with the shipment of products from refineries to bulk plants in an expanding market and the problem of devising long-range and short-range delivery schedules from bulk plants to service stations.

2. Model of a Producing Complex

Let us now turn our attention to the first problem on the list—a model of a producing complex. We are indebted to the Field Research Laboratory of Magnolia Petroleum Company and to Arabian American Oil Company for contributing this application. This problem will be discussed in more detail in a forthcoming publication by A. S. Lee and J. S. Aronofsky of Magnolia. Consider N

oil fields or reservoirs ($i = 1, 2 \cdots N$) as shown in Figure 3 which are producing at rates $Q_i(t)$ where t is the time. The total production of the N reservoirs is to be adjusted to meet a commitment $Q_c(t)$ (such as keeping a pipe line full or a refinery supplied). An outside source of crude oil is also available. Let the profit realizable per barrel be $c_i(t)$ and consider that the operation is to be run on this basis for a period of T years. Production limitations exist which require that the $Q_i(t)$ do not exceed certain values and that the pressures in the reservoirs do not fall below certain values. These limits may be functions of the time. We shall consider the case where these fields are relatively young so that development drilling activity will occur during the time period under consideration. The problem is to determine a schedule of $Q_i(t)$ such that the profit over T years is a maximum.

By splitting up the period T into time intervals ($k = 1, 2 \cdots K$) and bringing in the physics of the problem, it can be shown that the condition that the field pressures are not to fall below certain minimum values assumes the form:

$$\sum_{j=1}^{k} (f_{i,k-j+1} - f_{i,k-j})Q_{ij} \leqq P_{i0} - P_{i\,\min} \tag{1}$$

for all i and k. The f's describe the characteristics of the fields and are known. The righthand side is the difference between the initial and the minimum permissible pressure of the i'th field. The variable is Q_{ij} which is the production rate of the i'th field during the j'th period. Additional constraints on the Q_{ij}'s are that the total production for any time period plus the crude oil possibly purchased from the outside source, Q_j, be equal to the commitment for that time period:

$$\sum_{i=1}^{N} Q_{ij} + Q_j = Q_{cj},\ j = 1, 2 \cdots K \tag{2}$$

Furthermore, production limitations exist such that:

$$Q_{ij} \leqq Q_{ij\max} \tag{3}$$

which are simple under bound constraints. The objective function expressing profit over the time period considered is:

$$\sum_{j=1}^{K} \sum_{i=1}^{N} c_{ij} Q_{ij} + \sum_{j=1}^{K} c_j Q_j = \max \tag{4}$$

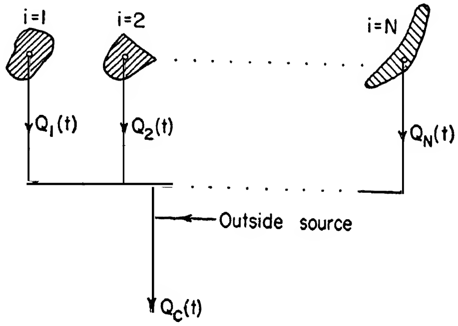

Fig. 3

which completes the formulation of the linear programming problem. The coefficients c_{ij} and c_j are the profit per barrel of the i'th reservoir at time j and correspondingly for purchased crude oil.

Thus far, everything has been rather straightforward. But now, the time has come to clutter up the theory with facts. Let us take a closer look at the coefficients c_{ij}. If we plot revenue vs a particular production rate Q_{ij}, we get a straight line passing through the origin as shown in Figure 4. Cost vs Q_{ij} is also more or less a straight line which, however, does not pass through the origin. The cost function is discontinuous at the origin, corresponding to a set-up charge such as building a road, a pipe line or harbor facilities or installing a gas-oil separator. It drops to zero when $Q_{ij} = 0$ because this corresponds to not yet developing the field. Also shown on Figure 4 is profit vs Q_{ij} which is the difference between revenue and cost. The profit function thus is the straight line shown plus the origin. Hence, we can say that profit from Q_{ij} production is $c_{ij}Q_{ij} - s_{ij}$ where s_{ij} is zero if Q_{ij} is zero and s_{ij} is a constant if $Q_{ij} > 0$. This is a particularly difficult constraint. No general methods are available for handling this except a cut-and-try approach. This type of fixed set-up charge constraint occurs in many practical problems and we shall meet it again later on.

One other complicating feature should be mentioned. Consider that during a certain time period, Q_{ij} was at level "A" as shown in Figure 4 and that in the succeeding time period $Q_{i,\,j+1}$ has dropped to level "B". The profit at level "B" is not obtained by following the profit line to operating level "B" but rather by following a line as shown which is parallel to the revenue line. The reduction in level from "A" to "B" involves merely turning a few valves and essentially does not entail any reduction in operating costs. If, on the other hand, we go from "A" to "C" in succeeding time periods, then we do follow the profit line because an increase in production necessitates drilling additional wells assuming that all the wells at "A" are producing at maximum economic capacity. If we should go from "A" to "B" to "C" in succeeding time periods and if "A" was the maximum field development up to that time, then in going from "B" to "C" we would follow the broken path as shown in Figure 4.

This state of affairs can be handled by building the concept of "production capacity" into the model and requiring that production capacity never decreases with time. But this can be done only at the expense of enlarging the system appreciably.

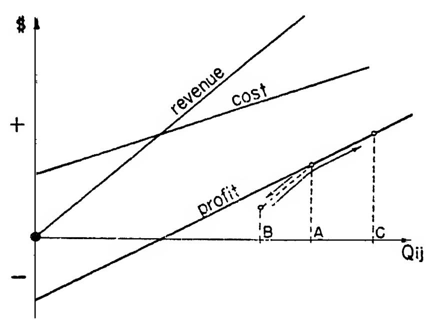

FIG. 4

There exist other factors and additional constraints which must be taken into account. As is so often the case, we are dealing here with a system which on the surface looks rather simple but which becomes considerably more complex as we get deeper into it to make it more realistic. Nevertheless, the simple system or modest extensions of it enables an entire producing complex to be studied thus providing a good basis upon which to build more realistic models.

3. Incremental Product Costs

Let us now leave the problems of petroleum production behind us and venture into the petroleum refinery. As was indicated before, a great deal of work has been done in this area. The few problems we shall discuss will be illustrative of what is going on in this field.

We shall consider at first a simple but nevertheless instructive example. We are indebted to Atlantic Refining Company for contributing this application (Birkhahn, Ramser and Wrigley, 1956). A refinery produces gasoline, furnace oil and other products as shown in Figure 5. The refinery can be supplied with a fairly large number of crude oils. The available crude oils have different properties and yield different volumes of finished products. Some of these crudes must be refined because of long-term minimum volume commitments or because of requirements for specialty products. These crudes are considered fixed and yield gasoline and furnace oil volumes V_G and V_F respectively. From the remaining crudes and from those crudes which are available in volumes greater than their minimum volume commitment must be selected those which can supply the required products most economically. These are the incremental crudes. Denote the gasoline and furnace oil volumes which result from the incremental crudes by ΔV_G and ΔV_F and the total volumes (fixed plus incremental) by V_{GT} and V_{FT}. The problem is to determine the minimum incremental cost of furnace oil as a function of incremental furnace oil production keeping gasoline production and general refinery operations fixed.

The formulation of this problem is straightforward:

$$\sum_1^N a_{Gi} V_i = V_{GT} - V_G = \Delta V_G \tag{5}$$

$$\sum_1^N a_{Fi} V_i = V_{FT} - V_F = \Delta V_F \tag{6}$$

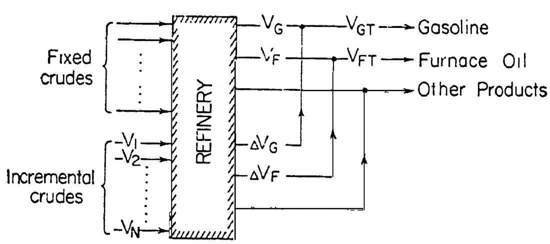

Fig. 5

$$V_i \leqq V_{i\max} \tag{7}$$

$$\sum_1^N c_i V_i = \min \tag{8}$$

where a_{Gi} and a_{Fi} are the gasoline and furnace oil yields of the i'th crude, V_i and $V_{i\max}$ are the volume and availability of the i'th incremental crude and c_i is the cost of producing incremental gasoline plus incremental furnace oil per barrel of the i'th crude. This cost is made up of the cost of crude at the refinery, the incremental processing costs and a credit for the by-products produced at the same time.

The procedure now consists of assuming a value for ΔV_F and obtaining an optimal solution. The shadow price of equation (6) will then be equal to the incremental cost of furnace oil because it represents the change in the functional corresponding to a change of one barrel in ΔV_F. The incremental cost thus obtained, however, is valid only over ranges of variation of ΔV_F which are sufficiently small so that the optimum solution remains feasible. Beyond that permissible range of ΔV_F the basis must be changed with a resulting change in the shadow price. For problems of this type, the so-called "parametric programming" procedure can be used. This procedure has been incorporated into the IBM 704 LP code. It starts with an optimal solution and then varies in an arbitrary but preassigned manner the constants on the right-hand side until one of the basic variables becomes zero. The computer then prints out the optimal solution which exists at that time, changes the basis to an adjacent extreme point which is also optimum and repeats this process until a termination is reached.

An actual problem was run with the model shown on Figure 5. Thirteen incremental crudes were available and incremental gasoline production was fixed at 14,600 barrels daily. The results are shown in Figure 6 which shows the minimum total incremental cost as a function of incremental furnace oil production. Ignore the dashed line for the moment. The circles represent points at which the optimum basis had to be changed. The functional is a straight line between these points. It turned out that incremental furnace-oil production was possible only in the range from about 7100 bpd to about 11200 pbd. Between the two extremes, the functional exhibits a minimum at about 8000 bpd. The reason for the minimum is to be found in the fact that near the two extremes of furnace oil production, little choice exists in the composition of the crude slate. Volume is the limitation and economics plays a secondary part. Away from the two extremes, however, we have greater flexibility in crudes run and thus have the freedom to pick the cheapest crude combination. Figure 7 shows the incremental cost of furnace oil as a function of furnace oil production. It is a staircase type function because the shadow price remains unchanged as long as the optimum basis remains feasible and jumps discontinuously whenever the basis is changed. At low levels of incremental furnace oil production, the incremental cost becomes negative because in that region it is *more* expensive to make *less* furnace oil.

If we now were to show our model and our results to the refiner, he would immediately detect a fly in the ointment. The negative incremental cost at low fur-

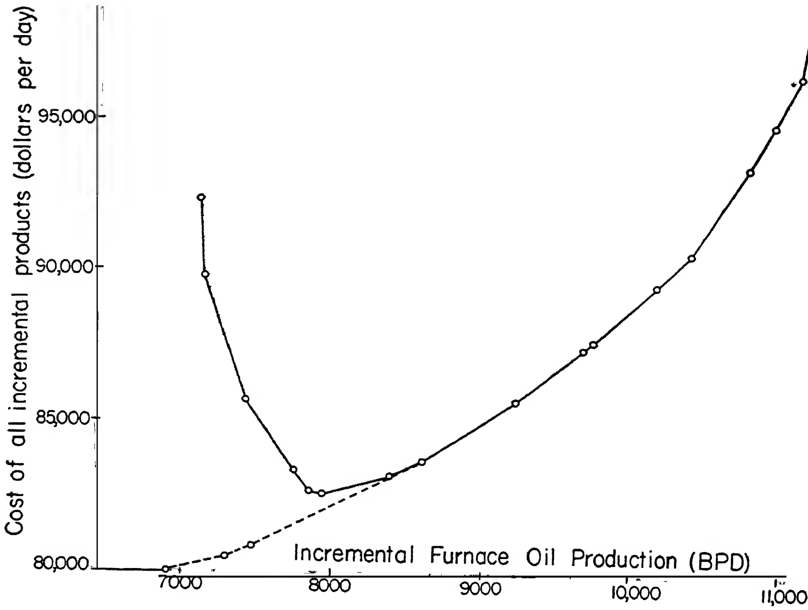

FIG. 6

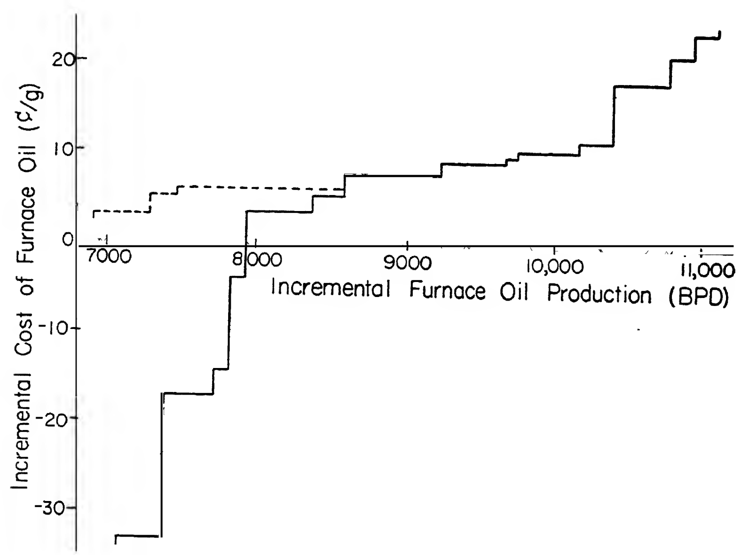

FIG. 7

nace oil production runs counter to his intuitive feeling for the problem. He would point out, and rightly so, that the formulation of our model is not complete. Common sense would dictate the making of the larger volumes of furnace oil at lower cost and disposing of the excess furnace oil in some manner. For example, this excess can be mixed into heavy fuel production. If all the heavy

fuel that is made can be sold, the net cost of the furnace oil over-production would be the negative of the value of heavy fuel indicating a credit we receive for increasing heavy fuel production.

We are tempted, therefore, to try the formulation shown in Figure 8 where we permit the diversion of some furnace oil to heavy fuel. The equation for gasoline production remains unchanged but the furnace oil equation now reads:

$$\sum_{1}^{N} a_{iF} V_i - s_1 = \Delta V_F \tag{9}$$

and the objective form is:

$$\sum_{1}^{N} c_i V_i - v_{HF} s_1 = \min \tag{10}$$

where s_1 is a slack variable indicating the volume of furnace oil diverted to heavy fuel and v_{HF} is the value per barrel of heavy fuel. It is not possible, however, to divert unlimited amounts of furnace oil into heavy fuel without violating heavy fuel's specifications. The upper limit on how much furnace oil can be mixed into heavy fuel depends on the volume of heavy fuel produced which in turn is related to the crude slate, and would depend also on the specifications of heavy fuel. Furthermore, if we bring heavy fuel into the picture explicitly, the cost coefficients used before must be modified. The problem is beginning to become more complex. To take these effects into account would form the basis of an entirely new study. For purposes of the present illustration, however, the situation can be handled roughly as follows. It turns out from experience and by considering the volumes involved that the excess furnace oil production should be less than or at most equal to about 15 per cent of the incremental furnace oil production if all the excess is to go to heavy fuel and specifications on heavy fuel are to be met. Therefore, the additional constraint

$$\sum_{1}^{N} a_{iF} V_i + s_2 = 1.15 \Delta V_F \tag{11}$$

was added to the system where s_2 is a slack variable. This constraint insures that no undue advantage is taken of the freedom introduced by excess furnace oil production.

The results for this second formulation of the problem are shown by the dashed lines in Figures 6 and 7. The abscissa now refers to that part of incremental fur-

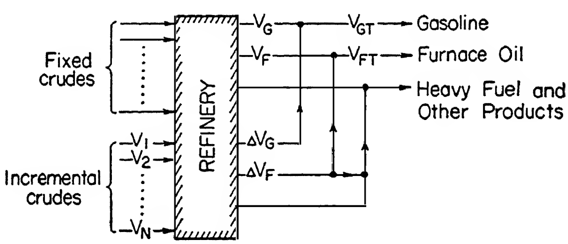

FIG. 8

nace oil production which leaves the refinery as furnace oil. Excess furnace oil is produced below incremental furnace oil production of about 8600 bpd. Above that level, it is not economic to produce more furnace oil than required and consequently, there is no difference between the two formulations of the problem. Constraint (11) is limiting for incremental furnace oil production below about 7500 bpd. Figure 9 shows the composition of the optimum crude slate for the second formulation as a function of incremental furnace oil production. This is useful information to have on hand. Note that no changes occur in the range of incremental furnace oil production from 7500 to 8600 bpd. In this range, actual incremental furnace oil production remains fixed at 8600 bpd with any excess going into heavy fuel.

The modern refinery is a complicated system with strong interdependence among the activities within it. The example just described illustrates this point and shows the importance of the refiners experience in correctly isolating portions of the refinery which can be separately considered.

4. Nonlinear Effect of Tetra-ethyl Lead

The next two applications are concerned with partially nonlinear systems. One of the most common types of nonlinearity encountered in refinery operations is connected with the effect of tetra-ethyl lead (TEL). TEL is added to gasoline to increase the gasoline's octane number. The increase in octane number, however, is not a linear function of the TEL concentration. The first cc of TEL has a pronounced effect on octane number, the second cc, however, has a smaller effect, and for the third cc the effect will be still smaller. The maximum concentration permitted in motor gasoline is 3 cc per gallon.

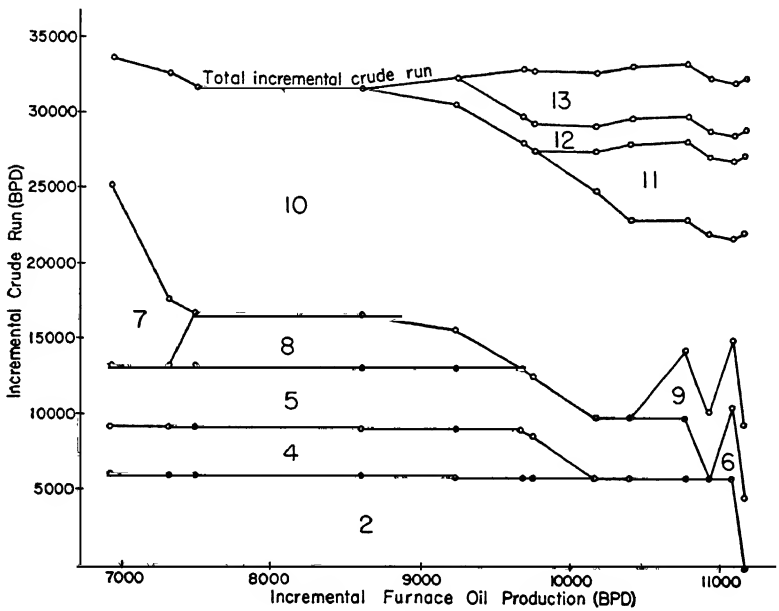

FIG. 9

A great deal of work has been done in the past few years on gasoline blending by linear programming. The problem is to blend the different stocks coming out of the refinery into gasolines having specification properties and to do it at minimum cost. In addition to octane number, other properties such as vapor pressure and various distillation points must be considered. All the important properties blend linearly on a volume basis except for the effect of TEL. To get around the TEL difficulty, it was usually assumed in setting up the linear programming model that the gasoline was shipped out at maximum TEL level of 3 cc per gallon or the TEL level was arbitrarily set at some lower value. In any event, TEL did not enter the system as a variable and thus was not permitted to seek its own level as determined by minimum cost. To get a feeling for the order of magnitude of money involved here, consider an average TEL concentration of 2 cc per gallon. At a price of TEL of about $2 per liter, a TEL bill of about $180,000 results for each million barrels of gasoline produced. Many companies produce of the order of tens of millions of barrels per year. Thus, even a reduction of only a few per cent in lead concentration begins to look big when translated into money savings.

Consider now the general blending problem shown in Figure 10. The streams coming out of the refinery are split three ways—to Premium grade gasoline, to Regular grade gasoline or to temporary storage. Additional stocks may be purchased from outside sources to go into gasoline. TEL is one such stock. The gasoline blends must satisfy a variety of quality specifications such as vapor pressure, distillation points and octane number.

In setting up this problem in linear programming language, we have first of all the usual types of linear constraints which relate the properties of the stocks and the fraction of their volumes to the desired properties of the blended gasoline. There is no difficulty here until we get to the octane condition. The relation we have is that:

$$\frac{\sum ON_{ci} V_i}{\sum V_i} + \Delta ON \geqq ON_s \tag{12}$$

where ON_{ci} is the "clear" octane number of the i'th stock (its octane number with no TEL in the stock), V_i is its volume, ON_s is the specified minimum octane number and ΔON is the octane increase due to lead. The first term on the left represents the "clear" octane number of the blend under the assumption of linear blending. Actually, clear octane numbers do not always blend linearly, but by using so-called "blending" octane numbers instead of actual ones, a sufficiently close linear approximation can be obtained.

Let us now take a closer look at the ΔON term. If, for a specified octane number of the blend, we plot the difference between the clear octane number of the blend and the specified octane number as a function of TEL concentration required to bring the blend up to specification, we obtain a family of curves as shown in Figure 11 where the parameter is a characteristic called "lead susceptibility". It is a measure of the ability of the blend to respond to TEL. Lead susceptibility can be considered to blend linearly with respect to volume. The curves are concave because of the saturation effects previously mentioned.

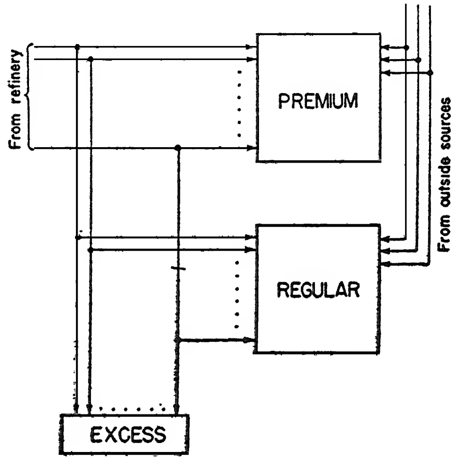

FIG. 10

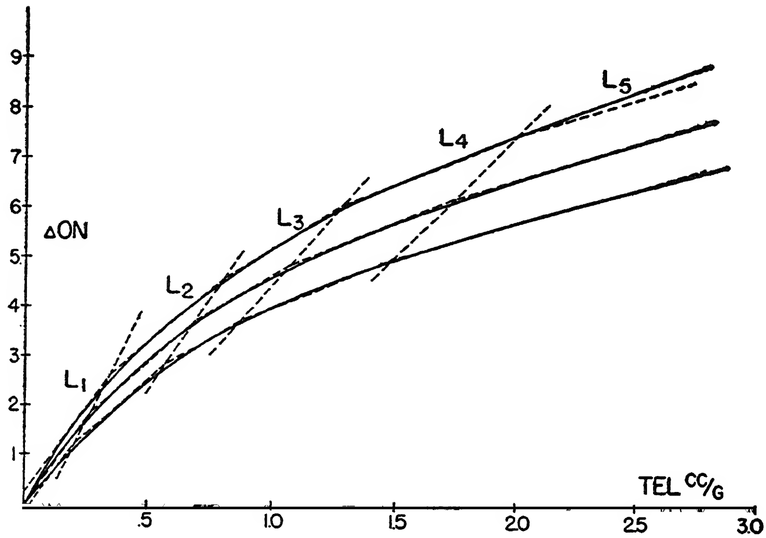

FIG. 11

From past experience, it is usually possible to estimate within reasonable limits what the lead susceptibility of the blend is going to be. We can then construct curves as shown in Figure 11 for the estimated lead susceptibility and for susceptibilities deviating from that value by, say, ±10 per cent. Data are available to do this. Let us now imagine that we split up the curves into, say, five bands as shown such that the curves within each band can be approximated by parallel and equidistant straight lines. This will always be posible by considering a sufficiently large number of bands and a sufficiently small range in lead susceptibility. The situation shown in Figure 11 was considered sufficiently accurate by us for our purposes. The bounding lines between bands are not required to

be parallel. The bands can be interpreted in the following manner. Instead of having only one type of TEL, we have, in this case, five fictitious types—TEL 1 through TEL 5. Each band corresponds to one type of lead. These fictitious leads have two important properties: they do not saturate, i. e., their effect on octane number is linearly related to the amount of each lead present, and the effect of lead susceptibility on ΔON is independent of the TEL concentration. They are not all equally effective, however, as far as increasing the octane number is concerned. In view of the concavity of the function, lead 1 is much more effective than lead 5. We can thus write:

$$\Delta ON = a + b \frac{\sum S_i V_i}{\sum V_i} + \frac{\sum m_j L_j}{\sum V_i} \tag{13}$$

where S_i is the lead susceptibility of the i'th stock, L_j is the amount of lead of type j present in the blend and a, b, and m_j are constants determined from the curves. Also, we know that:

$$m_{j+1} < m_j \, . \tag{14}$$

To insure that the fairy tale of the fictitious leads corresponds to reality, we must impose availability restrictions on the L_j's for otherwise we would satisfy the octane restrictions with L_1 because octane wise it is cheapest and, as a result, get way off the curve. As the straight lines within each band are equidistant, the maximum amount of each lead that can be put into the blend can be represented as a linear function of susceptibility, corresponding to the bounding straight lines between the bands. Hence, we can write:

$$\frac{L_j}{\sum V_j} \leqq d_j + e_j \frac{\sum S_i V_i}{\sum V_i} \tag{15}$$

where d_j and e_j again are constants determined from the curves. Substituting the expression for ΔON into equation (12) and multiplying through by $\sum V_i$, we obtain a system of linear relations which can be incorporated in the over-all linear programming formulation.

We are not yet quite through, however. Each grade of gasoline has two octane requirements which are called the F-1 and F-2 octane specifications. As we have two gasoline grades, this means that we have four octane specifications that must be met. Therefore, we have in reality four families of curves similar in shape to those shown in Figure 11, and all four must be represented by the procedure just discussed and added to the system. We must also distinguish not only among different TEL types but also between TEL going into Premium or Regular to meet the F-1 or F-2 octane requirement. If five fictitious leads are used for each gasoline grade and each octane, then we have a total of 20 fictitious leads which, as far as the matrix is concerned, are separate activities. From a physical point of view, we must impose two additional constraints because the fictitious leads are not completely independent. The total amount of TEL used in Premium to meet or exceed F-1 must be the same as the total amount of TEL used in Premium to meet or exceed F-2 because these two leads are physically identical.

They were separated in the matrix for mathematical reasons only. The same type of constraint applies to Regular. Hence, we must stipulate:

$$\sum L_{j\ \text{Premium}\ F\text{-}1} = \sum L_{j\ \text{Premium}\ F\text{-}2} \tag{16}$$

$$\sum L_{j\ \text{Regular}\ F\text{-}1} = \sum L_{j\ \text{Regular}\ F\text{-}2} \tag{17}$$

Finally, the objective will be of the form:

$$\cdots + C_L \sum (L_{j\ \text{Premium}} + L_{j\ \text{Regular}}) + \cdots = \min \tag{18}$$

where c_L is the unit cost of TEL and the dots indicate other terms whatever they may be.

It is clear that the optimum solution will make physical sense only if the fictitious leads for the limiting octanes are involved in the solution in a physically realizable way. Consider the situation where, let us say, lead 1 and 2 are at their upper bound, while lead 3 and 4 deviate from their upper bound and lead 5 is zero. This is not a physically realizable situation because of the gap existing between lead 3 and 4. This, however, could never occur in an optimal solution because of the concavity of the TEL response curve and because we are aiming to use as little TEL as possible. If a gap exists between L_j and L_{j+1}, it will always be more economic to reduce the level of L_{j+1} and push L_j up to its upper bound because L_j is more effective octane-wise than L_{j+1}. Therefore, we have the assurance that the fictitious leads for the limiting octanes always will be involved in the optimal solution in a physically realizable way. This will not necessarily happen, however, for those leads which belong to the nonlimiting octane specifications. As we have two octane specifications for each grade, there will in general be one octane in each grade which is limiting while there is give-away on the other two. The computer will have no incentive to meet or exceed physically realizably the octane rating for which there is give-away. It cannot make any money by it because the total amount of TEL already is fixed by the octane rating which is limiting as required by constraints (16) and (17). The computer simply picks that octane which is limiting, works on it to meet it most economically and lets the chips fall where they may, as far as the other octane rating is concerned. The optimal solution will still be perfectly satisfactory because the exact value of the give-away for the nonlimiting octane does not affect the solution.

Let us now briefly discuss the results for a case where this approach to the TEL problem was tried. The data were based on an actual situation that existed in one of our refineries a few years ago. In this case, gasoline production was fixed at a given level. The objective was to minimize cost of TEL minus credit for excess stocks. Two solutions of the same problem were available to which the linear programming solution could be compared. One was the solution that was actually used which was calculated in the refinery at the time when the problem arose. This solution, as is often the case, was prepared under severe time limitations. The availability of new blend stocks added to the difficulty of the problem. The other solution to the identical situation was obtained later by allowing sufficient time for a thorough analysis of the problem. In both these solutions, conventional hand blending procedures were used. Table I gives a comparison of TEL

levels between these two solutions and the one obtained by linear programming. The reduction in TEL is clearly evident. The solutions should, of course, not be compared merely on the basis of lead savings. As can be seen from the objective function, the credit for excess stocks must also be considered. The net savings of the linear programming solution still were substantial.

5. Nonlinear Effects of Variable Cut-Points

In the blending problem just discussed, the volumes and properties of the stocks coming out of the refinery were given and the problem was to blend these stocks to make certain end-products in the most economical way. The refinery as a whole was fixed and the optimum blending solution gave us little or no information about what the optimum refinery operation should be. This, of course, is a tremendous problem because the refinery abounds with nonlinearities and all types of mathematically peculiar constraints. One interesting step toward the over-all refinery optimization was discussed recently by Schrage (1956) where linear programming was combined with the method of steepest ascents.

The next application we would like to discuss is an attempt to reach back into the refinery just a little way and optimize with respect to gasoline blending a few of the operating conditions. The conditions we shall consider are the re-run still cut points. A re-run still is a unit within the refinery which separates a stock into light and heavy components. The operating temperature of the unit determines the "cut point" between the two components. The volumes and the properties of the "cuts" are nonlinear functions of the cut point. The cut point can be varied within limits and the question arises as to where the optimum cut point should be for any given gasoline blending situation.

One way of handling this problem is to introduce fictitious stocks as shown symbolically in Figure 12. We assume that instead of having only one cut point we have, say, three cut points corresponding to temperatures T_1, T_2, and T_3

TABLE I

TEL Content in cc/gal. of Blends

	Hand Blend No. 1	Hand Blend No. 2	Linear Prog. Blend
Premium	2.96	2.65	1.51
Regular	0.31	0.56	0.72

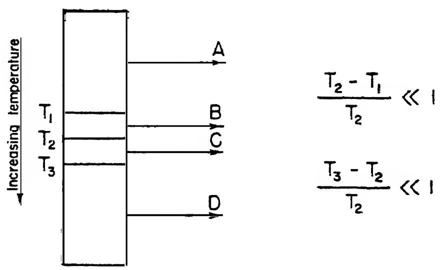

FIG. 12

which yield small fictitious cuts "B" and "C" and major segregations "A" and "D". The volumes and properties of the fictitious cuts are determined such that when they are combined linearly with the major segregations, correct volumes and properties result. Consequently, the fictitious cuts sometimes have abnormal properties when considered by themselves.

The major segregations and the fictitious cuts are now made available to the gasoline blend just as if they were actual stocks coming out of the refinery. The resulting optimal solution then is examined to see what happened to the fictitious cuts "B" and "C" in the shuffle. A number of things can occur as shown in Figure 13.

Because of the natural variation in properties with distillation temperature of the stock, it usually happens that in the optimum solution "A" goes entirely to Premium and "D" goes entirely to Regular. If "B" goes to Premium and "C" goes to Regular, we can conclude that the cut point should be at T_2. If both "B" and "C" go to Premium, the cut point should be at T_3 or at a higher temperature; of they both go to Regular, it should be at T_1 or at a lower temperature. There is nothing in the program that prevents "B" and "C" from splitting. If "B" splits and "C" goes to Regular, the cut point should be between T_1 and T_2. If "C" splits and "B" goes to Premium, the cut point should be between T_2 and T_3. These five situations are the normal ones encountered most of the time because of the normal progression to higher sulfur content and lower octane as the cuts get heavier. Occasionally, however, it may happen that both "B" and "C" split in such a way that the fraction of "C" going into Premium is greater than the corresponding fraction of "B" or that "B" goes entirely to Regular and "C" goes entirely to Premium. These are situations which are not realizable in practice because we have only one cut point in reality. To prevent such situations from occurring, additional constraints are imposed on the system which stipulate that the percentage of "B" going into Premium should be greater than the corresponding percentage of "C". These constraints will insure that the optimal solution will be physically realizable without too much trouble.

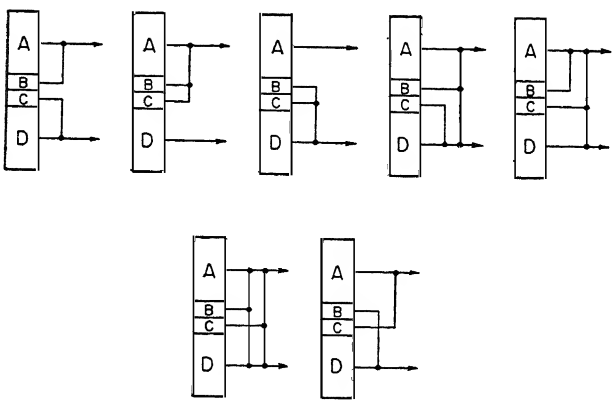

FIG. 13

6. Cost Coefficients

Before leaving the field of refining, let us consider the effect of cost coefficients on optimum gasoline blending. If the objective is an economic one, costs or values have to be determined for some of the stocks that are produced. This can be a complicated problem. In the case of the blending example discussed previously, the objective was to minimize lead costs minus credit for the excess stocks. This meant that a value had to be determined for each stock which was not required to be used up. The situation was complicated further by the fact that some excess stocks were earmarked for shipment to another refinery. This meant that their values had to be the values to that refinery which in turn depended on the local situation existing there during the time period of interest. These costs can be determined but they must be realistic for the solution to have meaning and a great deal of judgment and experience should go into their making.

As an illustration of the effect of the cost coefficients on the optimal solution, consider a hypothetical blending problem where the volumes of Premium and Regular are allowed to vary but their ratio is fixed. The objective was to maximize value of gasoline plus value of excess stocks minus TEL cost while meeting full quality requirements. Two cases were run which were identical in all respects except that in the second case the unit values of Premium and Regular were increased by a small amount. Stocks A through L were available for blending. The results are shown on Figures 14 and 15 where the composition, volume, and TEL content of the gasolines are compared for the two cases. As expected, the optimum gasoline production increased for case 2. Changes in composition also occurred. As far as Premium is concerned, it contains more B than before and contains C which did not enter Premium for case 1. Regular loses its B content and part of C and absorbs F which was not utilized at all in case 1. The gross effect of the change in the price structure is a shift of all of Regular's B and part of its C to

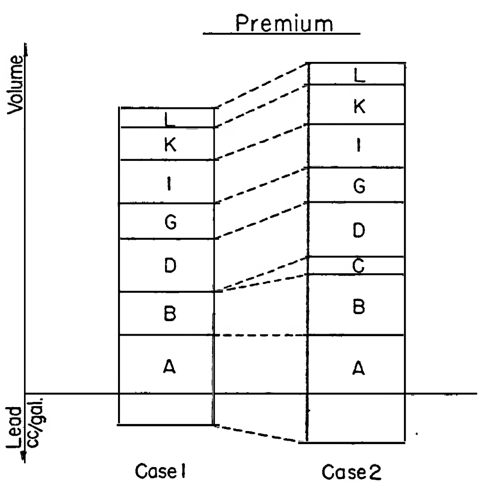

FIG. 14

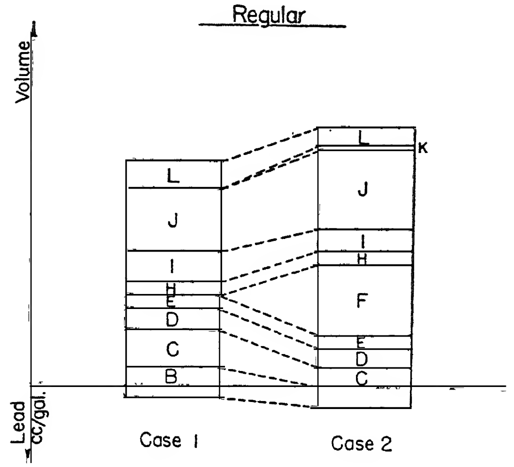

FIG. 15

Premium and extensive utilization of F in Regular. As the change in gasolii value was not drastic, it can be seen that we are dealing here with a systen which is rather sensitive to the price structure.

7. Distribution to an Expanding Market

Leaving the refinery with all its problems behind us we shall now turn to the area of marketing and distribution which has problems of its own. The classic example of a problem in this area is the transportation problem. A great deal of work has been done on this, particularly by oil companies. The first application of linear programming that we would like to discuss in this area is a type of transportation problem which, however, has some complicating features. We are indebted to Atlantic Refining Company for contributing this application.

Consider m refineries ($i = 1, 2 \cdots m$) and n bulk terminals or distribution centers ($j = 1, 2 \cdots\cdot n$) as shown in Figure 16. At the present time, the refineries are producing at levels P_i and the demands at the bulk plants are D_j. We may consider the sum of the P_i's to be equal to the sum of the D_j's so that all the demands are met. Assume now that we find ourselves in an expanding market. Projections are available for what the demand at the different bulk plants is going to be, say, five years from now. Denote these projected demands by D'_j. To try to meet the increased demands, we must expand refining capacity. Denote the increased production by $P_i + e_i$ where e_i is a variable denoting the amount of expansion. We must also increase the capacity of our bulk plants. The expansion of refining and bulk plant capacity costs money and an upper bound exists on how much can be spent on over-all expansion. This upper bound is such that it is impossible to meet the demand at all the distribution centers. The problem now is to determine which refinery and bulk terminal to expand, and by how much, so as to maximize the net return. The maximization of net return may not necessarily be the best objective but we shall use it here for purposes of illustration.

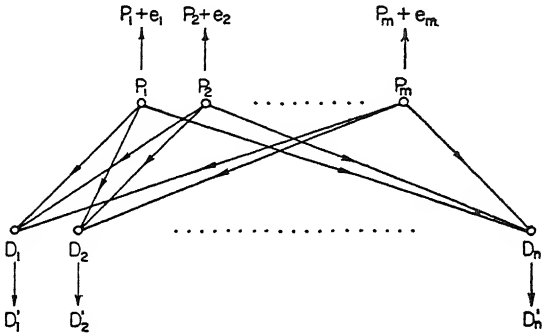

FIG. 16

This problem can be formulated as follows. The total production leaving the i'th refinery must be equal to the old production plus the expansion. Hence:

$$\sum_{j=1}^{n} x_{ij} = P_i + e_i, \qquad i = 1, 2, \cdots, m \tag{19}$$

where x_{ij} is the amount shipped from i to j. The amount received at the j'th bulk plant must be less than or can at most be equal to the projected demand in that area. Hence:

$$\sum_{i=1}^{m} x_{ij} \leqq D_j', \qquad j = 1, 2, \cdots, n \tag{20}$$

If c_{iR} is the unit cost of expanding refinery capacity at i, then the total cost of refinery expansion is $\sum c_{iR}e_i$. In considering the cost of bulk plant expansion, we must take account of the fact that the shipments to some bulk plants may actually be reduced while others expand so as to be able to take full advantage of shifts in the market with the limited amount of expansion capital. The expansion of a bulk plant does require capital but a "contraction" does not because it simply means that shipments to the bulk plant are reduced. To handle this situation, we add the relation

$$\sum_{i=1}^{m} x_{ij} - D_j = s_j^+ - s_j^-, \qquad j = 1, 2, \cdots, n \tag{21}$$

to the system where s_j^+ and s_j^- are non-negative variables. We also stipulate that:

$$\sum_{i=1}^{m} c_{iR}\, e_i + \sum_{j=1}^{n} c_{jB}\, s_j^+ \leqq M \tag{22}$$

where c_{jB} is the unit cost of bulk plant expansion and M is the maximum expansion capital available. The term on the left of (21) is the difference between the new shipments to j and the old shipments. If this difference is positive, then j expands, if it is negative then j "contracts". It can be shown that either s_j^+ or s_j^- but not both will be involved in the optimum basis. Hence, if j expands, then s_j^+ will be in the basis and there will be an expansion cost in view of the

last constraint. If j "contracts", then s_j^- will be in the basis and there will be no expansion cost.

Finally, the objective function is:

$$\sum_{i=1}^{m} \sum_{j=1}^{h} c_{ij} x_{ij} - \sum_{i=1}^{m} c_{iR}\, e_i - \sum_{j=1}^{n} c_{jB}\, s_j^{+} = \max \tag{23}$$

where c_{ij} is the profit per barrel shipped from i to j.

This formulation is satisfactory as long as the new shipments to the "contracted" bulk terminals do not fall below a certain value (they may even go to zero). This is a situation which is analogous to the one encountered in discussing the model of a producing complex. The plot of profit at the j'th bulk plant as a function of shipments to the bulk plant is again a straight line displaced from the origin because of a fixed overhead. The actual profit function is again the straight line plus the origin. Thus, our objective function should really be:

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} c_{ij} x_{ij} - \alpha_j \right) - \sum_{i=1}^{m} c_{iR}\, e_i - \sum_{j=1}^{n} c_{jB}\, s_j^{+} = \max \tag{24}$$

where:

$$\alpha_j = \begin{cases} 0 \quad \text{if} \quad \sum_i x_{ij} = 0 \\ \text{const.} \quad \text{if} \quad \sum_i x_{ij} > 0 \end{cases} \tag{25}$$

but no general method exists for handling situations of this type. However, if it turns out in the optimal solution that none of the bulk plant volumes contract by substantial amounts the solutions will be useful.

8. Service Station Deliveries—Long Range

Having considered the link of refinery to bulk terminal, let us now consider the last link in the chain—the flow of products from bulk terminal to service stations. Consider the situation shown in Figure 17. We are given the location of service stations and the roads connecting them. The small circles are the service stations, while the large circle denotes the bulk plant which supplies them by truck. Each service station, k, requires a delivery of D_k gallons of gasoline (for simplicity, let us assume only one grade of gasoline). Different truck types, denoted by the index s, are available for making the deliveries. The trucks differ in regard to carrying capacity and operating characteristics. We have a number of trucks of each type available for the operation. The problem is to devise a delivery schedule such that the transportation cost is minimized.

We are actually dealing here with two different types of problems depending on whether we look at this operation from the long-range or the short-range point of view. Let us consider the long-range point of view at first.

Assume that we look at this operation over an extended period of time so that the D_k represent total demands at the service stations during the period under consideration. Assume, furthermore, that the ratios of the D_k's to the gallon capacity of each of the trucks is sufficiently large so that many deliveries have to

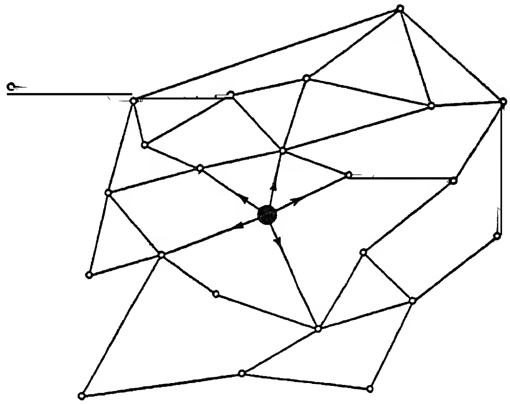

FIG. 17

be made to each station during that period in order to meet the demand. Under these conditions, the problem becomes a transportation type problem with trans-shipment of goods. The trans-shipment feature comes about through the fact that if a truck leaves the bulk plant and makes deliveries to, say, service stations 1, 2, and 3 in that order, then the gasoline destine for station 2 is trans-shipped via station 1 and the gasoline for station 3 is trans-shipped via stations 1 and 2.

Some work has been done on the trans-shipment problem, (Manne, 1954, Kalaba and Juncosa, 1956, Dwyer and Galler, 1956, Orden, 1956) in connection with aircraft scheduling and communication networks. Our problem here is slightly different but the general approach is the same. The key to the mathematical formulation lies in the use of triple indices. Adopt the convention that the first index refers to the point of departure, the second index to the intermediate destination and the third index to the ultimate destination. If y_{ijk} denotes the number of gallons shipped from i to j destined for k, then we can write:

$$\sum_i y_{ijk} = \sum_u y_{juk}, \qquad \text{all } j, k \text{ but } j \neq k \tag{26}$$

$$\sum_i y_{ikk} = D_k, \qquad \text{all } k \tag{27}$$

$$\sum_k \sum_j y_{0jk} = \sum D_k \tag{28}$$

The left side of (26) is the sum of what arrives at j from all points but destined for k while the right side is the sum of what leaves j for all points destined for k. These two must be equal because we do not wish to accumulate anything at j destined for k. Equation (27) states that the sum of what arrives at k from all points and is destined for k must be equal to the demand at k. Equation (28) states that the sum of what leaves the bulk plant (indicated by the index zero) must be equal to the total demand. These three conditions insure that we deliver the proper number of gallons where they are required and that they all originate at the bulk plant.

If x_{ijs} is the number of truck runs per period from i to j in the s-type truck (the index s denotes the type of truck and not ultimate destination), then we must also specify that:

$$\sum_i x_{ijs} = \sum_u x_{jus}, \qquad \text{all } j, s \tag{29}$$

which means that the number of s-type trucks which arrive at j is equal to the number of s-type trucks that leave j.

To insure that we have enough carrying capacity available for each i–j route, we must stipulate that:

$$\sum_k y_{ijk} \leqq \sum_s g_s x_{ijs}, \qquad \text{all } i, j \tag{30}$$

where g_s is the carrying capacity of the s-type truck. The left side represents the actual number of gallons of gasoline that are hauled from i to j, while the right side represents the maximum number that can be hauled. The slack in this relation is indicative of the fact that the trucks may have to run partially full or empty some of the time.

If we denote the time required by the s-type truck to go from i to j by h_{ijs} and let h_s be the time that an s-type truck can be used per period, then we have the additional constraint:

$$\sum_i \sum_j h_{ijs} x_{ijs} \leqq h_s, \qquad \text{all } s \tag{31}$$

which insures that the trucks are not run longer than possible during the time period under consideration.

The objective is:

$$\sum_i \sum_j \sum_s c_{ijs}\, x_{ijs} = \min \tag{32}$$

where c_{ijs} is the cost per trip of operating an s-type truck over the link i–j. Having determined the x's and y's a schedule can then be constructed from them.

For any actual problem, the number of constraints represented by equations (26) to (32) is rather frightening and can be beyond the capacity of even the largest computers if the standard Simplex procedure is employed. Fortunately the matrix involved here exhibits a great deal of structure. Efforts are under way in a number of places to exploit this structure (as the structure of the ordinary transportation problem was exploited) so as to reduce the computational labor.

The problem considered here represents a simplified situation but the type of analysis employed is representative of what is done for more sophisticated models. In any actual problem, the x's are limited to integral values so that the non-integral optimal solution must be adjusted to integral x's. If, as we assumed in the beginning, many trips are necessary to meet the demand, this imperfection may not be too serious. Unfortunately, no general methods exist at present for handling linear programming problems in which some or all of the variables are constrained to be integers.

9. Service Station Deliveries—Short Range

The problem just discussed permits us to look at the over-all situation on a long-range basis. From a short-range point of view, however, the problem is somewhat different and much more complicated. It becomes similar in type to the so-called "clover-leaf problem" or "farmer's daughter problem" which in turn is related to the classic problem of the traveling salesman. In the traveling salesman problem, we have a number of towns which the salesman desires to visit in a sequence such that the total distance traveled is a minimum. In the farmer's daughter problem, we have the same situation plus the additional constraint that the salesman wishes to return to, let us say, town "A" before a certain maximum time has elapsed. In the traveling salesman problem, the solution consists of a single loop while in the farmer's daughter problem, the solution consists of a number of loops, each originating and terminating at town "A". The farmer's daughter, of course, is at "A".

Returning now to our delivery problem and examining it from the short-range point of view, it turns out that it is similar to the farmer's daughter problem except for some additional complications of its own. On a daily basis, the dispatcher at the bulk terminal has a list of service stations to which deliveries of certain amounts must be made *today* because the service stations are on the verge of running out of gasoline. As before, he has trucks of different types at his disposal and his problem now is to devise routes for the trucks so that the deliveries are made at minimum transportation cost. These routes, of course, originate and terminate at the bulk plant. Thus, the bulk plant is equivalent to the farmer's daughter but instead of having only one boy friend she has as many as we have different types of trucks on the road. One of the important differences between the long-range formulation of this problem and the formulation on a daily basis is that in the former the individual trucks lose their identity except for the type to which they belong, while in the latter, each truck must be considered as an entity.

The Operations Research Group at Atlantic became interested in devising means for handling this problem on a daily basis. With the assistance of George Dantzig, a method was devised that is not guaranteed to lead to the optimum solution but will usually yield a solution rather close to it.

10. Conclusion

We have attempted in this paper to discuss some oil industry problems and to indicate how linear programming was or can be used to solve them. There can be no doubt that linear programming has made a place for itself in the oil industry, particularly in the manufacturing phase. It is beginning to be appreciated by management as an important help in making complicated decisions. It must be realized, however, that not everything in this world is linear and that occasionally we come across constraints which are mathematically pathological types. This is good in a way because if ever a method is devised that solves all problems, life would become rather dull. Much still remains to be done. We need

a great deal more basic research on optimization methods in the universities and industrial research laboratories.

It should be pointed out that the successful application of linear programming to practical problems was made possible by the advent of large, high-speed computers and by the existence of an efficient linear programming code. If digital computers were nonexistent, the answers would be many years too late. We would like to express our thanks to William Orchard-Hays and Leola Cutler of the RAND Corporation and to Harold Judd of IBM for the excellent code which they developed for the IBM 704, and made available to industry.

We would like to thank the management and personnel of Magnolia Petroleum, Esso Research and Engineering, Atlantic Refining, Arabian American Oil, Richfield Oil and Shell Development for their assistance and cooperation in the preparation of this paper.

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IV–9

QUADRATIC PROGRAMMING AS AN EXTENSION OF CLASSICAL QUADRATIC MAXIMIZATION*

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The article describes a procedure to maximize a strictly concave quadratic function subject to linear constraints in the form of inequalities. First the unconstrained maximum is considered; when certain constraints are violated, maximization takes place subject to each of these in equational (rather than inequality) form. The constraints which are then violated are added in a similar way to the constraints already imposed. It is shown that under certain general conditions this procedure leads to the required optimum in a finite number of steps. The procedure is illustrated by an example while also a directory of computations is given.

1. Introduction

The problem of quadratic programming consists of maximizing a quadratic function of a vector x,

$$Q(x) = a'x - \tfrac{1}{2}x'Bx, \tag{1.1}$$

subject to the condition that none of the components of x be negative:

$$x \geqq 0, \tag{1.2}$$

and possibly also subject to certain additional linear constraints:

$$C^{*\prime}x \leqq d^*. \tag{1.3}$$

It is assumed that the matrices a, B, C^* and d^* are given and that B is a positive-definite $n \times n$ matrix. Further, it will prove convenient to combine the constraints (1.2) and (1.3) such that they are written as

$$C'x \leqq d, \tag{1.4}$$

which implies $C = [-I \quad C^*]$, $d' = [0 \quad d^{*\prime}]$. Here C' is an $N \times n$ matrix with $N \geqq n$.[2]

Usually, a quadratic programming problem is solved by starting with some x

* Received March 1960.

[1] The authors are indebted to Mr. P. J. M. van den Bogaard of the Econometric Institute for his detailed comments on an earlier version of this paper.

[2] The procedure to be proposed is also applicable when none or only some of the non-negativity constraints (1.2) are imposed. In that case we may have $N < n$.

which satisfies the constraints but which does not necessarily maximize Q, after which this x is replaced by a series of other x's—all of which satisfy the constraints—until the required solution is found.[3] Quite a different procedure is that of maximizing Q without taking account of the constraints, to see whether the resulting solution does or does not satisfy the constraints, and to use this information as a basis for further computations. This approach will be followed here; it was inspired by the work done by one of the authors [9] on quadratic criterion functions in economics, and it is based on the consideration that maximizing a quadratic function subject to linear equations is so much simpler than maximizing subject to linear inequalities. An outline of the procedure is given in Section 2, a more rigorous analysis in Section 3. Section 4 contains an example, Section 5 some concluding remarks.

2. Outline of the Procedure

2.1. *Information Supplied by the Unconstrained Maximum*

The vector which maximizes Q without regarding the constraints is

$$x^0 = B^{-1}a, \tag{2.1}$$

as is easily verified by straightforward differentiation. Clearly, if x^0 satisfies the constraints (i.e., if $C'x^0 \leqq d$), then x^0 is the required solution because a constrained maximum can never exceed the unconstrained maximum. The interesting possibility is therefore the one in which x^0 violates one or more constraints. Consider then Fig. 1 which deals with the simple case $n = 2$ in which the only constraints are nonnegativity constraints. The condition $x_2 \geqq 0$ is violated by x^0; so we have to be satisfied with a lower Q-value than $Q(x^0)$. Now given the fact that B is positive-definite, the locus of constant Q-values is a family of concentric ellipses (ellipsoids if $n \geqq 3$) around x^0. The optimal solution will then be such that it lies in the admissible region (the positive quadrant) and on the ellipse which is nearest to x^0. Clearly, this is the point where an ellipse touches the horizontal axis; in algebraic terms, it is the vector which is obtained by maximizing Q subject to $x_2 = 0$, i.e., subject to the second nonnegativity constraint written in the form of an equation instead of an inequality. We shall say in such a case that this constraint is *satisfied exactly* by the vector considered, or that it is *satisfied in equational form.* Since the vector considered is the optimal vector (to be written as $\hat{x}$ from now on), the result of the example of Fig. 1 can simply be described as follows: The vector of the unconstrained maximum violates one of the constraints, and the constrained maximum (the optimal vector) satisfies the same constraint in equational form.

Next, consider Fig. 2 in which x^0 violates both nonnegativity constraints. Given our experience with the case of Fig. 1, the obvious approach seems to be to impose the two constraints in equational form one after the other. Thus, when imposing $x_2 = 0$ we obtain a vector $x^{(2)}$ which violates the constraint $x_1 \geqq$

[3] For other contributions to the problem of quadratic programming, see the list of references at the end of this paper.

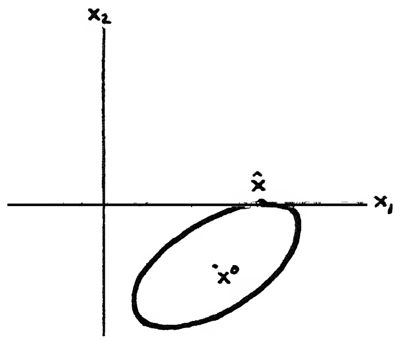

FIG. 1

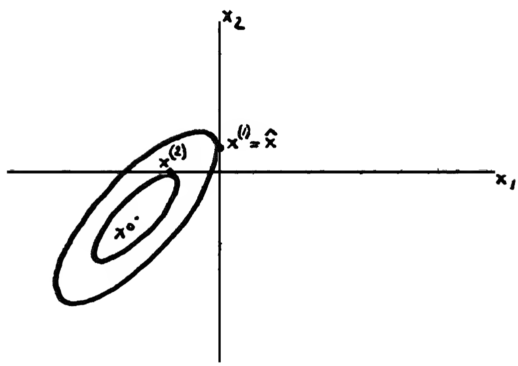

FIG. 2

0;[4] and if we impose $x_1 = 0$ we obtain $x^{(1)}$ which violates no constraint and which is obviously—as graphical inspection shows—the optimal vector $\hat{x}$. This result suggests that the rule mentioned at the end of the preceding paragraph should be extended as follows: If the vector of the unconstrained maximum violates one or more constraints, one of these is satisfied exactly by the optimal vector.

Further investigations show that the expression "one of these" is to be replaced by "at least one of these." For example, if we consider Fig. 3 and note that x^0 violates the constraint $x_2 \geqq 0$, then the obvious approach is to impose $x_2 = 0$. But the resulting vector $x^{(2)}$ violates $x_1 \geqq 0$, and it is easily seen that the optimal vector is the origin, $x^{(1,2)}$, which satisfies both constraints exactly. The situation is precisely the same when we have additional constraints besides the nonnegativity constraints. This case is illustrated in Fig. 4 for two nonnegativity constraints (indicated by 1 and 2) and two additional constraints (3 and 4). The shaded area is the admissible region. It is seen that x^0 violates constraints 3 and 4, that maximizing subject to 3 or to 4 in equational form does

[4] We shall introduce the notation $x^{(2)}$ for the vector x which is obtained by maximizing Q subject to constraint 2 in equational form, $x^{(2,4)}$ for the vector obtained by maximization subject to constraints 2 and 4 in equational form, etc.

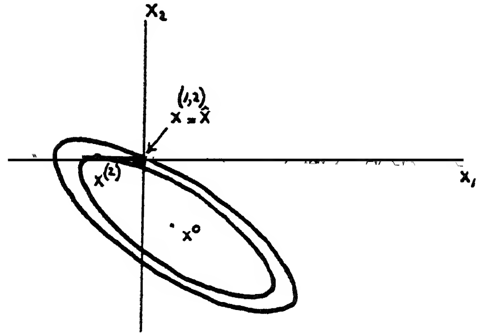

FIG. 3

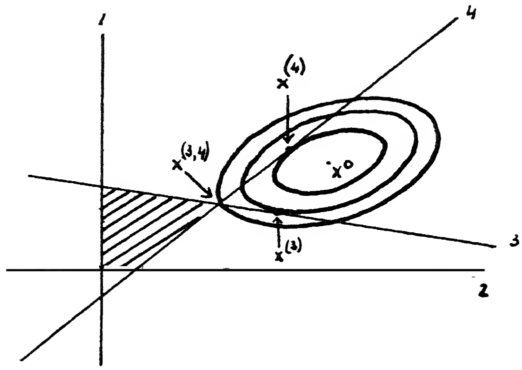

FIG. 4

not lead to $\hat{x}$ [because both $x^{(3)}$ and $x^{(4)}$ violate one constraint, viz., 4 and 3 respectively], and that $\hat{x}$ satisfies both 3 and 4 exactly. Here, therefore, all constraints violated by x^0 are satisfied exactly by $\hat{x}$. All these cases discussed so far are covered by the following Rule, the proof of which will be given in Section 3:

RULE 1. *If x^0 (the vector of the unconstrained maximum) violates certain constraints, then $\hat{x}$ (the optimal vector) satisfies at least one of these exactly.*

2.2. *Maximization Subject to Subsets of Constraints in Equational Form*

The preceding discussion shows that $\hat{x}$ is found by maximizing Q subject to a certain subset S of the N constraints $C'x \leqq d$ written in equational form ($C'x = d$).[5] Of course, in general we do not know this particular S and, in fact, our main task will be to find it. Even so, it is important to observe at this stage that we can easily derive for any S the vector x^S which maximizes Q subject to S in equational form. Let us therefore arrange the N constraints such that those of

[5] This subset may be the empty set, viz., when the vector of the unconstrained maximum does not violate any of the constraints ($x^0 = \hat{x}$). Note also that in some cases the process of maximization subject to S in equational form is trivial, viz., when S is such that only one x satisfies S in equational form. This is the case in Figs. 3 and 4.

S are the first ones, and let us denote by T the set of constraints not in S. Then the coefficient matrices of (1.4) can be partitioned according to

$$C = [C_S \, C_T]\,; \qquad d = \begin{bmatrix} d_S \\ d_T \end{bmatrix}. \tag{2.2}$$

Consider then the well-known Lagrangean expression

$$a'x - \tfrac{1}{2}x'Bx - \lambda'_S(C'_Sx - d_S), \tag{2.3}$$

where λ_S is a vector of Lagrange multipliers and $C'_Sx - d_S = 0$ are the constraints in S written in equational form. Differentiating (2.3) with respect to x, we obtain

$$x^S = x^0 - B^{-1}C_S\lambda_S\,. \tag{2.4}$$

Premultiplying (2.4) by C'_S gives

$$C'_SB^{-1}C_S\lambda_S = C'_Sx^0 - d_S\,, \tag{2.5}$$

because $C'_Sx^S = d_S$. If we now define

$$E = C'B^{-1}C = \begin{bmatrix} C'_S B^{-1}C_S & C'_S B^{-1}C_T \\ C'_T B^{-1}C_S & C'_T B^{-1}C_T \end{bmatrix} = \begin{bmatrix} E_S & F' \\ F & E_T \end{bmatrix}, \text{ say;} \tag{2.6}$$

$$e = C'x^0 - d = \begin{bmatrix} C'_S x^0 & - & d_S \\ C'_T x^0 & - & d_T \end{bmatrix} = \begin{bmatrix} e_S \\ e_T \end{bmatrix}, \text{ say,} \tag{2.7}$$

it is easily seen that (2.5) implies

$$\lambda_S = E_S^{-1}e_S\,, \tag{2.8}$$

which expresses λ_S in known quantities; the existence of E_S^{-1} is ensured if the rank of C_S equals the number of constraints in S. Furthermore, by premultiplying (2.4) by C'_T we find

$$C'_Tx^S = C'_Tx^0 - C'_TB^{-1}C_S\lambda_S\,, \tag{2.9}$$

and the right-hand side should be $\leqq d_T$ in order that x^S satisfies the constraints in T. Applying (2.6), (2.7) and (2.8), we find that this condition can be written in the simple form

$$FE_S^{-1}e_S - e_T \geqq 0. \tag{2.10}$$

As will appear below, the left-hand side of (2.10) is the only thing that needs to be computed for the relevant subsets S of the constraints. It is also easily verified that the corresponding Q-value is

$$Q(x^S) = \tfrac{1}{2}x^{0\prime}Bx^0 - \tfrac{1}{2}e'_SE_S^{-1}e_S = \tfrac{1}{2}a'B^{-1}a - \tfrac{1}{2}e'_SE_S^{-1}e_S\,, \tag{2.11}$$

because $Q(x^S) = \frac{1}{2}x^{0\prime}Bx^0 - \frac{1}{2}(x^S - x^0)'B(x^S - x^0)$ follows from (1.1) and (2.1), and $(x^S - x^0)'B(x^S - x^0) = e'_SE_S^{-1}e_S$ from (2.4).

2.3. *Further Steps of the Computation*

The above shows that we can in principle derive $\hat{x}$ by considering x^S for all subsets of the N constraints (1.4), viz., by verifying whether they satisfy the constraints and by computing their Q-values, see (2.10) and (2.11). However, this is far from efficient for several reasons. Firstly, it follows from Rule 1 that we can confine our attention to those sets S that contain at least one constraint violated by x^0. Secondly, it obviously makes no sense to consider a set S of which the constraints are contradictory when written in equational form. For example, if two of the N constraints are

$$x_1 + x_2 \geqq 1 \qquad \text{and} \qquad x_1 + x_2 \leqq 5,$$

then no x^S exists for any S containing these two constraints. Thirdly, we can extend Rule 1 in a manner which may be described as follows. Suppose x^0 violates constraints 1, 2, and 3. It seems obvious to compute in the next round x^S for those sets S which consist of one constraint. We can then confine ourselves to $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ [corresponding to $S = (1)$, (2), (3), respectively], because $x^{(i)} = \hat{x}$ for $i \geqq 4$ is impossible in view of Rule 1. Suppose then that none of the three vectors $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ satisfies all constraints; more precisely, that $x^{(1)}$ violates constraints 3 and 5, that $x^{(2)}$ violates 3, and that $x^{(3)}$ violates 1 and 2. The question arises what to do next. The answer is supplied by

RULE 2. *Suppose that two or more constraints are satisfied exactly by $\hat{x}$ and partition the set of these constraints into two subsets, S and S', containing at least one constraint each. Then x^S (the vector which maximizes Q subject to S in equational form) violates at least one constraint which is an element of S'.*[6]

Applying this Rule to our example, we observe first that it is indeed applicable because $\hat{x}$ has to satisfy at least two constraints exactly: both x^0 and $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ violate at least some constraint, so Rule 1 does not admit the possibility of an $\hat{x}$ which satisfies less than two constraints in equational form. Suppose now for a moment that constraint 1 is one of the constraints which is satisfied exactly by $\hat{x}$. Then Rule 2 states that $x^{(1)}$ violates some constraint which is satisfied exactly by $\hat{x}$, which means that $\hat{x}$ must satisfy exactly, not only constraint 1, but also 3 or 5; and so in the next round, when considering all relevant x^S's for two-element constraint sets S, we should take $S = (1, 3)$ and $(1, 5)$. However, this argument is based on the assumption that constraint 1 is one of the constraints satisfied exactly by $\hat{x}$; and we cannot be sure that this is true. The only thing we can be sure about is that $\hat{x}$ satisfies either 1 or 2 or 3 in equational form, because these are the constraints violated by x^0. Hence we must repeat the same argument under the alternative assumptions that $\hat{x}$ satisfies 2 or 3 exactly. Assuming then that constraint 2 is one of the constraints satisfied by $\hat{x}$, we find that $S = (2, 3)$ is the two-element constraint set to be considered in the next round, since $x^{(2)}$ violates constraint 3. Similarly, assuming that 3 is one of the constraints satisfied by $\hat{x}$ in equational form, we find that $S = (3, 1)$ and $(3, 2)$ are to be considered. As a whole, therefore, five two-element con-

[6] For an exception to this Rule, see the last paragraph of Section 2.3.

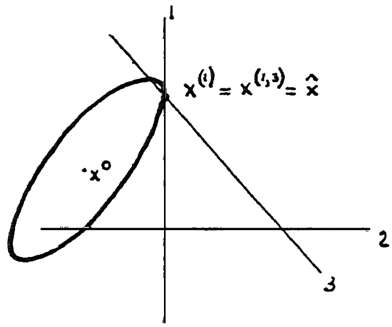

FIG. 5

straint sets appear: (1, 3), (1, 5), (2, 3), (3, 1), (3, 2), from which however the last two can be deleted since they are identical with the first and the third respectively. For each of the remaining three we have to verify whether their x^S does or does not violate certain constraints. If none of these x^S's satisfies all constraints, we have to proceed to three-element constraint sets. This does not lead to any novel features, as will be seen in Section 4; Rule 2 is then applicable just as it was here, its vector x^S being then interpreted as corresponding to a two-element rather than a one-element set S.

It is to be noted that there is one exception to Rule 2. It may happen that x^S coincides with $\hat{x}$ even though x^S is imposed to satisfy fewer constraints than $\hat{x}$ does. This is a degenerate case in which x^S happens to satisfy in equational form one or more constraints which do not fall under S. An example is given in Fig. 5: maximizing Q subject to constraint 1 in equational form leads to a vector $x^{(1)}$ which happens to satisfy constraint 3 exactly. Hence $x^{(1)} = x^{(1,3)}$ $(= \hat{x})$ in this case. In this article it will be assumed that there are no such problems of degeneracy.[7]

2.4. *Completing the Computation: Verification of the Solution*

The procedure described above amounts to considering first the vector of the unconstrained maximum, then the vectors which maximize Q subject to certain one-element subsets of constraints in equational form, then vectors corresponding to two-element subsets, and so on. At a certain point we shall arrive at a vector x^S which violates no constraint, and the question then arises whether this is the vector $\hat{x}$ which we look for. It would be convenient if this would always be true, but unfortunately this is not the case as can be shown by means of the example illustrated in Fig. 6. There x^0 violates 3 and 4, so that in the next round we consider $x^{(3)}$ and $x^{(4)}$. Doing so, we find that $x^{(3)}$ violates 4 [implying that $x^{(3,4)}$ is to be considered next] while $x^{(4)}$ violates 2 [so that $x^{(2,4)}$ is to be considered]. Now $x^{(3,4)}$ violates no constraints, so that it might be the optimal vector; but $x^{(2,4)}$ does not violate any constraint either and hence it is clear that a special

[7] Some partial results on the problem of degeneracy have been obtained, but they are not reported here. We expect to come back to it in a later publication.

rule is necessary in order to find out whether or not such an x^S equals $\hat{x}$. This is provided by

RULE 3. *Suppose that for some subset S of the constraints, x^S exists and violates none of the constraints; then $x^S = \hat{x}$ if and only if every x^{S^h} violates the h-th constraint where S^h is the set of all constraints satisfied exactly by x^S excluding the h-th.*

The application of this Rule to our example runs as follows. Considering $x^{(3,4)}$ we observe that (3, 4) is the set of constraints that are satisfied in equational form, so the sets S^h to be analyzed are the one-element set (3)—obtained by excluding constraint $h = 4$—and the one-element set (4), obtained by excluding $h = 3$. Hence we have to verify whether it is true that $x^{(3)}$ violates constraint 4 and that $x^{(4)}$ violates constraint 3. An inspection of Fig. 6 shows that this is the case as far as $x^{(3)}$ is concerned, but not for $x^{(4)}$: this vector violates constraint 2, not 3. Next, consider $x^{(2,4)}$; it satisfies constraints 2 and 4 exactly, so we have to consider $x^{(4)}$ and $x^{(2)}$ and to verify whether $x^{(4)}$ violates constraint 2 and $x^{(2)}$ violates constraint 4. The answer is positive as Fig. 6 shows and we can therefore conclude that $x^{(2,4)} = \hat{x}$. Of course, this result is immediately obvious by graphical inspection, but a graphical device works only when the number of variables is very small. When we deal with more than a few variables, an algebraic device like that of Rule 3 cannot be avoided.

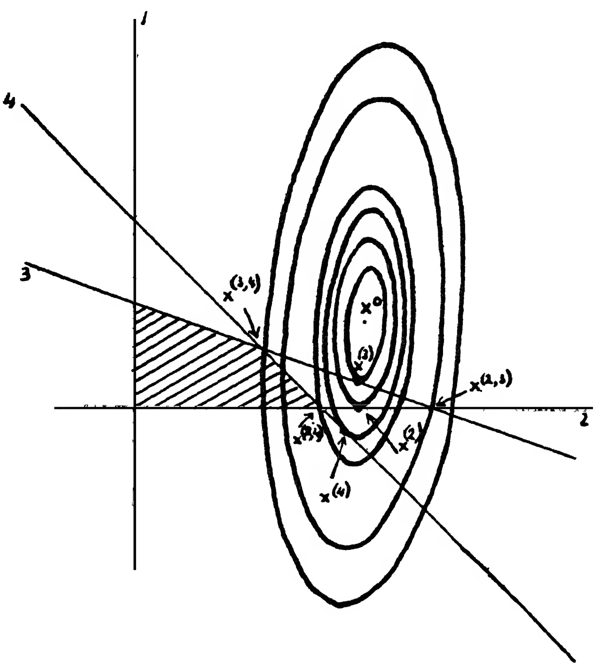

FIG. 6

3. Analysis of the Procedure

PROBLEM. *Maximize* (1.1) *subject to* (1.4), *where all vectors and matrices are real-valued, the vectors* x, a, d *containing* n, n, N *elements respectively, the matrix* B *being of order* $n \times n$ *and* C *of order* $n \times N$. *The vectors and matrices* a, B, C, d *are known.*

It will prove very convenient to apply the language of set theory to the sets of N constraints $C'x \leqq d$. In particular, we shall consider subsets consisting of constraints that are satisfied exactly by some x ($c'_h x = d_h$ for some $h = 1, \cdots, N$) as well as subsets of constraints that are violated by x ($c'_h x > d_h$). Further, we shall write 0 for the empty set,[8] $S = S'$ if the sets S and S' are identical, $S \neq S'$ if they are not, $S \subset S'$ if all elements of S are also elements of S', $h \in S$ if h is an element of S, SS' for the set of elements both in S and in S', $S + S'$ for the set of elements either in S or in S' (with the understanding that there are no elements both in S and in S', i.e., $SS' = 0$), and $S - S'$ for the set of elements in S but not in S'.

DEFINITION 1. *For each vector* x *of* n *elements,* $U(x)$ *is the subset of the* N *constraints* $C'x \leqq d$ *that are satisfied exactly by* x *(in equational form), and* $V(x)$ *is the subset of the* N *constraints violated by* x.

DEFINITION 2. *Subject to conditions of existence and uniqueness,* $\hat{x}$ *is the vector which maximizes* (1.1) *subject to* (1.4), *and* x^S *is the vector which maximizes* (1.1) *subject to some subset* S *of the* N *constraints* $C'x \leqq d$ *in equational form.*

We shall make three assumptions, the first of which is made in order to ensure that a solution exists (which requires that there is at least one x satisfying the constraints) and that there is maximization in a nontrivial sense (which requires that at least two different x's satisfy the constraints). The second assumption ensures that the solution is unique, the third that there are no problems of degeneracy. The last assumption implies that maximizing Q subject to any subset S of the constraints in equational form leads to a vector x^S which satisfies only the constraints of S exactly.

ASSUMPTION 1. *There exist at least two vectors* x *and* x^* *of* n *elements each such that* $x \neq x^*$ *and* $V(x) = V(x^*) = 0$.

ASSUMPTION 2. *The matrix* B *is positive-definite.*

ASSUMPTION 3. *For each subset* S *of the* N *constraints* $C'x \leqq d$ *for which* x^S *exists and is unique,* $U(x^S) = S$.

We shall first prove two lemmas and then proceed to the main theorems.

LEMMA 1. *For any two vectors* x *and* x^* *of* n *elements each,*

$$Q(y) > \operatorname{Min}\,[Q(x), Q(x^*)] \qquad \textit{where } y = \theta x + (1 - \theta)x^*, \tag{3.1}$$

provided that $x \neq x^*$ *and* $0 < \theta < 1$.

[8] As a general rule, the symbol 0 will occur in this section only as the empty set. There is only one exception to this rule: it also occurs in the expressions like $0 < \theta < 1$ to indicate that θ is a positive number smaller than 1. No confusion is likely to arise.

Proof. The quadratic function Q is strictly concave in every n-dimensional interval of its arguments (Assumption 2), hence

$$Q\{\theta x + (1-\theta)x^*\} > \theta Q(x) + (1-\theta)Q(x^*)$$
$$= Q(x) + (1-\theta)\{Q(x^*) - Q(x)\}$$
$$= Q(x^*) + \theta\{Q(x) - Q(x^*)\}$$

whenever $x \neq x^*$ and $0 < \theta < 1$. The inequality (3.1) follows immediately.

LEMMA 2. *For any vector x of n elements,*

$$U(x)V(x) = 0. \tag{3.2}$$

For any two vectors x and x^* of n elements each, if $y = \theta x + (1-\theta)x^*$, then

(3.3) $U(x)U(x^*) \subset U(y)$ *for any* θ;

(3.4) $V(y) = 0$ *if* $V(x) = V(x^*) = 0$ *and* $0 \leqq \theta \leqq 1$;

(3.5) $V(y) = 0$ *for some* θ $(0 < \theta < 1)$ *if* $V(x) = U(x)V(x^*) = 0$.

Proof. (3.2): $c'_h x = d_h$ and $c'_h x > d_h$ contradict each other.

(3.3): If $h \in U(x)U(x^*)$, then $c'_h x = c'_h x^* = d_h$ and hence

$$c'_h y = \theta c'_h x + (1-\theta)c'_h x^* = \theta d_h + (1-\theta)d_h = d_h .$$

(3.4): $V(x) = V(x^*) = 0$ implies $c'_h x \leqq d_h$, $c'_h x^* \leqq d_h$ for all h. Hence $c'_h y = \theta c'_h x + (1-\theta)c'_h x^* \leqq d_h$ for all h provided that $0 \leqq \theta \leqq 1$.

(3.5): If $V(x^*) = 0$, then it is given that $V(x) = V(x^*) = 0$ and hence $V(y) = 0$ for *all* θ in $(0, 1)$ according to (3.4). Assume next $V(x^*) \neq 0$; this implies $x \neq x^*$, for $x = x^*$ is contradicted by $V(x) = 0 \neq V(x^*)$. If $h \in V(x^*)$, then $c'_h x^* > d_h$ and $c'_h x < d_h$ because $c'_h x > d_h$ is excluded by $V(x) = 0$ and $c'_h x = d_h$ is excluded by $U(x)V(x^*) = 0$. But $c'_h x < d_h$, $c'_h x^* > d_h$ implies that there exists a θ_h such that $\theta_h c'_h x + (1-\theta_h)c'_h x^* = d_h$ and $0 < \theta_h < 1$. For all $h \in V(x^*)$, write $\theta' = \text{Max}_h\, \theta_h$; then $V(y) = 0$ if $\theta' \leqq \theta \leqq 1$.

THEOREM 1 (existence and uniqueness). *There is exactly one vector $\hat{x}$ satisfying Definition 2, and for each subset S of the N constraints $C'x \leqq d$ there is either exactly one vector x^S satisfying Definition 2 or none at all.*

Proof. Assumptions 1 and 2 ensure that there exists at least one vector which maximizes Q. Suppose that there are two different vectors $\hat{x}$ and $\hat{x}^*$ which both maximize Q subject to the constraints. Then for any $y = \theta\hat{x} + (1-\theta)\hat{x}^*$ with $0 < \theta < 1$, $Q(y) > Q(\hat{x}) = Q(\hat{x}^*)$ because of Lemma 1 and $V(y) = 0$ because $V(\hat{x}) = V(\hat{x}^*) = 0$ [Lemma 2, (3.4)]. This contradicts the assumption that both $\hat{x}$ and $\hat{x}^*$ maximize Q subject to the constraints.

Suppose similarly that both x^S and $y^S \neq x^S$ maximize Q subject to S in equational form. Then for any $y = \theta x^S + (1-\theta)y^S$ such that $0 < \theta < 1$, we have $Q(y) > Q(x^S) = Q(y^S)$ because of Lemma 1 and $S = U(x^S)U(y^S) \subset U(y)$ because of Lemma 2, (3.3), which contradicts again the assumption made. Note that we cannot exclude the possibility that no x^S exists because S may be contradictory in equational form.

COROLLARY 1. *For any vector x of n elements,*

$$V(x) = 0 \text{ implies either } x = \hat{x} \text{ or } Q(x) < Q(\hat{x}). \tag{3.6}$$

For any x and any subset S of the N constraints $C'x \leqq d$ such that x^S exists,

$$S \subset U(x) \text{ implies either } x = x^S \text{ or } Q(x) < Q(x^S). \tag{3.7}$$

Proof. Trivial, given the existence and uniqueness of $\hat{x}$ and the uniqueness of x^S if it exists.

THEOREM 2 (exploring the attainable summit). *For any subset S of the N constraints $C'x \leqq d$ such that x^S exists, exactly one of the following possibilities applies:*

I. (approaching the summit). $U(x^S) \subset U(\hat{x}) \neq U(x^S)$ *implying*

$$U(\hat{x})V(x^S) \neq 0.$$

II. (reaching the summit). $U(x^S) = U(\hat{x})$ *implying* $x^S = \hat{x}$.

III. (leaving the summit). $U(\hat{x}) \subset U(x^S) \neq U(\hat{x})$ *implying* $Q(x^S) < Q(\hat{x})$.

IV. (missing the summit). $U(\hat{x}) \neq U(\hat{x})U(x^S) \neq U(x^S)$ *implying either* $V(x^S) \neq 0$, *or* $V(x^S) = 0$ *and* $U(x^S)V(x^{S^h}) = 0$ *for some* $S^h = U(x^S) - (h)$ *where* (h) *is a one-element subset of the constraints satisfying*

$$h \in U(x^S) - U(\hat{x})U(x^S).$$

Proof. The possibilities listed are the only ones, because the intersection $U(\hat{x})U(x^S)$ is either identical with both sets (II), or identical with one of them and a proper subset of the other (I, III), or a proper subset of both (IV).

I. We have $\hat{x} \neq x^S$, because $\hat{x} = x^S$ would imply $U(\hat{x}) = U(x^S)$. Then $S = U(x^S) \subset U(\hat{x})$ implies $Q(\hat{x}) < Q(x^S)$ according to Corollary 1, (3.7); and this implies $V(x^S) \neq 0$ according to (3.6). Assume $U(\hat{x})V(x^S) = 0$. Then some θ exists such that $0 < \theta < 1$ and $V(y) = 0$ where $y = \theta\hat{x} + (1 - \theta)x^S$ [Lemma 2, (3.5)] and $Q(y) > \text{Min }[Q(\hat{x}), Q(x^S)] = Q(\hat{x})$ (Lemma 1). This is a contradiction; hence $U(\hat{x})V(x^S) \neq 0$.

II. We have $S = U(x^S) = U(\hat{x})$, so either $\hat{x} = x^S$ or $Q(\hat{x}) < Q(x^S)$ in view of (3.7). Assume $\hat{x} \neq x^S$, in which case $V(x^S) \neq 0$ [because if $\hat{x} \neq x^S$ and $V(x^S) = 0$, then $Q(x^S) < Q(\hat{x})$ according to (3.6); and this is impossible in view of our previous conclusion: either $\hat{x} = x^S$ or $Q(\hat{x}) < Q(x^S)$]. Now $U(\hat{x})V(x^S) = U(x^S)V(x^S) = 0$ [Lemma 2, (3.2)]; and hence, noting that $V(\hat{x}) = 0$ by definition and applying Lemma 2, (3.5), we conclude that some θ exists such that $0 < \theta < 1$ and $V(y) = 0$ where $y = \theta\hat{x} + (1 - \theta)x^S$. But also $Q(y) > \text{Min }[Q(\hat{x}), Q(x^S)] = Q(\hat{x})$ according to Lemma 1. So the assumption $\hat{x} \neq x^S$ leads to a contradiction, hence $\hat{x} = x^S$.

III. Here $x^S \neq \hat{x}$ because $x^S = \hat{x}$ would imply $U(x^S) = U(\hat{x})$. Also

$$U(\hat{x})V(x^S) \subset U(x^S)V(x^S) = 0$$

in view of Lemma 2, (3.2). Then, for some θ such that $0 < \theta < 1$, $V(y) = 0$ and $Q(y) > \text{Min }[Q(\hat{x}), Q(x^S)]$ where $y = \theta\hat{x} + (1 - \theta)x^S$; see Lemma 2, (3.5), and Lemma 1. Hence necessarily $Q(x^S) < Q(\hat{x})$.

IV. We have either $V(x^S) \neq 0$ or $V(x^S) = 0$. Since the statement specifies

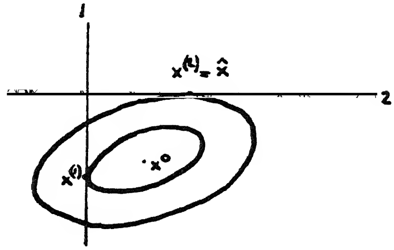

FIG. 7

nothing with respect to the first possibility,[9] we consider $V(x^S) = 0$. Further, we take $V(x^{S^h}) \neq 0$ for all $S^h = U(x^S) - (h) = S - (h)$ [where

$$h \in U(x^S) - U(\hat{x})U(x^S)]$$

because the assertion $U(x^S)V(x^{S^h}) = 0$ for some S^h is trivially true if

$$V(x^{S^h}) = 0$$

for some S^h. Then $\hat{x} \neq x^S$ [because $\hat{x} = x^S$ would imply $U(\hat{x}) = U(x^S) = U(\hat{x})U(x^S)$] and $x^{S^h} \neq \hat{x}$, $x^{S^h} \neq x^S$ for all S^h because

$$V(x^{S^h}) \neq V(\hat{x}) = V(x^S) = 0.$$

Assume $U(x^S)V(x^{S^h}) \neq 0$ for all S^h and consider $y = \theta\hat{x} + (1 - \theta)\sum \theta_h x^{S^h}$ where $0 < \theta$, $\theta_h < 1$ and $\sum \theta_h = 1$.[10] Write also $c'_k\hat{x} = d_k - \delta_k$ and

$$c'_k x^{S^h} = d_k + \epsilon_{kh}$$

for $k \in U(x^S) - U(\hat{x})U(x^S)$. Then $\delta_k > 0$ [because $\delta_k < 0$ is excluded by $V(\hat{x}) = 0$, and $\delta_k = 0$ by $(k)U(\hat{x}) = 0$], $\epsilon_{kh} = 0$ if $k \neq h$ [because $(k)U(x^{S^h}) = (k)U(x^S) - (k)(h) = (k)U(x^S) = (k)$ if $k \neq h$], and $\epsilon_{kh} > 0$ if $k = h$ [because $\epsilon_{kh} = 0$ if $k \neq h$, $\epsilon_{kh} \leqq 0$ if $k = h$ for all k and fixed h would contradict $U(x^S)V(x^{S^h}) \neq 0$]. Applying this δ, ϵ-notation, we have

$$c'_k y = \theta(d_k - \delta_k) + (1 - \theta)\textstyle\sum_h \theta_h(d_k + \epsilon_{kh})$$

$$= d_k - \theta\delta_k + (1 - \theta)\theta_k\epsilon_{kk}$$

for any $k \in U(x^S) - U(\hat{x})U(x^S)$. Now if we choose θ, θ_k such that

$$\theta\delta_k = (1 - \theta)\theta_k\epsilon_{kk}\,,$$

[9] This possibility is a real one, see Fig. 7. There we have $S = U(x^S) = (1)$, $U(\hat{x}) = (2)$, $V(x^S) = (2)$.

[10] In the special case when S is a one-element constraint set, h takes only one value and θ_h must be taken as 1. The case $S = 0$ is excluded in Possibility IV, because $S = 0$ implies $U(x^S) = U(\hat{x})U(x^S)$ $(= 0)$.

this means $U(x^S) - U(\hat{x})U(x^S) \subset U(y)$.[11] But also $U(\hat{x})U(x^S) \subset U(y)$, as follows from repeated application of Lemma 2, (3.3);[12] and hence $U(x^S) \subset U(y)$. However, $Q(y) > \text{Min}_h [Q(\hat{x}), Q(x^{S^h})]$, as follows from repeated application of Lemma 1;[13] and $Q(\hat{x}) > Q(x^S)$ because $V(x^S) = 0$ and $\hat{x} \neq x^S$ [see (3.6)] and $Q(x^{S^h}) > Q(x^S)$ because $S^h \subset U(x^S)$ and $x^{S^h} \neq x^S$ [see (3.7)]. Hence

$$Q(y) > Q(x^S).$$

But we just derived $S = U(x^S) \subset U(y)$, which implies either $y = x^S$ or

$$Q(y) < Q(x^S),$$

both of which contradict $Q(y) > Q(x^S)$. Hence $U(x^S)V(x^{S^h}) = 0$ for some S^h such that $h \in U(x^S) - U(\hat{x})U(x^S)$.

COROLLARY 2. *$\hat{x} = x^S$ for some subset S of the N constraints $C'x \leqq d$.*

Proof. Write $\hat{S} = U(\hat{x})$ and consider $x^{\hat{S}}$. We have $U(\hat{x}) = \hat{S} = U(x^{\hat{S}})$ which implies that we are in Possibility II; hence $\hat{x} = x^{\hat{S}}$.

COROLLARY 3. *$\hat{x} = x^0$ if and only if $V(x^0) = 0$.*

Proof. The necessity of the condition $V(x^0) = 0$ is obvious, so we confine ourselves to the sufficiency. Applying (3.6), we find that $V(x^0) = 0$ implies either $x^0 = \hat{x}$ or $Q(x^0) < Q(\hat{x})$; applying (3.7), we find that $0 \subset U(\hat{x})$ implies

[11] When S is a one-element constraint set, we take $\theta_k = 1$ and $\theta = \epsilon_{kk}/(\delta_k + \epsilon_{kk})$; see footnote 10. In the general case, when the index k takes p values (say), the θ_k are to be specified such that

$$\frac{\theta_1 \epsilon_{11}}{\delta_1} = \cdots = \frac{\theta_p \epsilon_{pp}}{\delta_p}$$

and such that $\sum \theta_k = 1$. Given the positive signs of the δ's and ϵ's involved, this leads to unique positive values of the θ_k. Finally, $\theta = \theta_k\epsilon_{kk}/(\delta_k + \theta_k\epsilon_{kk})$ for any k.

[12] We have $U(\hat{x})U(x^S) \subset U(\hat{x})$ and $U(\hat{x})U(x^S) \subset U(x^{S^h})$ for all S^h such that $h \in U(x^S) - U(\hat{x})U(x^S)$. Hence $U(\hat{x})U(x^S)$ is a subset of the intersection of $U(\hat{x})$ and all relevant $U(x^{S^h})$; and the statement made in the text is proved when it is shown that all constraints of this intersection are satisfied exactly by any linear combination y of $\hat{x}$ and the x^{S^h}. This can either be proved directly [by means of a trivial extension of the proof of (3.3)] or by repeated application of (3.3), as follows. Write the vectors $\hat{x}$, x^{S^h} as $z_1, z_2, \ldots$ and $y = \sum \epsilon_i z_i$ with $\sum \epsilon_i = 1$. Supposing that i takes 3 values, we can write

$$y = \epsilon_1 z_1 + (1 - \epsilon_1) \frac{\epsilon_2 z_2 + \epsilon_3 z_3}{\epsilon_2 + \epsilon_3},$$

so that $U(z_1)U(z^*) \subset U(y)$ according to (3.3), where $z^* = (\epsilon_2 z_2 + \epsilon_3 z_3)/(\epsilon_2 + \epsilon_3)$. But $U(z_2)U(z_3) \subset U(z^*)$, hence $U(z_1)U(z_2)U(z_3) \subset U(y)$. This is easily extended to the case in which i takes more values.

[13] In the notation used at the end of footnote 12, we have to prove $Q(y) > \text{Min}[Q(z_1), Q(z_2), Q(z_3)]$. Now $Q(y) \geqq \text{Min}[Q(z_1), Q(z^*)]$ and $Q(z^*) \geqq \text{Min}[Q(z_2), Q(z_3)]$, where the equality sign holds if and only if the two vectors between square brackets (z_1, z^* and z_2, z_3) are equal. Hence the statement made holds except when the vectors z_1, z_2, z_3 are all equal. This exception does not occur here, since $\hat{x}$ differs from the x^{S^h} as follows from $V(\hat{x}) = 0 \neq V(x^{S^h})$. The extension to the case of a larger number of vectors is equally simple.

either $\hat{x} = x^0$ or $Q(\hat{x}) < Q(x^0)$. Hence $\hat{x} = x^0$ (which means that we are in Possibility II of Theorem 2 for $S = 0$).

COROLLARY 4 (Rules 1 and 2).[14] *If* $S + S' = U(\hat{x})$ *and* $S' \neq 0$, *then* x^S *exists and* $S'V(x^S) \neq 0$.

Proof. The existence of x^S follows from the fact that the constraints of S are not contradictory in equational form; for if they were contradictory, so would those of $S + S' = U(\hat{x})$ be, implying the non-existence of $\hat{x}$ (which is ruled out by Theorem 1). Considering Possibility I of Theorem 2, we find that it implies $U(\hat{x})V(x^S) = (S + S')V(x^S) = S'V(x^S) \neq 0$ because

$$SV(x^S) = U(x^S)V(x^S) = 0$$

in view of Lemma 2, (3.2). The other possibilities of Theorem 2 are all excluded, because $U(x^S) = S \subset S + S' = U(\hat{x}) \neq U(x^S)$ leads to Possibility I only.

COROLLARY 5 (Rule 3). *Suppose that for some subset* S *of the* N *constraints* $C'x \leqq d$, x^S *exists and* $V(x^S) = 0$. *Then* $x^S = \hat{x}$ *if and only if*

$$U(x^S)V(x^{S^h}) \neq 0$$

for all $S^h = S - (h)$ *where* (h) *is a one-element subset of the constraints such that* $h \in S$.

Proof. (a) *Necessity:* We have to prove for any x^{S^h} that if $x^S = \hat{x}$ [in which case it is necessarily true that $V(x^S) = 0$], $U(x^S)V(x^{S^h}) \neq 0$. Suppose

$$U(x^S)V(x^{S^h}) = U(\hat{x})V(x^{S^h}) = 0$$

for some x^{S^h}. Consider then $y = \theta\hat{x} + (1 - \theta)x^{S^h}$ for $0 < \theta < 1$; since

$$U(x^{S^h}) = S^h \neq S = U(x^S) = U(\hat{x})$$

we have $\hat{x} \neq x^{S^h}$ and hence $Q(y) > \text{Min}\,[Q(\hat{x}), Q(x^{S^h})]$ in view of (3.1); also $Q(\hat{x}) < Q(x^{S^h})$ because $S^h \subset U(x^S) = U(\hat{x})$ and $x^{S^h} \neq \hat{x}$, see (3.7). Hence $Q(y) > Q(\hat{x})$. However, $V(y) = 0$ for some θ such that $0 < \theta < 1$, as follows from $U(\hat{x})V(x^{S^h}) = 0$. This involves a contradiction, hence $U(x^S)V(x^{S^h}) \neq 0$ for each x^{S^h}.

(b) *Sufficiency:* We have to prove that if $V(x^S) = 0$ and if $U(x^S)V(x^{S^h}) \neq 0$ for each x^{S^h}, then $x^S = \hat{x}$. Considering Possibility I first, we observe that it must be ruled out because its implication $U(\hat{x})V(x^S) \neq 0$ is contradicted by $V(x^S) = 0$ which is given. The same applies to Possibility IV, because it implies $U(x^S)V(x^{S^h}) = 0$ for some x^{S^h} if $V(x^S) = 0$, which is contradicted by $U(x^S)V(x^{S^h}) \neq 0$ for each x^{S^h}. As to Possibility III, consider

$$y = \theta\hat{x} + (1 - \theta)\sum \theta_h x^{S^h}$$

where $0 < \theta, \theta_h < 1$ and $\sum \theta_h = 1$,[15] the constraints h over which summation takes place satisfying $h \in U(x^S) - U(\hat{x})$. We have $Q(y) > \text{Min}_h\,[Q(\hat{x}), Q(x^{S^h})]$

[14] Rule 1 deals with the case $S = 0$, Rule 2 with $S \neq 0$.

[15] Except that one must take $\theta_h = 1$ if S is a one-element constraint set (in which case $\hat{x} = x^0$, given that Possibility III is assumed to apply to S here).

as follows from repeated application of Lemma 1;[16] also $Q(\hat{x}) > Q(x^S)$ because this is the implication of Possibility III; and $Q(x^{S^h}) > Q(x^S)$ for all x^{S^h} (all $h \in S$) because of (3.7) [the possibility $x^{S^h} = x^S$ being excluded because

$$U(x^{S^h}) = S^h \neq S = U(x^S)].$$

Hence $Q(y) > Q(x^S)$. Further, we have $U(\hat{x}) \subset U(x^{S^h})$ for all x^{S^h} considered here [because $U(\hat{x}) \subset U(x^S)$ and $h \in U(x^S) - U(\hat{x})$], so $U(\hat{x}) \subset U(y)$.[17] But in addition to this, we can choose θ, θ_h such that $U(x^S) - U(\hat{x}) \subset U(y)$,[18] in which case $U(x^S) \subset U(y)$. This, however, contradicts $Q(y) > Q(x^S)$ according to (3.7). Hence Possibility III is also ruled out. So only Possibility II remains implying $x^S = \hat{x}$.

4. An Example; Directory of Computations

The computational procedure will be illustrated by means of an example used by Houthakker [4] for the illustration of his capacity method. He considers a monopolist who faces four linear demand functions for his four products:

$$(4.1) \qquad \begin{cases} x_1 = 18.239 - 2.086p_1 + 0.255p_2 + 1.033p_3 - 0.374p_4 \\ x_2 = 1.898 + 0.255p_1 - 0.499p_2 - 0.129p_3 + 0.217p_4 \\ x_3 = -4.916 + 1.033p_1 - 0.129p_2 - 0.759p_3 + 0.254p_4 \\ x_4 = 7.923 - 0.374p_1 + 0.217p_2 + 0.254p_3 - 0.512p_4 \end{cases}$$

where the x's are the quantities produced and sold and the p's prices. The problem is to maximize total gross revenue subject to certain constraints. Total gross revenue is of the form $\sum p_i x_i$, which is a quadratic form in the prices given that the x's are linear in the prices, see (4.1). But we may also express the programming problem in quantities instead of prices by solving the system (4.1) for the p's, which leads to four "inverted demand equations" which are linear in the x's; and this, in turn, makes $\sum p_i x_i$ quadratic in the x's. As long as there is no problem of uncertainty about the numerical values of the coefficients of the problem, it does not matter whether we work with p's or x's. Following Houthakker, we shall use the x-approach, which leads to the objective function

$$(4.2) \quad Q(x) = \sum_1^4 p_i x_i = 18x_1 + 16x_2 + 22x_3 + 20x_4$$
$$- \tfrac{1}{2}\{6x_1^2 + 2x_1x_2 + 16x_1x_3 + 10x_2^2 + 2x_2x_3 + 8x_2x_4 + 17x_3^2 + 6x_3x_4 + 11x_4^2\},$$

[16] See footnote 13. We have $V(\hat{x}) = 0 \neq V(x^{S^h})$, hence $\hat{x} \neq x^{S^h}$; so the exception mentioned in that footnote occurs neither here nor there.

[17] The proof is entirely similar to that of footnote 12.

[18] The proof is entirely similar to that of footnote 11 and the accompanying text of the proof of Possibility IV of Theorem 2.

or in the matrix notation of (1.1):

$$(4.3)\qquad a = \begin{bmatrix} 18 \\ 16 \\ 22 \\ 20 \end{bmatrix}; \qquad B = \begin{bmatrix} 6 & 1 & 8 & 0 \\ 1 & 10 & 1 & 4 \\ 8 & 1 & 17 & 3 \\ 0 & 4 & 3 & 11 \end{bmatrix}.$$

There are two types of constraint subject to which maximization takes place. First, there is the requirement that none of the quantities be negative:

1. $x_1 \geqq 0$
2. $x_2 \geqq 0$
3. $x_3 \geqq 0$
4. $x_4 \geqq 0.$

Second, there are constraints due to the limited availability of certain factors of production. Thus, the production of each unit of x_1, x_2, x_3 or x_4 requires 1 unit of a factor A of which the supply is limited to $1\frac{2}{3}$ units; and there is a factor B which is used for x_1 and x_3, and a factor C which is used for x_2 and x_4, both of which are in limited supply. So we have three additional constraints which are specified as

$$\begin{array}{lrcrcrcrcl} 5. & x_1 & + & x_2 & + & x_3 & + & x_4 & \leqq & 1\frac{2}{3} \\ 6. & 5x_1 & & & + & 10x_3 & & & \leqq & 2 \\ 7. & & & 4x_2 & & & + & 5x_4 & \leqq & 3. \end{array}$$

Combining these seven constraints, we arrive at the general form $C'x \leqq d$ when we specify

$$(4.4)\qquad C' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 5 & 0 & 10 & 0 \\ 0 & 4 & 0 & 5 \end{bmatrix}; \qquad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1\frac{2}{3} \\ 2 \\ 3 \end{bmatrix}.$$

The computational procedure can then be described conveniently in terms of the following three steps:

Initial Step. Compute the vector e defined in (2.7), i.e.,

$$e = C'x^0 - d = C'B^{-1}a - d.$$

If all elements of e are nonpositive, then the unconstrained maximum $x^0 = B^{-1}a$ satisfies all constraints and no further computations are necessary. If one or more elements of e are positive, compute E defined in (2.6), i.e., $E = C'B^{-1}C$; proceed then to the Intermediate Steps.

In our example, we have for $-e$:[19]

$$-e = \{4.560 \quad 0.475 \quad -1.229 \quad 1.981 \quad -4.119 \quad -8.508 \quad -8.802\},$$

[19] The computations have been made in five decimal places, but they are reported here in three only.

which shows that x^0 violates constraints 3, 5, 6, 7. So we have to compute E, which is the symmetric matrix

$$\begin{bmatrix} 1.043 & -0.128 & -0.516 & 0.187 & -0.586 & -0.051 & -0.426 \\ -0.128 & 0.250 & 0.064 & -0.108 & -0.078 & -0.007 & -0.457 \\ -0.516 & 0.064 & 0.379 & -0.127 & 0.200 & -1.211 & 0.377 \\ 0.187 & -0.108 & -0.127 & 0.256 & -0.208 & 0.333 & -0.846 \\ -0.586 & -0.078 & 0.200 & -0.208 & 0.673 & 0.936 & 1.352 \\ -0.051 & -0.007 & -1.211 & 0.333 & 0.936 & 12.363 & -1.636 \\ -0.426 & -0.457 & 0.377 & -0.846 & 1.352 & -1.636 & 6.056 \end{bmatrix}.$$

Intermediate Steps, No. 1. Set up a "sign table of quadratic programming" which is a rectangular array of signs (plus, minus, or zero) the rows of which correspond to the N constraints, the columns to vectors x^S obtained by maximizing Q subject to a subset S of these constraints in equational form; these signs are for any x^S the signs of the successive elements of $FE_S^{-1}e_S - e_T$, which should be nonnegative in order that x^S satisfies the constraints, see (2.10). Indicate then the signs of $-e$ for x^0 in the first column (we have $-e = FE_S^{-1}e_S - e_T$ if $S = 0$) and write the constraint numbers corresponding to negative signs in the headings of the next columns. After this, compute $FE_S^{-1}e_S - e_T$ for all constraint sets S which consist of the single elements in the headings just-mentioned (in accordance with Rule 1); indicate the signs of the successive elements of $FE_S^{-1}e_S - e_T$ in the relevant place of their column, a dot being used for those signs which are imposed to be zero. As soon as a column emerges which has no minus signs, the further parts of this step are to be omitted and one should proceed to the Final Step immediately; when all columns have at least one minus sign, one has to proceed to Intermediate Step No. 2.

The signs of the first column are supplied immediately by the Initial Step, and so we write 3, 5, 6, 7 in the headings of the next columns (see the Sign Table below). We then have to consider $FE_S^{-1}e_S - e_T$ for S = (3), (5), (6),

SIGN TABLE OF QUADRATIC PROGRAMMING:
HOUTHAKKER'S MONOPOLIST

Constraint	*	One-element constraint sets				Two-element constraint sets								Three-element constraint sets								
		3	5	6	7	3,5	3,6	3,7	5,2	5,6	5,7	6,7	7,2	3,5,2	3,5,6	3,5,7	3,6,7	3,7,2	5,2,6	5,2,7	5,6,7	6,7,2
1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2	+	+	−	+	−	−	+	−	·	+	−	−	·	·	+	−	+	·	·	·	−	·
3	−	·	−	−	−	·	·	·	−	−	−	−	−	·	·	·	·	·	−	+	−	−
4	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
5	−	−	·	−	−	·	−	−	·	·	·	−	−	·	·	·	+	−	·	·	·	−
6	−	−	−	·	−	−	·	−	−	·	−	·	−	−	·	−	·	−	·	−	·	·
7	−	−	−	−	·	−	−	·	−	−	·	·	·	−	−	·	·	·	−	·	·	·

* No constraint imposed in equational form.

(7). For $S = (3)$ e.g. this is

$$\begin{bmatrix} -0.516 \\ 0.064 \\ -0.127 \\ 0.200 \\ -1.211 \\ 0.377 \end{bmatrix} [0.379]^{-1}(1.229) - \begin{bmatrix} -4.560 \\ -0.475 \\ -1.981 \\ 4.119 \\ 8.508 \\ 8.802 \end{bmatrix} = \begin{bmatrix} 2.886 \\ 0.684 \\ 1.570 \\ -3.473 \\ -12.430 \\ -7.582 \end{bmatrix},$$

and the six resulting signs are specified in the second column of the Sign Table; the dot in the third row indicates that the third constraint is imposed in equational form. It is seen that each of the four vectors $x^{(3)}$, $x^{(5)}$, $x^{(6)}$, $x^{(7)}$ violates at least three constraints, so we have to proceed to Intermediate Step No. 2.

Intermediate Steps, No. 2. Indicate in the headings of the open columns next to the columns of the one-element constraint sets which were prepared in Intermediate Step No. 1, the two-element sets which are to be considered next; do so in accordance with Rule 2, viz., by combining the constraint which is imposed in equational form with each of the violated constraints. Compute then

$$FE_S^{-1}e_S - e_T$$

for each of the resulting (two-element) sets S. As soon as a column emerges without minus signs, the further parts of this step should be omitted and one should proceed to the Final Step immediately; otherwise one has to proceed to Intermediate Step No. 3.

In our case we have to consider 8 two-element sets; we note that six additional sets, viz., (5, 3), (6, 3), (6, 5), (7, 3), (7, 5), (7, 6), need not be analyzed separately because they occur in reverse order also [like (3, 5), (3, 6), etc.]. We then compute $FE_S^{-1}e_S - e_T$ for each of these eight S's. For example, taking $S = (3, 5)$, we have

$$\begin{bmatrix} -0.516 & -0.580 \\ 0.064 & -0.078 \\ -0.127 & -0.208 \\ -1.211 & 0.936 \\ 0.377 & 1.352 \end{bmatrix} \begin{bmatrix} 0.379 & 0.200 \\ 0.200 & 0.673 \end{bmatrix}^{-1} \begin{bmatrix} 1.229 \\ 4.119 \end{bmatrix} - \begin{bmatrix} -4.560 \\ -0.475 \\ -1.981 \\ 8.508 \\ 8.802 \end{bmatrix} = \begin{bmatrix} 0.962 \\ -0.002 \\ 0.707 \\ -2.809 \\ -0.527 \end{bmatrix},$$

and the resulting signs (with dots inserted in the third and the fifth place) are specified in the column under (3, 5). It is seen that in each of the eight columns there are at least two negative entries, so we proceed to Intermediate Step No. 3.

Intermediate Steps, No. 3. In accordance with Rule 2, indicate in the headings of the open columns next to the columns of the two-element constraint sets which were prepared in Intermediate Step No. 2, the three-element sets S which are to be considered next. Compute for each of these $FE_S^{-1}e_S - e_T$ but proceed to the Final Step immediately as soon as such a vector contains nonnegative elements only; otherwise proceed to Intermediate Step No. 4 which deals with four-element constraint sets in the same way.

Our example requires the consideration of 9 three-element sets. Considering in particular the fourth, (3, 6, 7), we find for $FE_S^{-1}e_S - e_T$

$$\begin{bmatrix} -0.516 & -0.051 & -0.426 \\ 0.064 & -0.007 & -0.457 \\ -0.127 & 0.333 & -0.846 \\ 0.200 & 0.936 & 1.352 \end{bmatrix} \begin{bmatrix} 0.379 & -1.211 & 0.377 \\ -1.211 & 12.363 & -1.636 \\ 0.377 & -1.636 & 6.056 \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} 1.229 \\ 8.508 \\ 8.802 \end{bmatrix} - \begin{bmatrix} -4.560 \\ -0.475 \\ -1.981 \\ 4.119 \end{bmatrix} = \begin{bmatrix} 0.400 \\ 0.233 \\ 0.414 \\ 0.620 \end{bmatrix}.$$

All elements of this vector are positive, so we proceed to the Final Step. For completeness' sake, the columns of the five remaining vectors $[x^{(3,7,2)}, \cdots, x^{(6,7,2)}]$ are also specified; but this is not required since we can proceed to the Final Step immediately after the fourth vector.

Final Step. As soon as an Intermediate Step has led to an x^S which violates none of the constraints, verify the hypothesis $x^S = \hat{x}$ by considering all vectors x^{Sh}. If each x^{Sh} violates constraint h, then the hypothesis is correct. If some x^{Sh} does not violate constraint h, the hypothesis is not correct. In that case one has to take up the Intermediate Steps again and to proceed until a new x^S is found which violates none of the constraints, after which the Final Step is applied to this x^S; and so on.

In our case there are three sets S^h to be considered, viz., (3, 6), (3, 7), and (6, 7). It happens that the corresponding vectors have all been considered in Intermediate Step No. 2: $x^{(3,6)}$ violates constraint 7 (and also 5), $x^{(3,7)}$ violates 6 (and also 2 and 5), and $x^{(6,7)}$ violates 3 (and also 2 and 5). The conclusion is $x^{(3,6,7)} = \hat{x}$; the numerical value of this vector is

$$\hat{x} = \{0.400 \quad 0.233 \quad 0 \quad 0.414\},$$

as follows from the numerical specification given in Intermediate Step No. 3.[20] The corresponding Q-value is [see (2.11)]:

$$Q(\hat{x}) = \tfrac{1}{2}\,[18\ 16\ 22\ 20] \begin{bmatrix} 6 & 1 & 8 & 0 \\ 1 & 10 & 1 & 4 \\ 8 & 1 & 17 & 3 \\ 0 & 4 & 3 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 16 \\ 22 \\ 20 \end{bmatrix}$$

$$-\tfrac{1}{2}\,[1.229\ 8.508\ 8.802] \begin{bmatrix} 0.379 & -1.211 & 0.377 \\ -1.211 & 12.363 & -1.636 \\ 0.377 & -1.636 & 6.056 \end{bmatrix}^{-1} \begin{bmatrix} 1.229 \\ 8.508 \\ 8.802 \end{bmatrix} = 17.037.$$

5. Concluding Remarks

Whether the present method is or is not computationally efficient compared with other methods of quadratic programming is a question that does not admit

[20] When nonnegativity constraints are imposed on each of the elements of x, the vector $FE_S^{-1}e_S - e_T$ (completed with zeros at appropriate places) gives x^S immediately. If this is not the case, x^S is to be found from

$$x^S = x^0 - B^{-1}C_S E_S^{-1} e_S = B^{-1}(a - C_S E_S^{-1} e_S),$$

where C_S is the submatrix of C corresponding to the constraints of S. This result follows directly from (2.4) and (2.8).

a unique answer applicable to all cases.[21] To take an extreme example, Houthakker's capacity method works in the simplest conceivable way when $\hat{x} = 0$ (because this method "starts in the origin") but it is much worse off when the optimal vector is that of the unconstrained maximum. On the other hand, the present method is simplest when the latter alternative applies, while it is much poorer when $\hat{x} = 0$ because this implies that as many as n constraints are satisfied in equational form. Generally, the method is simple as long as the constrained maximum satisfies few constraints exactly. This follows directly from the fact that the successive intermediate steps require the inversion of matrices the order of which increases successively.

There is one important situation in which the method seems to be very advantageous. Suppose that a quadratic programming problem has been solved (by one method or another) and that one is interested in the sensitivity of the solution for changes in the constraints or in the coefficients of the objective function. As long as such changes are small, there is a good chance that the set S of constraints which the new optimum satisfies in equational form is the same as the similar set of the old optimum. This hypothesis can be tested in a straightforward fashion by means of Rule 3, which means that the initial step and all intermediate steps can be deleted if the test turns out to be positive; and the final step which is carried out gives then the new optimum immediately.

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[21] There is a computational advantage of the present method which seems worth-while, viz., that there is no problem of accumulation of rounding errors in the successive intermediate steps. The reason is that at every step we turn back to the basic matrix E and vector e.

IV–10

NOTES ON QUADRATIC PROGRAMMING: THE KUHN-TUCKER AND THEIL-VAN DE PANNE CONDITIONS, DEGENERACY, AND EQUALITY CONSTRAINTS*

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1. Introduction

Let us consider the quadratic programming problem

$$(1.1)\qquad \text{Max } Q(x) = a'x - \frac{1}{2}x'Bx \qquad \left(\sum_{i=1}^{n} a_i x_i - \frac{1}{2}\sum_i \sum_j x_i x_j b_{ij}\right)$$

under the side conditions

$$(1.2)\qquad C^{*\prime}x \leqq d^* \qquad (\sum_i c_{ki}^* x_i \leqq d_k^*;\ k = 1, \cdots, m)$$

and the non-negativity conditions

$$(1.3)\qquad x \geqq 0 \qquad (x_i \geqq 0;\ i = 1, \cdots, n).$$

The inequality conditions (1.2) and (1.3) will be referred to as constraints. For any n-vector x, constraints may either be *amply satisfied* (i.e., the strict inequality holds), or *binding* (i.e., the strict equality holds), or *violated.* If no constraint is violated, the x-vector concerned is called *feasible.* It is assumed that at least one feasible vector exists. The matrix B is supposed to be positive definite, which is a sufficient condition to ensure that $Q(x)$ is bounded above within the constraints.[2] Moreover, it then follows easily that $Q(x)$ is strictly concave, and hence the solution vector is unique.

The purpose of this note is to prove that the rules on which Theil and Van de Panne [6] based their recent method follow straightforward from the well-known Kuhn-Tucker conditions [5]; to propose a method for handling degeneracy in quadratic programming, and to consider the case in which there are linear equality constraints besides the inequalities (1.2)–(1.3). In the next section the Kuhn-Tucker conditions will be quoted; Section 3 will summarize the rules of Theil and Van de Panne, and in Section 4 these rules will be proved using the Kuhn Tucker conditions. Section 5 will proceed by discussing degeneracy; this is followed by a numerical illustration of degeneracy in Section 6. The final section considers the case in which some of the constraints take the form of equations.

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[2] The condition is a little stronger than necessary. See [2].

2. The Kuhn-Tucker Conditions

The well-known Kuhn-Tucker conditions assert that $\hat{x}$ is a solution of the problem if and only if there exists a vector $\hat{u}$ such that, apart from (1.2) and (1.3),

$$\hat{u} \geqq 0 \qquad (u_k \geqq 0;\ k = 1, \cdots, m) \tag{2.1}$$

$$\hat{u}'d^* - \hat{u}'C^{*\prime}\hat{x} = 0 \tag{2.2}$$

$$B\hat{x} + C^*\hat{u} - a \geqq 0 \tag{2.3}$$

and

$$\hat{x}'B\hat{x} + \hat{x}'C^*\hat{u} - \hat{x}'a = 0. \tag{2.4}$$

For our purposes it will prove useful slightly to rewrite the above conditions, following the development of Barankin and Dorfman [2].
Definitions:

$$v = Bx + C^*u - a \geqq 0 \qquad \text{[compare (2.3)]} \tag{2.5}$$

$$y = d^* - C^{*\prime}x \geqq 0 \qquad \text{[compare (1.2)].} \tag{2.6}$$

With these definitions, the new, equivalent formulation of the Kuhn-Tucker conditions may now be stated as follows: find vectors x, u, v and y, all $\geqq 0$, such that

$$\begin{bmatrix} B & C^* & -I & 0 \\ C^{*\prime} & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \\ v \\ y \end{bmatrix} = \begin{bmatrix} a \\ d^* \end{bmatrix}, \tag{2.7}$$

and such that $v'x + u'y$ is minimized; this minimum value has to equal 0 [see (2.2) and (2.4)]. Vectors x, u, v and y satisfying all these conditions satisfy all the Kuhn-Tucker conditions, and vice versa.

Let us also combine, for the sake of symmetry, (1.2) and (1.3), and let us agree to write

$$C' = \begin{bmatrix} C^{*\prime} \\ -I \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} d^* \\ 0 \end{bmatrix}. \tag{2.8}$$

Here C' and d are $(m + n) \times n$ and $(m + n) \times 1$ matrices respectively. We can henceforth simply write

$$C'x \leqq d \tag{2.9}$$

instead of conditions (1.2) and (1.3). Moreover, we can write

$$y'u + x'v = [d' \; - x'C] \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.10}$$

and this is precisely the expression to be minimized.

Let us pause a moment to interpret result (2.10). It says that with all $(n + m)$ constraints (2.9) we associate a real number such that either the constraint is exactly satisfied, or the associated number equals 0. Hence, at most $(n + m)$ of the $2(n + m)$ elements of u, v, x and y are positive. Degeneracy, by definition, will be said to occur when fewer than $(n + m)$ elements will be positive. From (2.7) we see that this special case happens when the right-hand vector (a, d) is linearly dependent on fewer than $(n + m)$ columns of the matrix

$$\begin{bmatrix} B & C^* & -I & 0 \\ C^{*\prime} & 0 & 0 & I \end{bmatrix}.$$

Barring degeneracy for the time being, it can be concluded that exactly $(m + n)$ elements of u, v, x and y will be positive.

3. The Theil-Van de Panne Method

In this section the approach of Theil and Van de Panne will be briefly and heuristically explained. They reason that the inequality constraints (2.9) are the source of all trouble in the maximizing problem. Indeed, maximizing a function under equality constraints is not much of a problem, and can be solved quite generally and rather simply with the device of Lagrangean multipliers. Hence the procedure they suggest for finding the solution vector $\hat{x}$ amounts to the problem of finding the set S out of the $(n + m)$ constraints (2.9), such that, when (1.1) is maximized with the constraints belonging to S binding, a vector x^S results which is feasible and optimal; x^S is defined as the vector maximizing (1.1) with all constraints belonging to S binding.[3]

They generate this set S in the following way. First, maximize (1.1), never mind the constraints (2.9). The resulting vector, say x^0,[4] either will violate some constraints, or else it will obviously be the solution vector. If x^0 does violate some constraint(s), then they proceed to prove that $\hat{x}$, the solution vector, binds at least one of the constraints violated by x^0. Or, in other words, if the supremum is taken on at some constrained point, then at least one of the constraints binding in this point will be violated by x^0. This is Rule 1. Hence, they next consider all one-element sets S consisting of equations violated by x^0. If no resulting x^S is feasible, all two-element sets S are considered, of which the first element is one that was violated by x^0, say constraint h, and the second element some constraint violated by x^h, say constraint k. And so on. The successive steps of this procedure are determined by the authors' Rule 2, which states that if $x^S = \hat{x}$, and if $S' \subset S$, but $S' \neq S$, $x^{S'}$ will violate at least some constraint in $S - S'$; that is, if the constraints S' which are imposed to be binding are indeed binding for the solution vector and if there are some constraints which are not imposed to be binding, but which are nevertheless binding for the solution vector, viz. $S - S'$, then $x^{S'}$ will violate at least some constraint of the latter set.

[3] Such a vector exists provided that the constraints belonging to S are not inconsistent when written in equational form.

[4] Note that x^0 also fits the definition of x^S if one interprets the "0" as the empty set.

Suppose, next, that we have succeeded to find a set S in this way such that x^S is feasible. By hypothesis, such a set S exists, and by our procedure and Rules 1 and 2 we are bound to hit upon it at some stage. The question remains, whether this feasible x^S is the optimum vector such that

$$x^S = \hat{x}. \tag{3.1}$$

Rule 3 of [6] then states that, under condition (3.1), for any $h \in S$, the vector x^{S-h} is such that upon substitution in (2.9) the constraint h is violated: $\sum_i c_{hi} x_i^{S-h} > d_h$. Here, the vector x^{S-h} is the vector maximizing (1.1) with all constraints belonging to S, apart from the deleted constraint h, binding.

4. The Proof

In this section we will present a mathematical proof of the Theil-Van de Panne rules, using the Kuhn-Tucker conditions. In the next section, following a discussion on degeneracy, a more intuitive argument will be used to prove the same.

For any consistent set S, we can easily compute x^S in a self-explanatory notation. Write

$$Q(x, \lambda) = a'x - \tfrac{1}{2}x'Bx - \lambda'_S(C'_S x - d_S), \tag{4.1}$$

then

$$\frac{dQ(x, \lambda)}{dx} = a - Bx - C_S \lambda_S \tag{4.2}$$

gives an expression for x^S upon equating (4.2) to 0:

$$x^S = B^{-1}a - B^{-1}C_S\lambda_S . \tag{4.3}$$

Since $x^0 = B^{-1}a$, we have

$$x^S = x^0 - B^{-1}C_S\lambda_S . \tag{4.4}$$

To obtain an expression for the vector λ_S, the Lagrangeans associated with the strict equalities, we premultiply (4.4) with C'_S:

$$C'_S B^{-1} C_S \lambda_S = C'_S x^0 - d_S , \tag{4.5}$$

where use has been made of the fact that $C'_S x^S = d_S$. Hence

$$\lambda_S = (C'_S B^{-1} C_S)^{-1}(C'_S x^0 - d_S). \tag{4.6}$$

Now we have seen in Section 2 that the Kuhn-Tucker conditions say that when $x^S = \hat{x}$, all components of λ_S are positive (in the absence of degeneracy). Define

$$P_S = C'_S B^{-1} C_S , \tag{4.7}$$

then P_S is a strictly positive definite matrix when the columns of C_S are independent.[5] This can safely be assumed. In fact, it follows from the assumption of

[5] See, e.g. Zurmühl [7, p. 133].

nondegeneracy. Rule 1 follows directly from (4.5), for

$$\lambda'_S P_S \lambda_S = \lambda'_S(C'_S x^0 - d_S) > 0.$$

Since λ_S is a strictly positive vector, at least one element of $C'_S x^0 - d_S$ is positive, and hence at least one constraint binding in the solution is violated by x^0. This, at the same time, proves Rule 2. For the set S' which is imposed to be binding in $x^{S'}$ and which is also binding in $x^S = \hat{x}$ can be used to eliminate an appropriate set of variables from the objective function, after which the new objective function is to be maximized subject to the old constraints except that those in S' are deleted; and in this new problem $x^{S'}$ plays the role of x^0. In Section 7 we will use a similar approach.

As for Rule 3, still assuming $x^S = \hat{x}$, we will show that deletion of one constraint from S would lead to its violation in the resulting vector x^{S-h}. Assume constraint 1 $(1 \in S)$ is deleted and let us partition P_S as follows:

$$C'_S B^{-1} C_S = P_S = \begin{bmatrix} p_{11} & q' \\ q & P_{11} \end{bmatrix}, \tag{4.8}$$

where p_{11} is the leading element of P_S. We need to prove that

$$C'_1 x^{S-1} - d_1 > 0, \tag{4.9}$$

or, with (4.4) and (4.8)

$$C'_1 x^0 - d_1 - q'\lambda_{S-1} > 0, \tag{4.10}$$

or, with the first element of (4.5) and (4.8)

$$[p_{11} \quad q']\lambda_S - q'\lambda_{S-1} > 0. \tag{4.11}$$

From (4.6) and (4.8) we immediately get

$$\lambda_{S-1} = P_{11}^{-1}(C'_{S-1}x^0 - d_{S-1}) \tag{4.12}$$

or, in view of (4.5) again

$$\lambda_{S-1} = P_{11}^{-1}[q \quad P_{11}]\lambda_S = [P_{11}^{-1}q \quad I]\lambda_S . \tag{4.13}$$

Upon substitution of (4.13) into (4.11) we obtain for the left-hand side of (4.11)

$$[p_{11} \quad q']\lambda_S - q'[P_{11}^{-1}q \quad I]\lambda_S . \tag{4.14}$$

Partition $\lambda_S = \begin{bmatrix} \lambda_S^1 \\ \lambda_S^{S-1} \end{bmatrix}$, then we can write (4.14):

$$p_{11}\lambda_S^1 + q'\lambda_S^{S-1} - q'P_{11}^{-1}q\lambda_S^1 - q'\lambda_S^{S-1} = (p_{11} - q'P_{11}^{-1}q)\lambda_S^1. \tag{4.15}$$

This expression should be positive, given that λ_S^1 is positive. But, clearly, $p_{11} - q'P_{11}^{-1}q > 0$. For we have, for any column $x \neq 0$ of appropriate order, $x'P_S x > 0$ because P_S is positive definite; take $x' = [-1 \quad q'P_{11}^{-1}]$, then

$$0 < x'P_S x = p_{11} - q'P_{11}^{-1}q;$$

which means that Rule 3 holds whenever Kuhn–Tucker's $\lambda_S > 0$.

Conversely, it follows from $x^S = \hat{x}$ and $C'x_k^{S-k} > d_k$ (for all $k \in S$) that all λ_S^k are positive. For we can reverse the proof immediately. We know that the left-hand side of (4.9) is positive, hence (4.15)—which is nothing else than (4.9) rewritten—is also positive; in these expressions 1 can be replaced by any $k \in S$. Also, inequalities of the type $p_{11} - q'P_{11}^{-1}q > 0$ hold generally, hence $\lambda_S^k > 0$.

The result of all this is that Rule 3 of [6] is equivalent to the Kuhn-Tucker condition, and hence may be reformulated: $x^S = \hat{x}$, if x^S is feasible and $\lambda_S > 0$. All these results were derived under the assumption of nondegeneracy. Let us now consider degeneracy.

5. Degeneracy

5.1. Degeneracy and Perturbations

In the previous discussion we have explicitly excluded the possibility of degeneracy. Fortunately, this is not a serious limitation, for we can easily allow for degeneracy. In such a case we bring to bear a fundamental theorem, proved in [2] and [3], asserting that if the original problem is feasible, then there will always exist another problem, with slightly different parameters a^* and d^*, whose solution is arbitrarily close to the solution of the original problem ($|\hat{x}^* - \hat{x}| < \epsilon$) and which is such that not fewer than $(n + m)$ elements of u, v, x and y are positive.

All the same, it may be useful to indicate the very special character of degeneracy in some more detail. Since under degeneracy fewer than $(n + m)$ elements of u, v, x and y are positive, there is at least one pair (v_i, x_i) or (u_k, y_k) which is equal to $(0, 0)$. We can give two alternative, essentially identical, interpretations of such an occurrence. Either, some $x_i = 0$ or $y_k = 0$, though x_i was not imposed to be 0, or the kth constraint of (1.2) was not imposed to be binding. Or alternatively, for some x_i or y_k imposed to be binding the associated Lagrangean equals 0. In general, we should then write for any S such that $x^S = \hat{x}$:

$$\left(\frac{\partial Q(x, \lambda)}{\partial d_h}\right)_{h \in S} = \lambda_S^h \geqq 0. \tag{5.1}$$

In accordance with the latter interpretation, let us assume that $1 \in S$, but $\lambda_S^1 = 0$. Then we will first show that $x^S = x^{S-1} = \hat{x}$, which implies that the solution vector remains unique, even though the set S generating this solution need not be unique. From (4.4) we have

$$x^S = x^0 - B^{-1}[C_1\, C_{S-1}]\begin{bmatrix}\lambda_S^1 \\ \lambda_S^{S-1}\end{bmatrix} = x^0 - B^{-1}C_{S-1}\lambda_S^{S-1}, \tag{5.2}$$

using the partitioned notation introduced before, and $\lambda_S^1 = 0$. Also from (4.4)

$$x^{S-1} = x^0 - B^{-1}C_{S-1}\lambda_{S-1}\,. \tag{5.3}$$

Hence

$$x^S = x^{S-1}, \tag{5.4}$$

when it is true that

$$\lambda_S^{S-1} = \lambda_{S-1}, \tag{5.5}$$

given that $\lambda_S^{\ 1} = 0$. To prove (5.5), write, using (4.5)

$$C'_{S-1}B^{-1}[C_1\ C_{S-1}]\begin{bmatrix}\lambda_S^{\ 1}\\ \lambda_S^{S-1}\end{bmatrix} = C'_{S-1}x^0 - d_{S-1}. \tag{5.6}$$

Hence

$$\lambda_S^{S-1} = (C'_{S-1}B^{-1}C_{S-1})^{-1}(C'_{S-1}x^0 - d_{S-1}), \tag{5.7}$$

and from (4.6) we immediately see that the right-hand side of this expression equals λ_{S-1}.

It may be useful to illustrate all this by means of a simple picture, taken from [6]; see Fig. 1. The two alternative interpretations in this concrete example now take the following form. First, taking $S = \{1\}$, we obtain $x^1 = \hat{x}$, a vector which violates no other constraint, though it happens to satisfy constraint 3 exactly. Second, taking $S = \{1, 3\}$, we obtain $x^{1,\,3} = \hat{x}$, which has as associated vector

$$\lambda_{13}^1 > 0$$

$$\lambda_{13}^3 = 0.$$

Let us compare these two events with the "normal" situation. In the absence of degeneracy, with the strict inequality holding in (5.1) for all $h \in S$,—and, it should be added, $\Sigma c_{hi}x_i^{\ S} = d_h$ for no $h \notin S$—we have the simple result that

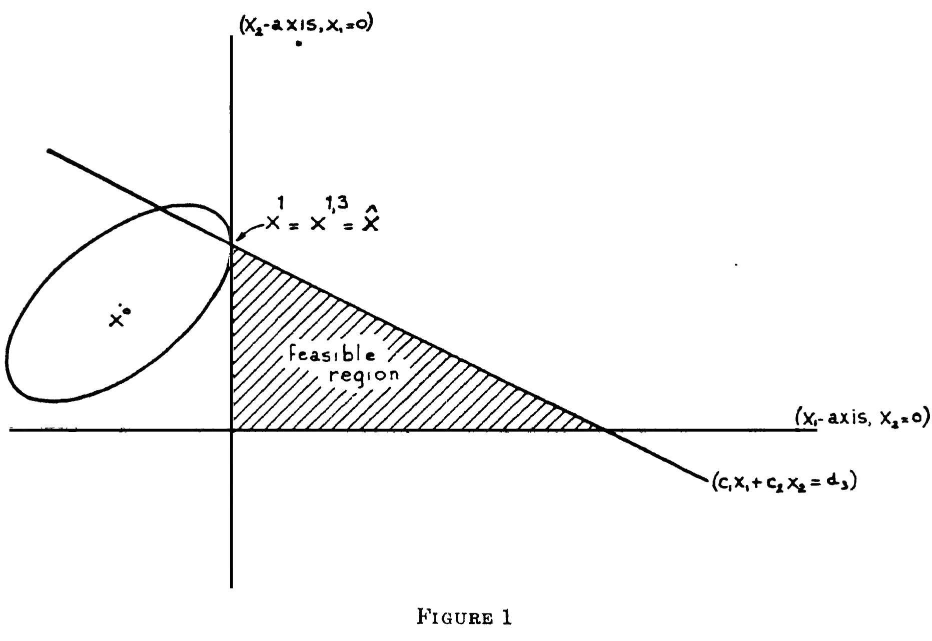

FIGURE 1

a small increase in d_h (for any $h \in S$) increases the value of $Q(x^S)$, while a small decrease "hurts" the value of $Q(x^S)$. This follows from (5.1). Moreover, no change in d_h, if small enough, influences the feasibility of x^S, as follows from the fact that for all constraints $h \notin S$ there was some "spare" room. Under conditions of degeneracy these simple statements no longer hold. First, considering x^1 again, we see that a slight change in d_1 may impair its feasibility. This follows from the fact that $\Sigma_i c_{3i} x_i^1 = d_3$, and hence the smallest change in d_1, leading to the minutest change in x^1, could make $\Sigma_i c_{3i} x_i^1 > d_3$. Second, considering $x^{1,3}$, note that *any* change in d_3—the constant term of the binding constraint with associated Lagrangean equal to 0—, "hurts" the value of $Q(x^{1,3})$. This may be shown as follows. Since maximizing a function under more constraints never increases the value of the maximand, we have $Q(x^{1,3}) \leqq Q(x^1)$. The equality holds for the very special case that, whether explicitly stipulated or not, constraint 3 is binding anyway, which happens when λ_{13}^3 equals 0. However, from (4.6) we have

$$\frac{\partial \lambda_{13}^3}{\partial d_3} = -C'_3 B^{-1} C_3,$$

where the right-hand side is a negative constant. Hence λ_{13}^3 equals 0 for but one value of d_3, and even the smallest change "hurts" by resulting in a positive or negative value of λ_{13}^3. It is worth-while to add that the figure also illustrates that a slight change in d_1 or d_3 "dedegenerates" the situation again. Our results are summarized in Table 1.

Notice in particular that the resulting changes in $\hat{x}$ can be made arbitrarily small, and that after a perturbation the set S giving the solution vector is unique in all 4 cases.

5.2. *Further Comment on the Proof*

It may be useful to give an intuitive argument for the proof in Section 4 using the discussion on degeneracy. Let us exclude the case of degeneracy. Then the set S such that $x^S = \hat{x}$ is unique, and all $\lambda_S^h (h \in S)$ are strictly positive. Consider then (4.6) again, and write, substituting (4.7):

$$\lambda_S = P_S^{-1}(C'_S x^0 - d_S) \tag{5.8}$$

and hence

$$\frac{\partial \lambda_S}{\partial d_S} = -P_S^{-1}. \tag{5.9}$$

TABLE 1

Results of Small Perturbations on Degeneracy

Perturbation	Vector x^1	Vector $x^{1,3}$	Value $Q(\hat{x})$ after perturbation
Increase d_1	solution	lower $Q(x)$ value	greater
Decrease d_1	not feasible	solution	smaller
Increase d_3	solution	lower $Q(x)$ value	no change
Decrease d_3	not feasible	solution	smaller

Since P_S is positive definite, we have, for all $h \in S$,

$$\frac{\partial \lambda_S^h}{\partial d_h} = -p^{hh} = \text{a negative constant,} \tag{5.10}$$

where in the usual manner the superscripts denote elements from the inverse of P_S. Equivalently

$$\frac{\partial Q(x, \lambda)}{\partial d_h} = \lambda_S^h > 0 \quad \text{and} \quad \frac{\partial^2 Q(x, \lambda)}{\partial (d_h)^2} = \frac{\partial \lambda_S^h}{\partial d_h} < 0. \tag{5.11}$$

Hence, increasing $d_h (h \in S)$ increases the value of $Q(x^S)$ at a decreasing rate. Furthermore, according to (5.10), there will always be an increase in d_h such that λ_S^h becomes 0. Call this increase $d_{h0} > 0$. Now we have indicated in our discussion on degeneracy, that after adding d_{h0} to d_h, so that $\lambda_S^h = 0$, it does not make any difference whether we maximize $Q(x)$ under constraints belonging to set S including h or excluding h, see (5.4). Disregarding constraint h from the original set S, giving the vector x^{S-h}, does increase $Q(x)$ in exactly the same way as an increase of d_h with d_{h0} would, while keeping constraint h as an equality ($\sum c_{hi} x_i = d_h + d_{h0}$). They produce the same vector x^{S-h}. But now:

$$C'_h x^{S-h} = d_h + d_{h0} > d_h, \tag{5.12}$$

and hence we see immediately that for each $h \in S$ maximization of $Q(x)$ while disregarding constraints h leads to violation of the hth constraint.

Finally, it is of some interest to observe from (4.8) and (5.10), that

$$\frac{\partial \lambda_S^h}{\partial d_i} = \frac{\partial \lambda_S^i}{\partial d_h}. \tag{5.13}$$

Interpreting the λ_S^h as the shadow price of the "source" d_h [this well-known interpretation is essentially based on (5.1)], this equality says that the influence of an increase in the supply of the ith source on the price of source d_h is equal to the influence of an increase in source d_h on the price of source d_i. Unfortunately, there are no general rules on the sign of this expression.

6. Numerical Illustration of Degeneracy

As a numerical example of the Theil-Van de Panne approach, consider the following example borrowed from Houthakker [4]. He considered a problem[6] which arose in the context of a monopolist facing linear demand functions for each of his four products. When we consider the monopolist as a quantity-adaptor it is useful to invert the demand functions, to get

$$p = a + \tfrac{1}{2} B x, \tag{6.1}$$

giving the price vector as a linear function of the quantities. Assume the monopolist strives to maximize total revenue $p'x$, but that his freedom of action is limited by the fact that three factors of production are scarce, apart from the obvious limitation that no negative quantities can be produced. Under these conditions, we can specify the matrices a and B of (1.1) and (6.1) as follows:

[6] The problem is slightly changed to incorporate degeneracy.

$$(6.2) \qquad a = \begin{bmatrix} 18 \\ 16 \\ 22 \\ 20 \end{bmatrix}; \qquad B = \begin{bmatrix} 6 & 1 & 8 & 0 \\ 1 & 10 & 1 & 4 \\ 8 & 1 & 17 & 3 \\ 0 & 4 & 3 & 11 \end{bmatrix},$$

and the matrices C' and d of (2.9):

$$(6.3) \qquad C' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 0 & 10 & 0 \\ 0 & 4 & 0 & 5 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \qquad d = \begin{bmatrix} 1392/1330 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These constraints will be numbered consecutively from 1 to 7. The matrix B is positive definite.

To prevent rounding errors all computations have been made without division. First, let us check whether x^0 is feasible:

$$(6.4) \qquad \begin{aligned} x^0 = B^{-1}a &= \frac{1}{2916}\begin{bmatrix} 1521 & -186 & -753 & 273 \\ -186 & 364 & 94 & -158 \\ -753 & 94 & 553 & -185 \\ 273 & -158 & -185 & 373 \end{bmatrix}\begin{bmatrix} 18 \\ 16 \\ 22 \\ 20 \end{bmatrix} \\ &= \frac{1}{2916}\begin{bmatrix} 13296 \\ 1384 \\ -3584 \\ 5776 \end{bmatrix}. \end{aligned}$$

Substitution in (6.2) shows that the constraints 1, 2, 3 and 6 are violated by x^0

$$(6.5) \qquad C'x^0 - d = \frac{1}{(2916)(1330)} \cdot \begin{bmatrix} 18 & 380 & 688 \\ 32 & 994 & 640 \\ 34 & 138 & 440 \\ -17 & 683 & 680 \\ -1 & 840 & 720 \\ 4 & 766 & 720 \\ -7 & 682 & 080 \end{bmatrix}.$$

Hence we try next x^1, x^2, x^3 and x^6. Should one of these vectors be feasible, we would have found the solution vector, for Rule 3 is complied with. However, computations show that not even one of these vectors is feasible. All necessary computations can be made with the aid of (6.5) and the matrix $C'B^{-1}C$:

$$(6.6) \quad C'B^{-1}C = \frac{1}{2916}\begin{bmatrix} 981 & 1365 & 1971 & -855 & -114 & 291 & -303 \\ 1365 & 18025 & -2385 & -75 & -10 & -1765 & 485 \\ 1971 & -2385 & 8829 & -621 & -666 & 549 & -1233 \\ -855 & -75 & -621 & 1521 & -186 & -753 & 273 \\ -114 & -10 & -666 & -186 & 364 & 94 & -158 \\ 291 & -1765 & 549 & -753 & 94 & 553 & -185 \\ -303 & 485 & -1233 & 273 & -158 & -185 & 373 \end{bmatrix}.$$

To check for feasibility first compute λ_S from (4.6). For example:

$$\lambda_1 = \frac{2916}{981} \cdot \frac{1}{(2916)(1330)} \cdot (18 \quad 380 \quad 688),$$

and similarly for all positive elements of (6.5). Since (6.6) is positive definite, all one-element Lagrangeans are positive. Then, for x^S, say, to be feasible, we should have for all constraints not in S, say $\bar{S}$;

$$C'_{\bar{S}} x^S \leqq d_{\bar{S}} \, ,$$

or, with (4.4)

$$C'_{\bar{S}} x^S = C'_{\bar{S}} x^0 - C'_{\bar{S}} B^{-1} C_S \lambda_S \leqq d_{\bar{S}} \, , \tag{6.7}$$

or again

$$C'_{\bar{S}} B^{-1} C_S \lambda_S - (C'_{\bar{S}} x^0 - d_{\bar{S}}) \geqq 0. \tag{6.8}$$

The matrix $C'_{\bar{S}} B^{-1} C_S$ can be read immediately from (6.6) and $C'_{\bar{S}} x^0 - d_{\bar{S}}$ from (6.5). And λ_S is computed from (4.6), as said above:

$$\lambda_S = (C'_S B^{-1} C_S)^{-1} (C'_S x^0 - d_S). \tag{6.9}$$

If any component of λ_S so computed is negative, the vector x^S, even though it may be feasible, is not the solution vector. It turns out that no vector is feasible. For example, x^3 violates 1, 2 and 6. Hence, consider all 2-element sets S with a first element equal to a constraint violated by x^0 (i.e., 1, 2, 3 or 6) and a second violated by the corresponding vector, in this case respectively x^1, x^2, x^3 or x^6. For example, consider $S = \{3, 6\}$. It turns out that $x^{3,6}$ violates 1, 2 and 5. Similarly, it turns out that no 2-element set S produces a feasible vector x^3. Continuing with 3 elements sets, we have to take, among many others, $S = \{2, 3, 6\}$. With the benefit of hindsight, we will consider this case in some detail. Formula (6.9), taking $S = \{2, 3, 6\}$, gives:

$$\lambda_{236} = \frac{2916}{(56 \quad 545 \quad 322 \quad 400)}$$

$$\cdot \begin{bmatrix} 4 \;\; 581 \;\; 036 & 349 \;\; 920 & 14 \;\; 273 \;\; 820 \\ 349 \;\; 920 & 6 \;\; 852 \;\; 600 & -5 \;\; 686 \;\; 200 \\ 14 \;\; 273 \;\; 820 & -5 \;\; 686 \;\; 200 & 153 \;\; 454 \;\; 500 \end{bmatrix} \times \frac{1}{2916} \cdot \begin{bmatrix} 24 \;\; 808 \\ 25 \;\; 668 \\ 3 \;\; 584 \end{bmatrix}$$

$$= \frac{1}{(56 \quad 545 \quad 322 \quad 400)} \cdot \begin{bmatrix} 173 \;\; 785 \;\; 458 \;\; 528 \\ 164 \;\; 194 \;\; 011 \;\; 360 \\ 758 \;\; 132 \;\; 472 \;\; 960 \end{bmatrix}$$

$$= \frac{1}{6650} \begin{bmatrix} 20 \;\; 438 \\ 19 \;\; 310 \\ 89 \;\; 160 \end{bmatrix}.$$

Checking feasibility with (6.8), using the relevant elements of (6.6) and (6.5):

$$\frac{1}{(6650)(2916)}\left(\begin{bmatrix}1365 & 1971 & 291\\ 75 & -621 & -753\\ 10 & -666 & 94\\ 485 & -1233 & -185\end{bmatrix}\begin{bmatrix}20 & 438\\ 19 & 310\\ 89 & 160\end{bmatrix} - \begin{bmatrix}91 & 903 & 440\\ -88 & 418 & 400\\ -9 & 203 & 600\\ -38 & 410 & 400\end{bmatrix}\right)$$

$$= \frac{1}{(6650)(2916)}\cdot\begin{bmatrix} & & 0\\ 10 & 822 & 260\\ 4 & 928 & 560\\ & 801 & 900\end{bmatrix}.$$

Hence $x^{2,3,6}$ provides the optimal solution: all Lagrangeans positive and (6.8) non-negative.

$$\hat{x} = x^{2,3,6} = x^0 - B^{-1}C_{236}\lambda_{236} = \begin{bmatrix}4/10\\ 31/133\\ 0\\ 55/133\end{bmatrix}.$$

That the solution happens to satisfy constraint 1 exactly, even though this was not stipulated, indicates degeneracy. In fact, considering $S = \{1, 2, 3, 6\}$, we get for λ_{1236} the exact vector:

$$\lambda_{1236} = \frac{1}{(6650)(123\ 974\ 556\ 480)}\cdot\begin{bmatrix} & & & & & 0\\ 2 & 533 & 791 & 985 & 338 & 240\\ 2 & 393 & 948 & 685 & 628 & 800\\ 11 & 053 & 571 & 455 & 756 & 800\end{bmatrix}$$

$$= \frac{1}{6650}\begin{bmatrix} & 0\\ 20 & 438\\ 19 & 310\\ 89 & 160\end{bmatrix},$$

TABLE 2

Solution Values of the Kuhn-Tucker Variables

Constraints	Lagrangeans	Product check
$y_1 = 0$	$u_1 = 0$	$y_1u_1 = 0$
$y_2 = 0$	$u_2 = \frac{20438}{6650}$	$y_2u_2 = 0$
$y_3 = 0$	$u_3 = \frac{19310}{6650}$	$y_3u_3 = 0$
$x_1 = \frac{532}{1330}$	$v_1 = 0$	$x_1v_1 = 0$
$x_2 = \frac{310}{1330}$	$v_2 = 0$	$x_2v_2 = 0$
$x_3 = 0$	$v_3 = \frac{89160}{6650}$	$x_3v_3 = 0$
$x_4 = \frac{550}{1330}$	$v_4 = 0$	$x_4v_4 = 0$

where we needed to invert the 4×4 matrix $C'_{1236}B^{-1}C_{1236}$. Obviously (5.5) holds, and similarly (5.4). In practice, degeneracy is likely to escape notice through rounding errors. Summing up the complete solution in Kuhn-Tucker notation we can collect our results as in Table 2.

7. How to Handle Equality Constraints

It may happen that there are a number of equalities (rather than inequalities) included among constraints (1.2) Let us assume that we have k equalities:

$$(7.1) \qquad Ex = f \qquad (\sum_i e_{hi}x_i = f_h\,; \qquad h = 1, \cdots, k).$$

The rank of E can be assumed to be equal to k; if it were not, either $(k - 1)$ equations would suffice to give the same information, or else the system would be inconsistent. Assuming the leading submatrix of order $k \times k$ to be nonsingular, the most straightforward approach appears to be to eliminate the first k variables $x_i (i = 1, \cdots, k)$, by expressing them in the remaining variables. Partitioning:

$$(7.2) \qquad (E_1 \; E_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f,$$

where E_1, E_2, x_1 and x_2 are of order $k \times k$, $k \times (n - k)$, $k \times 1$ and $(n - k) \times 1$ respectively. Solving for x_1:

$$(7.3) \qquad x_1 = E_1^{-1}f - E_1^{-1}E_2x_2\,.$$

Substituting these expressions in (1.1) and (2.9) has the disadvantage of losing the specially simple form matrices B and C may have in the original problem; as an advantage there are fewer variables. Another advantage is that this procedure always "works," since the new matrix B^*, say, of order $(n - k) \times (n - k)$ will always remain positive definite: substitute (7.3) in (1.1):

$$(7.4) \qquad a' \begin{Bmatrix} E_1^{-1}f - E_1^{-1}E_2x_2 \\ x_2 \end{Bmatrix} - \tfrac{1}{2}[f'(E_1^{-1})' - x'_2\,E'_2(E_1^{-1})' \qquad x'_2] \\ \times \begin{bmatrix} B_{kk} & B_{k,n-k} \\ B_{n-k,k} & B_{n-k,n-k} \end{bmatrix} \begin{Bmatrix} E_1^{-1}f - E_1^{-1}E_2\,x_2 \\ x_2 \end{Bmatrix}.$$

The latter part of this expression consists of a constant, an expression linear in x_2, and the quadratic form

$$x'_2\; [E'_2(E_1^{-1})' \;\; I] \begin{bmatrix} B_{kk} & B_{k,n-k} \\ B_{n-k,k} & B_{n-k,n-k} \end{bmatrix} \begin{bmatrix} E_1^{-1}E_2 \\ I \end{bmatrix} x_2\,.$$

The $(n - k) \times (n - k)$ matrix of this quadratic form will again be positive definite, since the rows of the premultiplying matrix and columns of post-multiplying matrix are linearly independent, see footnote 6 above.

Alternatively, we could proceed in line with the Theil-Van de Panne approach; but then we should not originally start at the point of the unconstrained maxi-

mum, x^0, but at x^E, say, where E is the set of k equalities. Formulae (4.1) to (4.6) remain valid. The situation is, in fact, very much similar to a situation we have in the Theil-Van de Panne approach after some steps have been made. Very much, but not quite. The difference is that when equalities are present the Lagrangeans associated with them need not necessarily all be positive. The situation is completely analogous to the unsymmetric dual problem in linear programming; as will be recalled, the dual of a primal subject to equality rather than inequality constraints amounts to finding a vector the elements of which are not restricted to be non-negative. Referring to Figure 1, suppose constraint 3 is imposed as an equality. The associated λ_3^3 is negative [for a decrease in d_3 would increase $Q(x)$]. λ_{13}^3 equals 0. λ_{13}^3 evaluated at $d_3 + \epsilon$ is negative and λ_{13}^3 evaluated at $d_3 - \epsilon$ is positive. Clearly, the associated Lagrangeans may change in sign as the set E is increased by more constraints, taken from the inequalities. However, the "E-Lagrangeans" may have, at any stage, any sign, while we specifically choose the constraints belonging to S such that $\lambda_S^i > 0$ for all i, at $x^S = \hat{x}$, as we pointed out in Section 4.

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IV–11

A METHOD OF SOLUTION FOR QUADRATIC PROGRAMS*

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This paper describes a method of minimizing a strictly convex quadratic functional of several variables constrained by a system of linear inequalities. The method takes advantage of strict convexity by first computing the absolute minimum of the functional. In the event that the values of the variables yielding the absolute minimum do not satisfy the constraints, an equivalent and simplified quadratic problem in the 'Lagrange multipliers' is derived. An efficient algorithm is devised for the transformed problem, which leads to the solution in a finite number of applications. A numerical example illustrates the method.

1. Introduction

We are concerned with the general quadratic programming problem posed in the following form: Minimize

$$-a^T x + \tfrac{1}{2} x^T Q_0 x, \tag{1}$$

over the set of all x satisfying:

$$A^T x \le d_0 . \tag{2}$$

Here matrix A has order $m \times n$; Q_0 is $m \times m$; a and x are $m \times 1$; and d_0 is $n \times 1$. Superscript 'T' denotes matric transposition. It is assumed that Q_0 is positive definite and symmetric.

Ignoring constraints (2) momentarily, the minimum of (1) is taken on at the unique point

$$x_0 = Q_0^{-1} a, \tag{3}$$

where the gradient $Q_0 x - a$ vanishes. If, further, x_0 satisfies the constraints (2), it must then solve the quadratic problem. If not, the optimum is taken on for some point x on the boundary of the convex polyhedron described by the constraints (2).

In a recent article [7], Theil and van de Panne appear to have been the first to utilize the non-singularity of Q_0, and, starting from knowledge of x_0, to systematically search out the optimal boundary point. As the authors noted, such a technique does not require a feasible point to initiate the calculations. Their technique is compared briefly at the end of this paper with the one described below. Actually our method appears to resemble more that proposed by Beale in 1955 [1].

In Section 2, we derive an equivalent problem, which is immediately feasible and permits a simple algorithm. Following Section 3, where the algorithm is

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described and necessary proofs given, an example, also used by Theil and van de Panne, is furnished.

2. Initial Transformation

The Kuhn-Tucker geometric conditions for an optimum state that a feasible point x is a solution if and only if the gradient $(Q_0x - a)$ at x is a non-positive combination of the outward-directed normals of those support hyperplanes, if any, containing x; that is, if and only if there is some y such that:

$$(4) \quad (Q_0x - a) + Ay = 0; \qquad y \geqq 0; \qquad \text{and} \quad y^T(d_0 - A^Tx) = 0;$$

the last condition ensuring that only those support planes containing x are considered. $y = 0$ would correspond to the point x_0, which would be the unique solution if it satisfied the constraints.

In the first part of (4) we may solve for x in terms of y uniquely:

$$(5) \qquad x = Q_0^{-1}(a - Ay),$$

from which the final solution may be computed, using an optimal set y of 'Lagrange multipliers'.

Using (5) we may eliminate x from (2). Then defining:

$$(6) \qquad Q = A^TQ_0^{-1}A; \qquad d = d_0 - A^Tx_0; \qquad \text{and} \quad z = d + Qy,$$

we may pose the following problem which, because of (5), is fully equivalent to the original:

Find a pair of vectors y and z which satisfy:

$$(7) \qquad -Qy + z = d; \qquad y, z \geqslant 0; \qquad \text{and} \quad y^Tz = 0,$$

where Q is symmetric and non-negative definite.

Let us now note that, by the Kuhn-Tucker conditions, this problem is entirely equivalent to the following quadratic problem in the variable point y:

Minimize

$$(8) \qquad f(y) = d^Ty + \tfrac{1}{2}y^TQy, \quad \text{subject to } y \geqslant 0,$$

where Q and d represent the given data, and Q is a symmetric and non-negative definite matrix.

It is this 'derived' problem, which has some interest in its own right, which shall be solved. Having an optimal y, the optimal x for the original problem is obtained as in (5).

Note that a 'feasible' point, i.e., one satisfying the constraints $y \geqslant 0$; namely $y = 0$, yielding $f(0) = 0$, is immediately available, and is indeed the only extreme point of the constraint set (the non-negative orthant), which is just a simple cone. $y = 0$ would yield the solution if and only if $d \geqslant 0$, as (7) shows.

Finally, with reference to the method to be described, note that no assumptions as to strict convexity of $f(y)$, or any assumptions as to 'degeneracy' usually required for linear programming methods are made. We require merely that a finite solution exists.

3. The Algorithm

Preliminaries

With reference to the conditions (7) for an optimum, y and z form an optimal pair if and only if (i) y is a point in the non-negative orthant; (ii) z, the gradient of f at y, points into the non-negative orthant; and (iii) y is perpendicular to its corresponding gradient z.

Throughout the calculations condition (i), starting with $y = 0$, is retained, and as the iterations proceed the functional f will be non-increasing. After a finite number of iterations, the calculations will end with conditions (ii) and (iii) satisfied.

Associated with an iteration is a point y and a set of n independent directions. As in the extreme point methods of linear programming, these directions appear as the rows of the inverse of an $n \times n$ 'current basis matrix'. In going from one iteration to the next, the current basis matrix is altered by replacing a single one of its columns by some other column, and the required 'current inverse' is obtained by algorithm from the inverse associated with the previous iteration. The original data Q and d are retained throughout, and all calculation is based on the computed inverse.

We adopt the following notation. If B is the current basis matrix we write:

$$B = (b_1, b_2, \cdots, b_n), \tag{9}$$

so that b_i denotes the i^{th} column of B. Further we write:

$$(B^{-1})^T = (b^1, b^2, \cdots, b^n), \tag{10}$$

so that b^i denotes the i^{th} row of B^{-1}, written in column form. The statement that $BB^{-1} = I$, the identity matrix of order n, is then equivalent to the n^2 scalar conditions:

$$b_i^T b^i = 1; \qquad b_i^T b^j = 0; \qquad \text{for} \quad i \neq j. \tag{11}$$

We denote the i^{th} column of I by e_i, which is thus a column with 1 as its i^{th} component and 0 as its other components. The constraints $y \geqslant 0$ may be expressed in scalar form as $e_i^T y \geqslant 0$; $i = 1, 2, \cdots, n$.

Now as to the composition of the current basis matrix B, during an iteration some components of the current feasible point y will be 0; that is, for some values of i, y will lie on the 'bounding hyperplane' $e_i^T y = 0$, with positive normal e_i. For some of these i (initially, all of them) the corresponding vector e_i will appear as a column of B. If for any i, we have e_i as a column of B and also $e_i^T y = 0$ we shall (following Beale [1]; see also [6]) call such a column of B a 'restricted' column. On the first iteration, all columns of B are restricted columns and in fact $B = I$. The other columns of B are 'free' columns, and are generated by the algorithm as described below.

Thus, for each iteration there will be a subset R of the set of integers from 1 to n which will specify which columns of B are restricted columns.

The current basis B will change by one column in going from one iteration to

the next. We will denote by b_r the column to be replaced, and by $\underline{b}_r$ the replacing column.

Finally, the criterion for continuing the iterations is based on the values of the current w expressed by:

$$w = B^{-1}z = B^{-1}(d + Qy). \tag{12}$$

One Iteration

At the start of the iteration one has the following computed data:

a. B^{-1} (whose i^{th} row is b^{iT})
b. $y \geqslant 0$ (the current feasible point)
c. z (the gradient of f at y)
d. R (specifying which columns of B are restricted).

The above set of data is obtained by algorithm from the previous iteration's data. In the following, underscored items will refer to the new iteration.

To initiate the calculations, $B = B^{-1} = I$, so that $b^i = b_i = e_i$; $y = 0$; $w = z = d$; and R is the whole set. An iteration consists of selecting a column of B to be replaced; selecting a column subsequently to replace it; and modifying the data.

Selecting b_r Compute: $w = B^{-1}z$.

If the pair of current solutions y and z are not optimal there are two cases:
Case I. Some component of w corresponding to a free column of B is not zero.
Case II. All components of w corresponding to free columns of B are zero, but $w \geqslant 0$ does not hold.

If neither of these cases hold we are finished, as will be shown. If either case holds there will be some value of i, which we label r, which singles out the r^{th} component ω_r of w. In Case I, ω_r is not 0, while b_r is a free column. In Case II, ω_r is negative, while b_r is a restricted column. In either case, b_r is selected as the vector to be replaced. In the event that more than one value of i qualifies for $i = r$, we may select any value. However, for the sake of definiteness we shall select that value of r for which ω_r has the largest absolute value.

Selecting $\underline{b}_r$. We seek a new feasible point of the form:

$$\underline{y} = y - \theta b^r, \tag{13}$$

where θ is so selected as to minimize f on that part of the line $y - \theta b^r$ which remains in the constraint set $y \geqslant 0$. The selection of θ will determine $\underline{b}_r$. Thus we determine θ as follows: Compute:

$$q = Qb^r, \tag{14}$$

Now $q = 0$ or not. First suppose $q \neq 0$, and compute:

$$\theta_0 = \omega_r/q^T b^r. \tag{15}$$

The functional is minimized for this value of θ. In fact one easily verifies that:

$$\begin{aligned} f(y - \theta b^r) &= f(y) - \tfrac{1}{2}q^T b^r[\theta_0^{\,2} - (\theta_0 - \theta)^2] \\ &\qquad (= f(y) - \theta\omega_r \quad \text{when} \quad q = 0). \end{aligned} \tag{16}$$

To retain feasibility, we then compute:

$$t_0 = \text{Min.} \frac{e_i^T y}{\theta_0 e_i^T b^r} \tag{17}$$

where the minimum is taken over those i for which the denominator is positive. If there is no such i we take t_0 as infinite.

We take θ equal to θ_0 if $t_0 > 1$, and θ equal to $t_0\theta_0$ if $t_0 \leq 1$. In the latter case it is possible that θ equals 0.

When $\theta = t_0\theta_0$ there is some value of i, which we label k, such that:

$$\theta = \frac{e_k^T y}{e_k^T b^r}. \tag{18}$$

If more than one value of k is possible, we select any one arbitrarily.

If $q = 0$, then:

$$f(y - \theta b^r) = f(y) - \theta(d^T b^r) = f(y) - \theta\omega_r,$$

so that any non-zero θ having the sign of ω_r will decrease f. Then, from our assumption of finite optimum, for some θ having the sign of ω_r, $y - \theta b^r$ will strike the boundary of the set $y \geqslant 0$. Thus, when $q = 0$ we take θ_0 in (17) as signum ω_r, and $\theta = t_0\theta_0$.

The value of θ then defines four cases and the replacing vector as follows:

Case Ia or Case IIa: $\theta = \theta_0$. Then $\underline{b}_r = Qb^r = q$

Case Ib or Case IIb: $\theta = e_k^T y / e_k^T b^r$. Then $\underline{b}_r = e_k$.

Modifying the data:

a. $\underline{B}^{-1}$, whose rows are given by the formulas:

$$\underline{b}^r = (1/\underline{b}_r^T b^r) b^r; \qquad \underline{b}^i = b^i - (\underline{b}_r^T b^i)\underline{b}^r \quad \text{for} \quad i \neq r. \tag{19}$$

b. $\underline{y} = y - \theta b^r$.

c. $\underline{z} = z - \theta q$.

d. $\underline{R}$ is obtained from R by (i) no change for Cases Ia or IIb; (ii) including r for Case Ib; and (iii) deleting r for Case IIa.

This completes the description of an iteration.

Proof of Convergence

Recall that we are considering the system (7), and that we may refer to the minimization of $f(y)$ as in (8). We note again that the only assumption made is that a solution exists (or equivalently, that the minimum of $f(y)$ for $y \geqslant 0$ is finite). The lines of proof follow quite analogously those given in [6], but for the sake of the differences are repeated here.

Now the minimum is taken on either in the interior of the positive orthant or on its boundary. It is taken on in the interior if and only if there is a point $y > 0$ for which the gradient $d + Qy$ at y vanishes. If in the course of the iterations some such point is arrived at we are finished. That is, if ever the set R associated with an iteration is the empty set, and Case II occurs (i.e., $z = 0$) then we are finished. If the set R is not empty, it specifies the boundary of the

positive orthant we are working with. Now (12) may be written as:

$$z = \sum_{i=1}^{n} \omega_i b_i, \tag{20}$$

expressing the gradient of f at y in terms of the current basis. Suppose that $\omega_i = 0$ for each i corresponding to a free column. Then (20) expresses z only in terms of columns of I, and further, $y^T w = 0$. Hence, by the Kuhn-Tucker conditions, (or in this case by the classical Lagrange multiplier conditions), the current y solves the problem:

$$\text{Min. } f(y) \text{ subject to } e_i^T y = 0, \quad \text{for all } i \text{ in } R; \tag{21}$$

that is, y yields the minimum over the face defined by R. If further we have $w \geqslant 0$ for the current w then, again, the Kuhn-Tucker conditions show that y solves the problem:

$$\text{Min. } f(y) \text{ subject to } e_i^T y \geqslant 0, \quad \text{for all } i \text{ in } R, \tag{22}$$

and hence, a fortiori, solves the problem (8). This will be the case when neither Case I nor Case II occurs. If $w \geqslant 0$ does not hold, then we have Case II. Thus, whenever Case II occurs the current y minimizes f on that face of the boundary specified by R. Thus, in particular, whenever Case II again occurs, and in the interim the functional f has been decreased, f will have been minimized on a different face, represented by a different R, (or, as we shall say, a better R). Now in going from one iteration to the next, f is not increased, so that one only goes to better R's. Since the number of sets R is finite, only a finite number of sequences of events: Case II—decrease in f—Case II is possible.

Convergence of the process will therefore follow when the following facts are demonstrated:

Lemma 1: When Case I occurs, Case II will occur in a finite number of iterations, unless the optimum is reached.

Lemma 2: When Case II occurs, there will be a decrease in f followed by a recurrence of Case II in a finite numbeı of iterations, unless the optimum is reached.

Proof of Lemma 1

When Case Ia occurs, the number s of free columns remains fixed. When Case Ib occurs, the number s of free columns is decreased by one. Therefore, if we show that when Case Ia occurs with s free columns it can continue to occur for at most s consecutive iterations, the recurrence of Case I can only continue to the situation where $s = 0$. But this corresponds to the initial iteration $y = 0$, with $f(0) = 0$. But, unless $d \geqslant 0$, in which case we are finished at the initial iteration, this situation is impossible. Thus we will have shown that consecutive occurrence of Case I must lead to Case II or the optimum.

Now when Case Ia occurs, with b_r as the vector being replaced, and with $\underline{b}_r = Qb^r$ as the replacing vector, we show that the r^{th} component of the new w is zero, and that when Case Ia continues to occur, those components of w which have become 0 in this way remain 0.

For this, consider a basis B with the property that for some i we have $b_i = KQb^i$ for some scalar K. Since for $j \neq i$ we have $0 = b_i{}^T b^j = b^{iT} Q b^j$, if for some $r \neq i$ we have a replacing vector $\underline{b}_r = Qb^r$, as in Case Ia, the new i^{th} row of the inverse remains unchanged: $\underline{b}^i = b^i - (\underline{b}_r{}^T b^i)\underline{b}^r = b^i$.

Now consider z as in (20). The new z is given by:

$$\underline{z} = z - \theta_0 Q b^r = \sum_{i=1}^{n} [\omega_i - \theta_0(b^{iT} Q b^r)] b_i \,. \tag{22}$$

The new coefficient of b_i in this expression is the i^{th} component of $\underline{w}$. The coefficient of b_r is 0 by definition of θ_0, and for each i such that we had (i) $\omega_i = 0$ and (ii) $b_i = KQb^i$ for some constant K we retain (i) $\underline{\omega}_i = 0$ and (ii) $\underline{b}^i = b^i$.

Thus, when Case Ia continues to occur on consecutive iterations, some additional component of w corresponding to a free column will become and remain 0. Hence Case Ia cannot continue to occur for more than s iterations. This proves the lemma.

Proof of Lemma 2

Suppose Case II occurs. Consider Case IIa. The functional is definitely decreased, since by (16) we then have:

$$f(\underline{y}) = f(y) - \tfrac{1}{2}(b^{rT} Q b^r)\theta_0{}^2. \tag{23}$$

Note that since Q is non-negative definite and symmetric, $a^T Q a = 0$ if and only if $Qa = 0$. Now on the next iteration either the optimum is obtained or not. If Case I occurs, Lemma 1 shows that either Case II occurs in a finite number of iterations or else the optimum is obtained. In either case, the sequence Case II—decrease in f—Case II occurs in a finite number of iterations, unless optimality is obtained.

Consider Case IIb. It is possible that in this case there is no decrease in f in proceeding to the next iteration. We show that Case I always occurs on the iteration following this case. With z expressed as in (20) we are supposing that all coefficients of b_i for i corresponding to a free column are 0. Since z does not change when θ is 0, and $\underline{b}_r = e_k$, we have:

$$z = \sum_{i=1}^{n} [\omega_i - (\omega_r / e_k{}^T b^r) e_k{}^T b^i] b_i + (\omega_r / e_k{}^T b^r) e_k \,, \tag{24}$$

expressing z in terms of the new basis, and specifying the new components of $\underline{w}$. Since e_k cannot be expressed in terms only of the restricted columns (which are all columns of I), some component $e_k{}^T b^i$ for which the i^{th} column of B is free is not 0, and the corresponding component of b_i in (24) is not 0. Hence Case I occurs on the next iteration.

Now if Case Ia occurs there is, by (23), a definite decrease in f, whereas if Case Ib occurs it is possible that still $\theta = 0$. But when Case Ib occurs, the number s of free columns is decreased by one. Now the only way to retain no decrease in f is to have a sequence of iterations with Case Ib and Case IIb only occurring, and each time with $\theta = 0$. But since Case Ib always follows Case IIb in this situation, and since s is decreased each time Case Ib occurs, this sequence must

terminate with either the optimum, a definite decrease (Case Ia or IIa), or $s = 0$, which, as noted in the proof of Lemma 1, is impossible.

Hence, in either case, when Case IIb occurs, either the optimum or a decrease in f will follow which will, by Lemma 1, be followed either by the optimum or by Case II in a finite number of iterations. This proves the lemma.

4. An Example

The following example is used by Theil and van de Panne [7]. We shall take advantage of their computation of Q. We first consider the problem (8), with Q and d as the given data, and then return to the original problem.

We are solving the problem (8), where:

$$Q = \begin{bmatrix} 1.043 & -0.128 & -0.516 & 0.187 & -0.586 & -0.051 & -0.426 \\ -0.128 & 0.250 & 0.064 & -0.108 & -0.078 & -0.007 & -0.457 \\ -0.516 & 0.064 & 0.379 & -0.127 & 0.200 & -1.211 & 0.377 \\ 0.187 & -0.108 & -0.127 & 0.256 & -0.208 & 0.333 & -0.846 \\ -0.586 & -0.078 & 0.200 & -0.208 & 0.673 & 0.936 & 1.352 \\ -0.051 & -0.007 & -0.211 & 0.333 & 0.936 & 12.363 & -1.636 \\ -0.426 & -0.457 & 0.377 & -0.846 & 1.352 & -1.636 & 6.056 \end{bmatrix}$$

and:

$$d^T = (4.560 \quad 0.475 \quad -1.229 \quad 1.981 \quad -4.119 \quad -8.508 \quad -8.802)$$

Thus, initially, we have:

a. $B = B^{-1} = I$

b. $y = 0$

c. $z = d$

d. $R = \{1, 2, 3, 4, 5, 6, 7\}$.

Iteration 1

Selecting b_r :

(i)
$$w = B^{-1}z = z = d.$$

Since no columns are free, *Case II* applies. Since $\omega_7 = -8.802$ is the most negative component of w, $b_r = e_7$.

Selecting $\underline{b}_r$:

(ii)
$$q = Qb^r = Qe_7 :$$

$$q^T = (-0.426 \quad -0.457 \quad 0.377 \quad -0.846 \quad 1.352 \quad -1.636 \quad 6.056)$$

(iii)
$$q^Tb^r = q^Te_7 = 6.056,$$

(iv)
$$\theta_0 = \omega_7/q^Tb^r = -1.45343.$$

Since $\theta_0 < 0$, and $y - \theta b^r = -\theta e_7 \geq 0$ for any $\theta < 0$, we take t_0 infinite. Hence *Case IIa* applies and $\theta = \theta_0$ and $\underline{b}_r = q$.

Modifying the data:

a.
$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0.07035 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0.07546 \\ 0 & 0 & 1 & 0 & 0 & 0 & -0.06225 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0.13970 \\ 0 & 0 & 0 & 0 & 1 & 0 & -0.22326 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.27015 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.16513 \end{bmatrix}$$

obtained using formulas (19).

b. $y = 0 - \theta_0 e_7$

$$y^T = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1.45345).$$

c. $z = d - \theta_0 q$

$$z^T = (3.94084 \quad -0.18922 \quad -0.68106 \quad 0.75140 \quad -2.15396 \quad -10.88581 \quad 0)$$

d. $R = \{1, 2, 3, 4, 5, 6\}$.

Iteration 2

Selecting b_r:

(i) $w = B^{-1}z = z$ (since the last component of z equals 0).

Since column 7 is the only free column, and the 7th component of w is 0, *Case II* again applies. Since $\omega_6 = -10.88581$ is the most negative component of w, $b_r = e_6$.

Selecting $\underline{b}_r$:

(ii)
$$q = Qb^6 \neq 0:$$

$$q^T = (-0.16608, \quad -0.13046, \quad -1.10915, \quad 0.10445, \quad 1.30123, \quad 11.92103, \quad 0).$$

(iii)
$$q^T b^6 = q^T e_6 = 11.92103,$$

(iv)
$$\theta_0 = \omega_6/q_6 = -0.91316.$$

Since $\theta_0 < 0$, $y - \theta b^r \geq 0$ again for any $\theta < 0$, and we take t_0 infinite. Hence *Case IIa* applies and $\theta = \theta_0$ with $\underline{b}_r = q$.

Modifying the data:

a.
$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0.01393 & 0.07411 \\ 0 & 1 & 0 & 0 & 0 & 0.01094 & 0.07842 \\ 0 & 0 & 1 & 0 & 0 & 0.09305 & -0.03712 \\ 0 & 0 & 0 & 1 & 0 & -0.00876 & 0.13733 \\ 0 & 0 & 0 & 0 & 1 & -0.10916 & -0.25275 \\ 0 & 0 & 0 & 0 & 0 & 0.08389 & 0.02266 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.16513 \end{bmatrix}$$

b. $y^T = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.91316 \quad 1.70012)$

c. $z^T = (3.78918 \quad -0.30835 \quad -1.69389 \quad 0.84678 \quad -0.96573 \quad 0 \quad 0)$
d. $R = \{1, 2, 3, 4, 5\}$.
Iteration 3

Selecting b_r :

(i) $w = B^{-1}z = z$ (again since the last two components of z are 0. This is due to the recurrence of Case IIa).

Since columns 6 and 7 of B are free columns, and the corresponding components of w are 0; and since w is not yet non-negative, *Case II* again applies. Since $\omega_3 = -1.69389$ is the most negative component of w, $b_r = e_3$.

Selecting $\underline{b}_r$:

(ii) $q^T = (-0.50494 \quad 0.08031 \quad 0.25223 \quad -0.06461 \quad 0.23690 \quad 0 \quad 0)$.

(iii) $$q^Tb^3 = q^Te_3 = 0.25223.$$

(iv) $$\theta_0 = -6.71566. = \omega_3/q^Te_3$$

Using formula (17) we have:

(v) $$t_0 = \frac{e_7{}^Ty}{\theta_0\, e_7{}^Tb^3} = 6.82 > 1,$$

so that *Case IIa* applies; $\theta = \theta_0$; and $\underline{b}_3 = q$.

Modifying the data:

a. $$B^{-1} = \begin{bmatrix} 1 & 0 & 2.00191 & 0 & 0 & 0.20021 & -0.00020 \\ 0 & 1 & -0.31840 & 0 & 0 & -0.01869 & 0.09024 \\ 0 & 0 & 3.96464 & 0 & 0 & 0.36891 & -0.14717 \\ 0 & 0 & 0.25616 & 1 & 0 & 0.01508 & 0.12782 \\ 0 & 0 & -0.93922 & 0 & 1 & -0.19655 & -0.21789 \\ 0 & 0 & 0 & 0 & 0 & 0.08389 & 0.02266 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.16513 \end{bmatrix}$$

b. $y^T = (0 \quad 0 \quad 6.71566 \quad 0 \quad 0 \quad 1.53805 \quad 1.45083)$.
c. $z^T = (0.39817 \quad 0.23098 \quad 0 \quad 0.41288 \quad 0.62521 \quad 0 \quad 0)$.
d. $R = \{1, 2, 4, 5\}$.
Iteration 4

Selecting b_r

(i) $w = B^{-1}z = z \geq 0$; hence the solution has been found in three iterations. The fact that in all of the iterations only Case IIa occurred before the optimum was reached is due to the simplicity of the problem.

The values of the functional $f(y)$ may be obtained by algorithm using formula (16), or the final value only may be computed directly when the optimum has been reached. In the latter case, noting (7) we may write:

(24) $$f(y) = \tfrac{1}{2}y^Td + \tfrac{1}{2}y^T(d + Qy) = \tfrac{1}{2}y^Td + \tfrac{1}{2}y^Tz = \tfrac{1}{2}y^Td.$$

The values of the functional for each iteration, starting with $f(0) = 0$ are 0, -6.397, -11.367, and -17.055.

We next return to the original problem which gave rise to the above example. This is an example of the problem (1) and (2), where

$$Q_0 = \tfrac{1}{2}\begin{bmatrix} 6 & 1 & 8 & 0 \\ 1 & 10 & 1 & 4 \\ 8 & 1 & 17 & 3 \\ 0 & 4 & 3 & 11 \end{bmatrix}, a = \begin{pmatrix} 9 \\ 8 \\ 11 \\ 10 \end{pmatrix}, \quad \text{and}$$

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 5 \end{bmatrix}$$

$$d_0^T = (0 \;\; 0 \;\; 0 \;\; 0 \;\; \tfrac{5}{3} \;\; 2 \;\; 3).$$

One initially computes

$$x_0 = Q_0^{-1}a = \begin{pmatrix} 4.560 \\ 0.475 \\ -1.229 \\ 1.981 \end{pmatrix},$$

and subsequently $d = d_0 - A^T x_0$, which is the d used above, and which shows that x_0, the absolute minimum of the functional does not satisfy the constraints. One then computes $Q = A^T Q_0^{-1} A$, and solves the problem as above. Q_0^{-1} in this case appears as the 4×4 matrix in the upper left of Q.

Finally, it remains to calculate the solution x via formula (5):

$$x = x_0 - (Q_0^{-1}A)y = \begin{pmatrix} 0.400 \\ 0.233 \\ 2 \\ 0.414 \end{pmatrix},$$

with functional value given by

$$-a^T x + \tfrac{1}{2} x^T Q_0 x = -\tfrac{1}{2} a^T x_0 + y^T d = -17.037.$$

5. Discussion

It appears to be well worthwhile to take advantage of the non-singularity of the strictly quadratic part of the functional, when such is the case, to examine first the point x_0 where the absolute minimum of the functional is taken on. This allows the transformation to the problem (8), which could hardly be simpler in form, and the extremely simple algorithm described above.

Note that (i) there is no need to seek out a 'first feasible solution' for the constraints, as required for the approaches suggested by Beale [1] and Wolfe [8], (ii) there is no need to consider any 'degeneracy' cases.

The formulation (8), involving Lagrange multipliers as it does, is a form of dual problem to the original, as described by Dorn [4]. It is in fact Dorn's 'Type

II' dual, after eliminating the set of variables not constrained to be non-negative.

As noted, the algorithm is similar to the efficient one proposed by Beale in 1955. Perhaps the chief difference is that the initial data, Q is retained in its original form throughout the calculations, which are based on a 'current basis matrix', as in the modified simplex method [3] for linear programming.

Perhaps the method here suggested is more efficient than that proposed by Theil and van de Panne, although a general statement to that effect is out of the question. Two possible objections to the Theil-van de Panne approach as compared with ours are that (1) they do not make efficient use of computed data, but compute afresh inverses of submatrices of Q_0, and that (2) for large-order problems, the amount of data they compute and refer to seems to grow somewhat combinatorially.

An acknowledgement is in order. Theil and van de Panne base their technique on some ingeniously derived rules, which were subsequently shown by J. C. G. Boot [2] to be derivable from the Kuhn-Tucker conditions. It was Boot's observation which provided the stimulus for the present method.

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V–12

SEQUENTIAL PRODUCTION PLANNING OVER TIME AT MINIMUM COST*

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Production of a given commodity is to be scheduled over time to meet known future requirements while minimizing total costs. The costs include both storage and production costs as functions of time. The unit production cost is an increasing function of the production rate.

Previous solutions to this problem have involved complicated iterative procedures. A new approach brings out the basic principle involved and leads to a surprisingly simple solution. This coincides with a common-sense technique sometimes used in business.

1. Introduction

Production planning models for determining the optimal production program over time of a single type item have been studied by numerous authors: Modigliani and Hohn [1], Hoffman and Jacobs [2], Dantzig and Johnson [3], Arrow and Karlin [4], Bellman [5], and others. For the case where the unit costs are either fixed or non-decreasing functions of the production rate, where there are storage costs, and where there is no cost for changing the rate of production, several authors have shown that this production model is equivalent to a transportation model. To solve such a model certain authors, Bowman [6] and Manne [7], have suggested use of the simplex method for transportation problems. Bishop [8], on the other hand, (following a similar approach used by Prager for the Caterer Problem) has developed a variant of the iterative simplex method to take advantage of the structure found in this problem.

This paper goes one step further by showing that the special features of the unit cost matrix lead to a simple, direct (non-iterative) solution. Indeed, the *fundamental principle is to satisfy in turn each requirement in due-date order by the cheapest available means.*

2. The Problem

We wish to schedule the production of a given commodity over n successive periods of time to meet known requirements while minimizing total costs. Requirements of R_k units are due at the end of the k-th period ($k = 1, 2, \cdots, n$).

We consider two kinds of costs: a unit production cost that is a nondecreasing function of the rate of production and is also a function of the period, and a unit storage cost that is a function of the period stored.

The key point of departure from previous analyses is that we identify each unit of production with its ultimate destination or period when it is to be used.

* Received February 1957.

Let

i = the period in which the item is produced,
j = the order of production of the item in that period.
k = the period when the item is to be used to satisfy a requirement,
C_{ijk} = the total cost of producing and storing the item.

We assume for fixed (i, k) that

$$C_{ijk} \leqq C_{i,j+1,k} \; ; \tag{1}$$

i.e., the unit production costs are nondecreasing functions of the production rate for each period, the marginal increase being added to the cost of the next item produced. It is clear that the j-th unit must be produced before the $(j + 1)$-st unit.

We assume for fixed (i, j) that

$$C_{ijk} = C_{iji} + A_{i+1} + A_{i+2} + \cdots + A_k \; ; \tag{2}$$

i.e., the unit storage costs depend only on the period stored and are added to the production costs.

The requirement R_k can be considered as R_k unit requirements each due at time k. The entire requirement schedule can be considered as a collection of R unit requirements ordered by their due dates. With this ordering, the optimal procedure is very simple to state:

Theorem. For a set of unit costs C_{ijk} satisfying (1) *and* (2), *and a known set of due dates for R units, the total cost is minimized if each unit requirement is met sequentially in order of its due date by assigning (producing and storing) the cheapest unit cost available at that stage.*

Proof. First note that by (1) this rule automatically satisfies the physical condition of the problem that a C_{ijk} must be chosen before a $C_{i,j+1,k}$.

Let $P_1 \geqq R_1$ items be produced in the first period. By (2) the total costs are unaffected if we assume that the first R_1 of these P_1 units are used to meet the requirement R_1, and that the remainder are used for the second and later periods. It is clear that the first R_1 units required have been assigned to production according to the rule stated in the theorem.

Since all production after the first R_1 units in the first period are for use in the second or later periods, they will all be stored for at least one period. Accordingly, one period of storage costs will now be added to unit production costs for all items that might be produced in the first period after the first R_1 units. Let $P_2 \geqq R_2$ be the number of units produced in the first two periods after the first R_1 units. These P_2 items can be arranged in increasing order of unit costs (including any storage), and the first R_2 units can be assumed to be used in order to satisfy the R_2 requirement.

If all P_2 units have unit costs greater than the cheapest unit that could be produced in either the first or second period (after the first R_1), then it would be cheaper to have the first of the R_2 units produced by the cheapest mode. Thus for an optimal solution the first of the R_2 units must be produced by the cheapest way to produce a unit either from the first period after the first R_1 units or from the second period. Similarly we can reason that for an optimal solution the second unit of R_2 must be produced by the cheapest means available after the

first unit of R_2 was produced in the cheapest way. Thus all R_2 units must be assigned according to our rule.

For all potential units of production not used in the first or second periods, add to production costs the cost of storage up to period 3. Let $P_3 \geqq R_3$ be the number of units produced in the first three periods after the first $R_1 + R_2$ units of requirements are met. Again these units can be arranged by increasing cost and the first R_3 units assigned to meet the next R_3 units of requirements. We now apply the same argument to show that each of the R_3 items in turn must be produced by the cheapest means remaining if the solution is to be optimal. The same inductive argument now applies for any number of periods.

This simple solution has many advantages. It is easy to state: merely satisfy each requirement sequentially in order of its due date as cheaply as possible. (This coincides with the common-sense intuitive approach sometimes used in industry.) It is easy to construct numerically or geometrically if marginal-cost curves are plotted. The production schedule for the next requirement is superimposed on the previous total production schedule.

The optimal production for $R_1, \cdots, R_n$ can be planned without knowing $R_{n+1}, \cdots$. Moreover, the optimal production level for the first period can be carried out knowing only that the subsequent requirements beyond some given period n have sufficiently small upper bounds so that no production for them in the first period is required.

The method can easily be extended to the case where initial inventory and upper bounds on the production rates are present. Also if there is a time lag of q periods from the time a unit starts in production to the time it is completed, this can be taken care of with a proper definition of cost.

It is not surprising that this simple solution will not extend to the case where the unit production costs are decreasing functions of the production rates, nor to the case where the cost of changing production rates from period to period is considered, unless restrictive assumptions are made.

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V-13

MATHEMATICAL PROGRAMMING AND SERVICE SCHEDULING*

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1. Statement of the Problem

Consider the problem of planning a service program over N successive unit time intervals, $i = 1, 2, 3, \cdots, N - 1, N$. We assume that some measure of the service level is available and that the service level requirements in each unit interval

$$r_i, \qquad i = 1, 2, \cdots, N$$

are given. A *feasible* program of service levels

$$x_i, \qquad i = 1, 2, \cdots, N$$

is one for which

$$x_i \geq r_i, \qquad i = 1, 2, \cdots, N \quad (1)$$

where x_i represents the potential service level in each unit interval.

There are many business situations where it might be necessary to keep the potential service level (preparedness) above requirements at least in some of the time intervals. For instance, a fleet of trucks (or cabs) or group of production machines, cannot be easily adjusted month by month and, therefore, during slack periods the potential service level may exceed requirements. In a manufacturing firm, there are certain functions to be performed, such as maintenance or clerical work (often overhead type functions), where again the potential service level is expensive to change. In fact, even in the case of production workers, the expense of hiring, training, firing, or the contractual obligations of guaranteed wage agreements may make it undesirable to change the level of employment during slack periods. The reader will readily find further illustrations of the type of planning problems we are describing here.

During the last few years, there has been a great deal of work in the theory of inventory control, and we find it convenient to bring our problem within the framework of more traditional types of inventory control problems. We consider service as a commodity which cannot be stored. Then we can say that our problem is to determine an optimum production program of a nonstorable (perishable) commodity under the condition that production requirements for this commodity in each time interval are given. In addition to (1), we shall, for convenience, suppose that the production is specified in the intervals $i = 0$ and $i = N + 1$ immediately preceding and following the planning period,

$$x_0 = r_0, \; x_{N+1} = r_{N+1}. \quad (2)$$

This assumption simplifies the presentation but is not essential (see Section 4). Thus, a production program is "feasible" if it satisfies both (1) and (2).

* Received April, 1956.

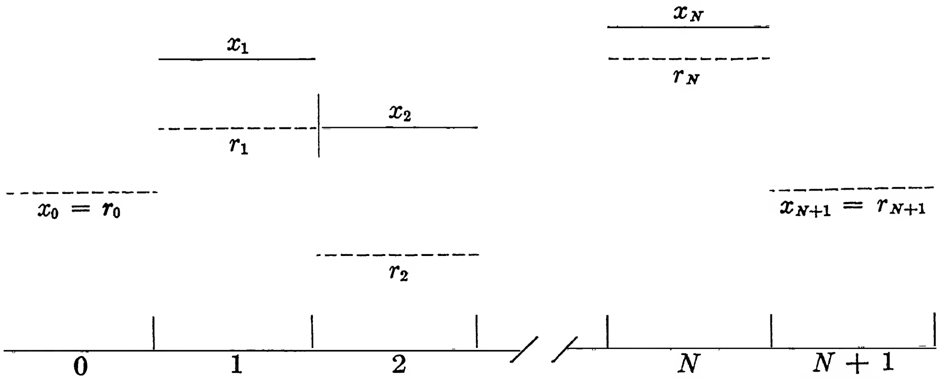
The requirements vector

$$r = (r_0, r_1, \cdots, r_N, r_{N+1})$$

and a feasible production vector

$$x = (x_0, x_1, \cdots, x_N, x_{N+1})$$

may be represented graphically as step functions, in the following manner:



It will be convenient to use geometric terminology appropriate to this type of geometric representation.

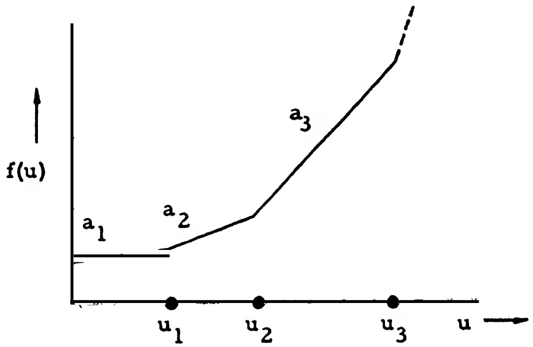
Customarily, three types of cost are considered for each interval i—cost of inventory, cost of production, cost of change of production. In the present context, the first type of cost does not enter. The second type of cost will be taken in the form of an increasing, continuous, convex, polygon function $f^{(i)}$ of the production x_i, the superscript i attached to f expressing that the cost function may be different for different unit intervals i. More explicitly, $f^{(i)}$ is taken to have the form (omitting the index i)

$$f(u) = \begin{cases} a_1 u + f(0) & \text{for } 0 \leq u \leq u_1 \\ a_2 (u - u_1) + f(u_1), & \text{for } u_1 \leq u \leq u_2 \\ a_3 (u - u_2) + f(u_2), & \text{for } u_2 \leq u \leq u_3 \\ \vdots & \vdots \\ a_n (u - u_{n-1}) + f(u_{n-1}), & \text{for } u_{n-1} \leq u \end{cases}$$

where

$$0 < a_1 < a_2 < \cdots < a_n .$$

The graph of f has the following appearance:



(In general, the parameters $f(0)$, u_j, a_j defining $f^{(i)}$ depend upon i and would be denoted by $f^{(i)}(0)$, $u_j^{(i)}$, $a_j^{(i)}$.) The cost of production in the i^{th} interval is then

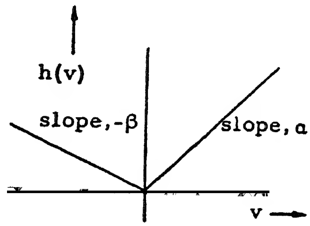
$$f^{(i)}(x_i). \tag{3}$$

The cost of changing the production level from x_{i-1} to x_i, we take proportional to the difference $(x_i - x_{i-1})$, with cost coefficient $\alpha \geq 0$ for increase in production and cost coefficient $\beta \geq 0$ for decrease in production. More fully, let

$$h(v) = \begin{cases} \alpha v, \text{ for } v \geq 0 \\ -\beta v, \text{ for } v \leq 0, \end{cases}$$

where

$$\alpha \geq 0, \beta \geq 0$$



Then the cost of change of production in the i^{th} interval is taken as

$$h(x_i - x_{i-1}) \tag{4}$$

or, alternatively,

$$\alpha \max \{x_i - x_{i-1}, 0\} + \beta \max \{x_{i-1} - x_i, 0\}.$$

Observe that the change of production cost function h is assumed to be independent of i.

The total cost of a feasible production program $x = (x_0, x_1, \cdots, x_N, x_{N+1})$ is then

$$C(x) = \sum_{i=1}^{N} f^{(i)}(x_i) + \sum_{i=1}^{N+1} h(x_i - x_{i-1}) \tag{5}$$

or, alternatively,

$$C(x) = \sum_{i=1}^{N} f^{(i)}(x_i) + \alpha \sum_{i=1}^{N+1} \max \{x_i - x_{i-1}, 0\} + \beta \sum_{i=1}^{N+1} \max \{x_{i-1} - x_i, 0\}$$

The problem is to determine a production program $\bar{x}$ satisfying (1) and (2) which provides $C(x)$ with a minimum among all programs x satisfying (1) and (2). This problem is a generalization of one proposed by Bellman, Glicksberg, and Gross in "The theory of dynamic programming as applied to a smoothing problem", J. Soc. Indust. Appl. Math., Vol. 2, pp. 82–88 (1954). In the present paper we shall describe an effective method for the construction of an optimal program $\bar{x}$.

2. Construction of Optimal Program

Let us understand henceforth that "program" or "vector" means "feasible program" or "feasible vector". By an *allowable deformation* of a vector

$$x = (x_0, x_1, \cdots, x_N, x_{N+1}),$$

we shall mean a transformation of x into some vector x^* in such a way that the cost C is not increased, i.e.,

$$C(x^*) \leq C(x).$$

We shall be dealing with three types of allowable transformations, to be numbered 1^0, 2^0, 3^0. The first and second will be described in this section, the remaining one in the next.

To describe the transformations certain notation is convenient. Given a vector x, and its graph, the symbol

$$I_k$$

will be used to denote any horizontal segment of the graph consisting of k successive intervals x_i (other than x_0 and x_{N+1}) at the same level:

$$x_j = x_{j+1} = \cdots = x_{j+k-1}, \qquad j \neq 0, j + k - 1 \neq N + 1.$$

The range of values of k is $1 \leq k \leq N$. Let

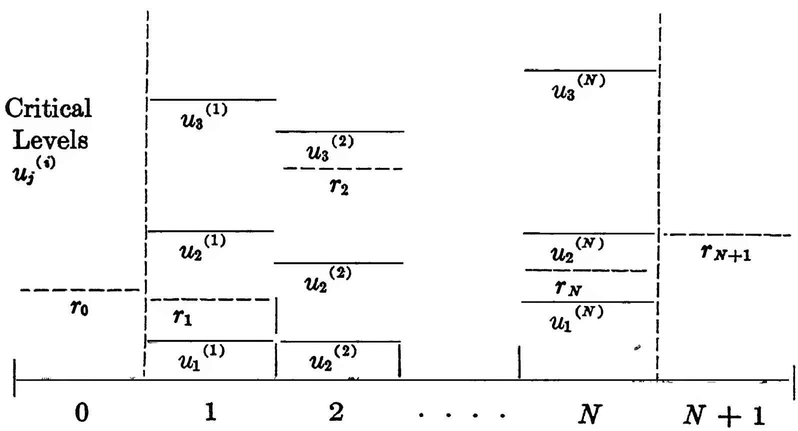
$$x_L \quad \text{and} \quad x_R$$

denote the leftmost and rightmost intervals of I_k, respectively:

$$L = j, \qquad R = j + k - 1.$$

We wish to study the effect on $C(x)$ of raising or lowering a given horizontal segment I_k of x. To do this we introduce the notion of critical level of x (or the graph of x) by which is meant any one of the levels

$$u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \cdots \qquad i = 1, 2, \cdots, N$$



The critical levels for each i, are simply the values u at which the derivative $Df^{(i)}$ of $f^{(i)}$ has a discontinuity. Let

$$D^+f, D^-f$$

denote, respectively, the right-hand derivative and the left-hand derivative of f. Except at a critical level, of course, these two derivatives are equal; between critical values they are constant and equal.

With a given horizontal segment I_k, we associate the quantities

$$^+A_k = \sum_{x_j} D^+f^{(j)}(x_j), \qquad ^-A_k = \sum_{x_j} D^-f^{(j)}(x_j) \tag{6}$$

(summed over the x_j belonging to I_k).

Observe that if a given vector x is deformed by *raising* one of its segments I_k an amount $\Delta u > 0$ (without crossing a critical level above I_k), then the production cost term (i.e., the first term of (5)) *increases* by the amount

$$^+A_k \cdot \Delta u.$$

If it is deformed by *lowering* I_k an amount $\Delta u > 0$, then this term *decreases* by the positive amount

$$^-A_k \cdot \Delta u$$

A final definition—the segment I_k is a *minimum* segment in case the immediate neighbors of I_k lie strictly above I_k, i.e.,

$$x_{L-1} > x_L \text{ and } x_R < x_{R+1}.$$

We are now prepared to formulate the first two allowable deformations.

1^0) If I_k is a minimum segment and

$$^+A_k \leq \alpha + \beta,$$

raise I_k to any level not above the lowest of x_{L-1}, x_{R+1}, and the critical levels above I_k.

2^0) If I_k is any horizontal segment and

$$^-A_k > \alpha + \beta,$$

lower I_k to any level not below the highest of the critical values below I_k.

To verify that 1^0) is an allowable deformation, i.e., decreases $C(x)$, notice that raising I_k by an amount Δu increases the first term of (5) by $^+A_k \cdot \Delta u$, but decreases the second term by $(\alpha + \beta) \cdot \Delta u$ (because I_k is minimum). Under the assumption $^+A_k \leq \alpha + \beta$, the net change is a decrease, as required. In the case of 2^0), lowering I_k by $\Delta u > 0$ decreases the first term by $^-A_k \cdot \Delta u$ and *at most* increases the second term by $(\alpha + \beta) \cdot \Delta u$. Thus, the net change is a decrease. Observe that 2^0 is a *strictly decreasing* transformation; 1^0 is strictly decreasing in case $\alpha + \beta$ is not equal to any sum of N or fewer a_j's (for, then, the inequality in 1^0 is strict).

The program $\bar{r}$. From the particular program

$$r = (r_0, r_1, \cdots, r_N, r_{N+1})$$

we shall construct another program

$$\bar{r} = (r_0, \bar{r}_1, \cdots, \bar{r}_N, r_{N+1})$$

by a succession of transformations of type 1^0. Consider the minimum segments of r ordered, say, from left to right. (If r has no such segments, we take $\bar{r} = r$; this occurs when r has the form

$$r_0 \leq r_1 \leq \cdots \leq r_j, \quad r_j \geq r_{j+1} \geq \geq r_{N+1}, \quad j = 0, 1, 2, \cdots, N + 1.)$$

Step 1: Consider the first minimum segment, say I_s, and the corresponding value ${}^+A_s$. If ${}^+A_s > \alpha + \beta$, step 1 is complete and we pass on to the next minimum segment. If ${}^+A_s \leq \alpha + \beta$ raise I_s to min $\{r_{L-1}, r_{R+1}, h\}$ where h is the lowest critical level above I_s. Having performed this deformation, now consider I_t, the largest I_k containing I_s in its new position (this will be I_s itself unless I_s has been raised to the level of one of its neighbors). If I_t is minimum and ${}^+A_t \leq \alpha + \beta$, raise I_t to the minimum of its neighbors and the critical levels above it. Continue in this way until the original minimum segment I_s has been deformed to an I_k which is either not a minimum segment, or is a minimum segment with ${}^+A_k > \alpha + \beta$. This completes step 1. Step 2 is to deform the second minimum segment of r in the same manner. This process is applied to all the minimum segments of r in turn. The result is a program $\bar{r}$ such that:

If I_k is a minimum segment of $\bar{r}$, then

$${}^+A_k > \alpha + \beta. \tag{7}$$

Theorem. The *vector* $\bar{\mathrm{r}}$ *is an optimal program. Furthermore* $\bar{\mathrm{r}}$ *lies above every optimal program. Also, if* $\alpha + \beta$ *is not the sum of* N *or fewer values* $\mathrm{a}_j^{(i)}$, *then* $\bar{\mathrm{r}}$ *is the unique optimal program.*

This is the main result of this paper. The details of the proof are given in the next section. Notice that the algorithm for constructing $\bar{r}$ is a simple one computationally. (The sense in which one program x lies above another y is that $x_i \geq y_i$, for all $i = 0, 1, 2, \cdots, N + 1$.)

3. Proof of Optimality

The proof is based on two lemmas; in the first lemma we show how to deform any program y into one lying above $\bar{r}$, and in the second lemma we show how to deform any program x lying above $\bar{r}$ into $\bar{r}$ itself.

Lemma 1. Every vector y *may be deformed into a vector* x *lying above* $\bar{\mathrm{r}}$ *by the use of deformation* 1^0 *alone.*

Proof. Consider the original requirements vector r and an arbitrary (feasible) vector y; the vector y lies above r. The process of deforming r into $\bar{r}$ consisted of a series of deformations 1^0 applied to minimum segments of r. Imagine repeating the process of deforming r into $\bar{r}$ with this addition—whenever, in the act of raising a minimum I_t of r, we encounter a segment I_j' of y, we carry this segment up with I_t. Such a segment I_j' is necessarily minimum, since y originally lay above r. Also ${}^+A_j' \leq \alpha + \beta$; for, (i) ${}^+A_t \leq \alpha + \beta$ and (ii) since every term in the sum defining ${}^+A_j'$ occurs in the sum defining ${}^+A_t$, ${}^+A_j' \leq {}^+A_t$. By car-

rying along segments of y into this way as we deform r into $\bar{r}$, we achieve a deformation of y into some vector x lying above $\bar{r}$, using a series of deformations 1^0.

Before proceeding to the next lemma, we formulate an additional allowable deformation.

3^0) Lower any x_i to a level not below the minimum of its neighbors, i.e., not below both x_{i-1} and x_{i+1}.

For, lowering x_i an amount $\Delta u > 0$ strictly decreases $f^{(i)}(x_i)$ in the first term of $C(x)$, and does not increase $h(x_i - x_{i-1}) + h(x_{i+1} - x_i)$ in the second term. Notice that 3^0 is a strictly decreasing deformation.

Lemma 2. Every vector x *which lies above* r *may be transformed into* $\bar{\text{r}}$ *by use of the deformations* 2^0 *and* 3^0 *alone.*

Proof. First we prove:

x can be deformed into a vector z lying above $\bar{r}$ such that z coincides with r at the endvalues $\bar{r}_0 = r_0$, $\bar{r}_{N+1} = r_{N+1}$ and at the minimum segments I_k of $\bar{r}$.

Coincidence at the end values holds by requirement (2) of feasibility. Now, consider any minimum segment I_k of $\bar{r}$, and those intervals x_i of x lying above I_k. Select a lowest such x_i, say x_s. By use of deformation 3^0 every x_i over I_k may be brought to the level x_s, producing thereby a segment I_k' belonging to (the deformed) x. If $I_k' = I_k$, the argument is complete; if I_k' is strictly above I_k, then

$$^{-}A_k' \geq {}^{+}A_k ,$$

which follows from the definition of these quantities. From (7)

$$^{-}A_k' > \alpha + \beta$$

Hence deformation 2^0 is available for lowering I_k' into coincidence with I_k, as required.

The graph of $\bar{r}$ can be decomposed into successive parts

$$r_0 , \bar{r}_1 , \cdots , \bar{r}_{i_1}$$

$$\bar{r}_{i_1} , \bar{r}_{i_1+1} , \cdots , \bar{r}_{i_2}$$

$$\bar{r}_{i_p} , \bar{r}_{i_p+1} , \cdots , r_{N+1}$$

such that each part is monotonic (increasing or decreasing) and has its minimum end value coinciding with r_0, r_{N+1}, or an interval $\bar{r}_i$ belonging to a minimum I_k of $\bar{r}$. Consider the corresponding parts of z, namely,

$$r_0 , z_1 , \cdots , z_{i_1}$$

$$z_{i_1} , z_{i_1+1} , \cdots , z_{i_2}$$

$$z_{i_p} , z_{i_p+1} , \cdots , r_{N+1}$$

These parts are not necessarily monotonic, but by the preceding paragraph the minimum of each such part of z coincides with the minimum end value of the corresponding part of r. To be definite, consider an increasing part of $\bar{r}$, say,

$$\bar{r}_j \leq \bar{r}_{j+1} \leq \cdots \leq \bar{r}_k$$

Then by the preceding paragraph

$$z_j = \bar{r}_j ,$$

and

$$z_j = \min \{z_j , z_{j+1} , \cdots , z_k\} ,$$

the latter being a consequence of z lying above $\bar{r}$. By successive applications of deformation 3^0 we may bring z_{j+1} into coincidence with $\bar{r}_{j+1}$, z_{j+2} , into coincidence with $\bar{r}_{j+2}$, $\cdots$, z_k into coincidence with $\bar{r}_k$. Dealing with the various parts of z in this way we ultimately deform all of z into $\bar{r}$. The lemma is thereby established.

The main theorem is now easily argued. By Lemmas 1 and 2, any feasible x can be allowably deformed into $\bar{r}$, i.e.,

$$C(x) \geq C(\bar{r}).$$

Hence $\bar{r}$ is an optimal program. To show that $\bar{r}$ lies above every optimal program, consider an optimal program y,

$$C(y) = C(\bar{r}),$$

which has at least one component, say y_i , lying strictly above $\bar{r}_i$. By Lemma 1 we may suppose that y has already been deformed to lie above $\bar{r}$. By Lemma 2 we may deform y into $\bar{r}$ by the application of 2^0 and 3^0 only. Since $y_i > \bar{r}_i$, at least one such application is required, and since 2^0 and 3^0 are strictly decreasing (as noted earlier) we would have

$$C(y) < C(\bar{r}).$$

The contradiction establishes the second statement of the theorem. The last statement follows from the fact that, under the given assumption on $\alpha + \beta$, deformation 1^0 is also strictly decreasing—thus any x different from $\bar{r}$ will require at least one deformation 1^0, 2^0, or 3^0 to be brought into coincidence with $\bar{r}$, so that $C(x) > C(\bar{r})$.

4. Modifications of the problem

It was pointed out earlier that the terminal conditions (2) were not essential to the applicability of the method used in this paper. More precisely, one may consider the class of programs

$$x = (x_1 , x_2 , \cdots , x_N)$$

satisfying (1), and search for one such that minimizes the cost function

$$\sum_{i=1}^{N} f^{(i)}(x_i) + \sum_{i=2}^{N} h(x_i - x_{i-1}).$$

This problem requires a slight modification of the foregoing procedure. In addition to segments I_k of x, for which both the leftmost interval x_L is not x_1 and the rightmost interval is not x_N , we distinguish initial segments H_k with x_L coinci-

dent with x_1 and terminal segments J_k with x_R coincident with x_N. Deformations 1^0 and 2^0 apply to segments H_k with $\alpha + \beta$ replaced by α, and to segments J_k with $\alpha + \beta$ replaced by β; deformation 3^0 remains unchanged. (Here, H_k is understood to be a minimum segment in case $x_R < x_{R+1}$, and J_k in case $x_{L-1} > x_L$.) The optimal program $\bar{r}$ is constructed by applying 1^0 to the successive minimum I_k of r, as described in Section 2 and, having performed these deformations, applying 1^0 to the initial minimum segment H_k and terminal minimum segment J_k (if any).

Perhaps the most realistic problem is the one that retains the first of the two conditions (2). Namely—among all programs

$$x = (x_0, x_1, x_2, \cdots, x_{N-1}, x_N)$$

satisfying (1) and

$$x_0 = r_0,$$

find one which minimizes the cost function

$$\sum_{i=1}^{N} f^{(i)}(x_i) + \sum_{i=1}^{N} h(x_i - x_{i-1}).$$

The method of the preceding paragraph solves this problem with the modification that the type of segment H_k is not introduced.

The results of this paper may be generalized in another direction. It was assumed that the production cost functions $f^{(i)}$ were quasi-linear functions, i.e., made up of linear parts. Essentially the same procedure produces an optimal program r in the more general case where the cost functions are arbitrary increasing, convex, continuous functions made up of a finite number of continuously differentiable parts.

AN ANALYTIC SOLUTION OF THE WAREHOUSE PROBLEM*

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1. Introduction

In a recent paper, [1], Bellman uses dynamic programming to establish a computational algorithm for the solution of the "warehouse" problem. [2]. The present paper also employs the dynamic-programming approach and shows that the structure of the solution can be determined analytically, with numerical results easily obtained via recursive formulas.

2. The Warehouse Problem

The problem considered here can be formulated as follows. Given a warehouse of fixed capacity, B, and an initial stock, v, of a certain product, subject to known seasonal fluctuation in selling price and cost, what is the optimal pattern of purchasing, storage, and sales?

3. Mathematical Formulation

Let the process continue for N periods. When i periods remain in the process, let

c_i = cost per unit
p_i = selling price per unit
x_i = amount bought
y_i = amount sold
$f_i(v)$ = the profit obtained over the remaining i periods, where initial stock is v, and an optimal policy is used.

As discussed in Bellman's paper [1], application of the "principle of optimality" of dynamic programming yields the functional equation

$$f_N(v) = \operatorname*{Max}_{x_N, y_N} [p_N y_N - c_N x_N + f_{N-1}(v + x_N - y_N)] \tag{3.1}$$

where the maximum is over the region

$$\begin{aligned} &\text{(a)} \quad && y_N \leqq v \\ &\text{(b)} && v + x_N - y_N \leqq B \\ &\text{(c)} && x_N, y_N \geqq 0 \end{aligned} \tag{3.2}$$

The analytic structure of the solution will be deduced from the above equation.

* Received February 1957.

4. A Transformation

Let u equal the level of inventory attained at the end of the period under investigation. The choice of u will be called the "policy" for that period and we wish to determine the optimal sequence $u_1, \cdots, u_N$.

Clearly $u = v + x - y$ and is less than or equal to B. Let us return to equation (3.1). With the new notation this becomes

$$f_N(v) = \underset{\substack{0 \leqq u \leqq B \\ y_N \leqq v \\ x_N, y_N \geqq 0}}{\text{Max}} [p_N y_N - c_N x_N + f_{N-1}(u)] \tag{4.1}$$

$$= \underset{0 \leqq u \leqq B}{\text{Max}} \left[\underset{\substack{v + x_N - y_N = u \\ x_N \leqq v \\ x_N, y_N \geqq 0}}{\text{Max}} (p_N y_N - c_N x_N) + f_{N-1}(u) \right] \tag{4.2}$$

$$= \underset{0 \leqq u \leqq B}{\text{Max}} \left[\phi_N(u, v) + f_{N-1}(u) \right] \tag{4.3}$$

where

$$\phi_N(u, v) = \underset{\substack{v + x_N - y_N = u \\ y_N \leqq v \\ x_N, y_N > 0}}{\text{Max}} (p_N y_N - c_N x_N) \tag{4.4}$$

In the determination of $\phi_N(u, v)$ we are faced with the maximization of a linear function over the points on a straight line, so that we need only investigate the end points.

When $0 \leqq u \leqq v$, the two points under consideration are $x_N = 0$, $y_N = v - u$ and $x_N = u$, $y_N = v$. In this region

$$\phi_N(u, v) = \text{Max} [p_N(v - u), p_N v - c_N u]. \tag{4.5}$$

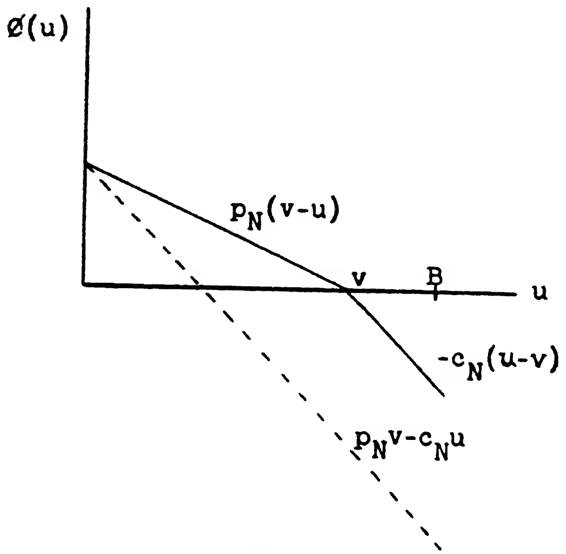

FIG. 1

Case 1: $c_N > p_N$

Arguing similarly, for $v \leqq u \leqq B$

$$\phi_N(u, v) = \text{Max}\ [-c_N(u - v),\ p_N v - c_N u] \tag{4.6}$$

For fixed v, $0 \leqq v \leqq B$, the result can be shown geometrically in two cases:
Case 1:

$$c_N > p_N$$

Case 2:

$$p_N > c_N$$

Having established the nature of the function $\phi_N(u, v)$, we shall proceed to the proof of a theorem.

5. Theorem

The structure of the function $f_N(v)$ defined in equation (4.3) is not immediately obvious. However, the following surprising property holds:

THEOREM 1: *The function $f_N(v)$ is linear in v, the coefficients being functions of $p_1, \cdots, p_N$ and $c_1, \cdots, c_N$,*

$$(1)\quad f_N(v) = K_N(p_1, p_2, \cdots, p_N;\ c_1, c_2, \cdots, c_N) + L_N(p_1, p_2, \cdots, p_N;\ c_1, c_2, \cdots, c_N)v.$$

Furthermore, the optimal policy, u, is independent of v, the initial stock, and depends only upon the selling prices and costs.

6. Proof of Theorem 1

The proof is by induction. Clearly $f_1(v) = p_1 v$ since $f_0(u)$, the zero stage return, is identically zero. Assume that $f_{N-1}(v) = K_{N-1} + L_{N-1}v$ where K_{N-1} and L_{N-1} are determined by the $(N - 1)$ prices and costs, $\{p_i, c_i\}$, $i = 1, 2, \cdots, N - 1$. We shall show that $f_N(v)$ has the same form where the coefficients now

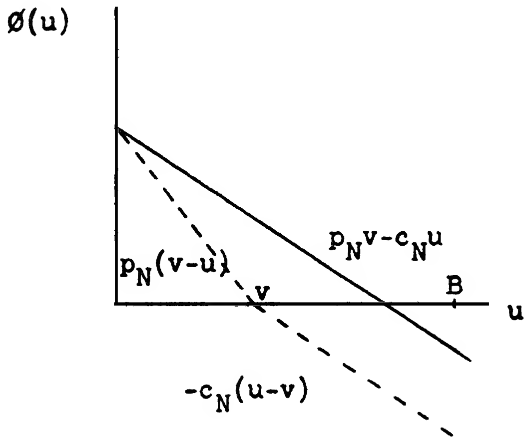

FIG. 2

Case 2: $p_N > c_N$

depend upon the sequence $\{p_i, c_i\}$, $i = 1, 2, \cdots, N$. As in case 1, let c_N be greater than p_N. Due to our hypothesis concerning the linearity of $f_{N-1}(u)$, the maximum must occur at one of three points: $u = 0$, $u = v$, or $u = B$. Thus,

$$f_N(v) = \text{Max}\,[\phi_N(0, v) + f_{N-1}(0), \phi_N(v, v) + f_{N-1}(v), \phi_N(B, v) + f_{N-1}(B)] \quad (6.1)$$

$$= \text{Max}\,[p_N v + K_{N-1}, K_{N-1} + L_{N-1}v, -c_N(B - v) + K_{N-1} + L_{N-1}B] \quad (6.2)$$

Since the third quantity in (6.2) is greater than the first two if and only if $c_N < L_{N-1}$ we have established the condition for a choice of $u = B$. The second quantity is maximum when $p_N < L_{N-1} < c_N$ and the first is largest when $L_{N-1} < p_N$. In all three cases it should be noted that the maximizing u is independent of v. If u is taken equal to B, then

$$f_N(v) = (K_{N-1} + L_{N-1}B - c_N B) + c_N v. \quad (6.3)$$

Hence $K_N = K_{N-1} + L_{N-1}B - c_N B$ and $L_N = c_N$. If $u = v$

$$f_N(v) = K_{N-1} + L_{N-1}v \quad (6.4)$$

whence $K_N = K_{N-1}$ and $L_N = L_{N-1}$. Finally $u = 0$ leads to

$$f_N(v) = K_{N-1} + p_N v. \quad (6.5)$$

Hence $K_N = K_{N-1}$, $L_N = p_N$. In each case $f_N(v)$ is a linear function of v with the new coefficients depending upon $p_1, \cdots, p_N$ and $c_1, \cdots, c_N$.

Case 2, $p_N > c_N$, remains to be considered. Since both $\phi_N(u, v)$ and $f_{N-1}(u)$ are linear we must investigate only two points, $u = 0$ and $u = B$. Here

$$f_N(v) = \text{Max}\,[\phi_N(0, v) + f_{N-1}(0), \phi_N(B, v) + f_{N-1}(B)] \quad (6.6)$$

$$= \text{Max}\,[p_N v + K_{N-1}, p_N v - c_N B + K_{N-1} + L_{N-1}B]. \quad (6.7)$$

We have a maximum at $u = 0$ if $L_{N-1} < c_N$ and at $u = B$ if the reverse inequality is true. In these cases

$$f_N(v) = K_{N-1} + p_N v, \quad (6.8)$$

if $u = 0$, with $K_N = K_{N-1}$, $L_N = p_N$. On the other hand,

$$f_N(v) = (K_{N-1} + L_{N-1}B - c_N B) + p_N v \quad (6.9)$$

if $u = B$, with $K_N = K_{N-1} + L_{N-1}B - c_N B$, $L_N = p_N$. This completes the proof.

7. Discussion

Let us now consider the economic interpretation of the problem, and investigate our mathematical results. Apparently, if $c_N > p_N$, we have three alternatives, dependent upon other parameters. Equations (6.3)–(6.5) have the following significance: (6.3) represents a purchase of enough goods to fill the warehouse and the current cost of this decision, $\phi_N(B, v)$, is $c_N(B - v)$. Equation (6.4) corresponds to doing nothing, with associated current cost of zero. Finally, (6.5) arises from selling all v items with which the period was entered and derives an immediate return of $p_N v$. Turning to equation (6.8), where $p_N > c_N$,

we have a slightly different interpretation. Here a policy dictating a final level of B means the sale of v and purchase of B, with associated cost $p_N v - c_N B$. A choice of $u = 0$, as above, means the sale of the entire stock, v, and returns $p_N v$. In all, we then have four distinct policies: sell, buy, sell and buy, and do nothing and in each case the policy is pursued up to the constraint of warehouse capacity or stock on hand.

8. A Numerical Example

Recalling the definition of p_i and c_i to be costs when i periods remain, let us consider the following 10 period process.

$$c_{10} = 8, \quad c_9 = 8, \quad c_8 = 2, \quad c_7 = 3, \quad c_6 = 4,$$
$$c_5 = 3, \quad c_4 = 3, \quad c_3 = 2, \quad c_2 = 5, \quad c_1 = 3$$
$$p_{10} = 3, \quad p_9 = 6, \quad p_8 = 7, \quad p_7 = 1, \quad p_6 = 4,$$
$$p_5 = 5, \quad p_4 = 5, \quad p_3 = 1, \quad p_2 = 3, \quad p_1 = 2.$$

We desire K_{10}, L_{10}, and u_{10}, the return coefficients and policy when ten periods remain, i.e. at the beginning of the process. Since $f_1(v) = p_1 v$, we conclude that

$K_1 = 0$, $L_1 = p_1$, $u_1 = 0$ so

$$K_1 = 0 \qquad L_1 = 2 \qquad u_1 = 0 \tag{8.1}$$

We note now that $c_2 > p_2$ so we refer to equations (6.3)–(6.5). Since $L_1 < p_2$, equation (6.5) is applicable and

$$K_2 = 0 \qquad L_2 = 3 \qquad u_2 = 0. \tag{8.2}$$

For the third from last period, $c_3 > p_3$ and $L_2 > c_3$ yields, from (6.3)

$$K_3 = B \qquad L_3 = 2 \qquad u_3 = B \tag{8.3}$$

Continuing this process,

$$\begin{aligned}
K_4 &= B & L_4 &= 5 & u_4 &= 0 \\
K_5 &= 3B & L_5 &= 5 & u_5 &= B \\
K_6 &= 4B & L_6 &= 4 & u_6 &= B \\
K_7 &= 5B & L_7 &= 3 & u_7 &= B \\
K_8 &= 6B & L_8 &= 7 & u_8 &= B \\
K_9 &= 6B & L_9 &= 7 & u_9 &= v \\
K_{10} &= 6B & L_{10} &= 7 & u_{10} &= v.
\end{aligned} \tag{8.4}$$

Our conclusions are, for this numerical example, that an optimal policy leads to a profit of $6B + 7v$, where v is the stock at the beginning of the 10-stage process and B the warehouse capacity and that the optimal policy requires no action during the first two periods, sell v and buy B during period 3, remain full

during the fourth and fifth periods, sell B and buy B during period 6, sell B during period 7, buy B during the 8th period, sell out during period 9.

9. Conclusions

We have established the following results:

1. The optimal N-stage return is a linear function of initial stock with coefficients dependent upon the costs and selling prices.
2. The optimal policy at any stage is *independent* of initial stock at that stage.
3. The optimal policy will always have the following structure: Do nothing for the first k stages (k may equal 0), and oscillate between a full and empty warehouse condition for the remainder of the process.
4. The policy and return can be calculated trivially using simple recurrence relations for the coefficients of the linear return function.

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V–15

DERIVATION OF A LINEAR DECISION RULE FOR PRODUCTION AND EMPLOYMENT

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An application of linear decision rules to production and employment scheduling was described in the last issue of this journal [2]. The hypothetical performance of these rules represented a significant improvement over the actual company performance as measured by independent cost estimates and other managerial measures of efficiency. The quadratic cost function which was used should be applicable to production and employment scheduling decisions in many other situations. Also the general approach of approximating decision criteria with quadratic functions and obtaining linear decision rules can usefully be extended to many decision problems.

In the present paper we will demonstrate a) how optimal (i.e., minimum expected cost) decision rules may be derived for a quadratic cost function involving inventory, overtime, and employment costs, and b) how the numerical coefficients of the rules may be computed for any set of cost parameters.

1. The Decision Problem

The costs to be minimized are represented by the following function of work force, W_t; aggregate production, P_t; net inventory, I_t; and ordered shipments, O_t (where the subscript, t, designates the time period):

$$C_N = \sum_{t=1}^{N} [(C_1 - C_6)W_t + C_2(W_t - W_{t-1} - C_{11})^2 + C_3(P_t - C_4 W_t)^2 + C_5 P_t + C_{12} P_t W_t + C_7(I_t - C_8 - C_9 O_t)^2 + C_{13}] \tag{1.1}$$

where, by definition, the excess of production over orders affects net inventory as follows:

$$P_t - O_t = I_t - I_{t-1}{}^{1} \qquad t = 1, 2, \cdots, N \tag{1.2}$$

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[1] We have not found it necessary to place bounds on the variables, such as non-negativity restraints on production, because for the type of problem with which we have been dealing the unconstrained solution can be expected to satisfy such constraints with but rare exceptions. Our general approach is to view certain actions, for example negative production and overcapacity operations, as being undesirable *because* they are expensive. In minimizing costs, these actions are automatically avoided so there is little or no need to place bounds on the solutions.

The existence of such a solution requires the satisfaction of the second order condition that the cost function be a positive definite quadratic form. We believe that this condition

The cost function above is somewhat more general than that presented in Equation 7 of [2], but the only important change is the recognition of a possible additional interaction between the size of the work force and the production level, i.e., $C_{12}P_tW_t$. The term $C_7(I_t - C_8 - C_9O_t)^2$ represents inventory carrying and run-out costs, the minimum of which varies with the rate of incoming orders. The terms

$$(C_1 - C_6)W_t + C_3(P_t - C_4W_t)^2 + C_5P_t + C_{12}P_tW_t$$

approximate the costs of regular and overtime hours for specified levels of the work force and rates of production.[2] Costs associated with hiring and firing, i.e., changes in the work force, are represented by the term $C_2(W_t - W_{t-1} - C_{11})^2$.[3] The constant cost term, C_{13}, is not changed by the scheduling decisions, and hence is irrelevant in their making.

The problem we then face is the following: To choose a decision rule (strategy) for making production and labor force decisions in successive time periods that will minimize the expected value of total costs over a large number of periods. Since costs are influenced by the interaction between current actions and future orders, forecasts of the future are indispensable even though such forecasts are subject to errors. The passage of time makes new information available which allows improvements in the accuracy of the forecasts. The design of an optimal decision rule should take these considerations into account.

In general, however, future orders are uncertain; that is to say, information about orders in each future period may be cast in the form of a probability distribution. H. A. Simon [6] has proved that the optimal solution for this uncertainty case can be obtained directly from the solution of the certainty case.[4] For this purpose we simply replace each period's probability distribution of orders with its mathematical expectation (the average of orders weighted by the probability distribution) and then proceed as though these expected values were certain. This procedure will yield a decision that is optimal for the first period. When new information is available at the end of the period, the forecasts should be revised and the process repeated. "Certainty equivalence" is extremely important, because it enables us to obtain a simple and tractable solution for

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normally will be met by the cost structures encountered in practice since in general costs rise when any decision variable (or a combination of them) is pushed to extreme values. It can be shown that an interior cost minimum exists if C_2, C_3, C_4 and C_7 are positive and $0 \le C_{12} \le 4C_3C_4$. These conditions are sufficient, but they are stronger than necessary.

[2] C_5 turns out to be irrelevant in making the scheduling decisions, because shifting production from one period to another will leave this component of cost unchanged. The costs of run-outs and of holding inventory prevent the cumulative of production from deviating for any length of time from the cumulative of orders received.

[3] The constant C_{11} is introduced to accommodate asymmetry in the costs of hiring and laying off; however, it proves to be irrelevant in obtaining optimal decisions. If this cost expression is expanded, we find that only the term, $-2C_{11}(W_t - W_{t-1})$, represents an interplay between the cost coefficient, C_{11}, and a decision variable. When costs are summed over many time periods, the expression $\Sigma_tW_t - \Sigma_tW_{t-1}$ is (virtually) zero.

[4] Similar conclusions have been obtained by H. Theil [7] for the case in which forecasts are not revised over time.

the general uncertainty decision problem. It should be noted, however, that the proof of certainty equivalence depends critically on the decision criterion function being a quadratic form. This is one reason for interest in quadratic criteria.

Because of the certainty equivalence property we can now re-state our problem as the following simpler one: To minimize (1.1), subject to the relations (1.2), with respect to the decision variables $(W_1, W_2, \cdots)$ and $(P_1, P_2, \cdots)$ for any given initial conditions (inventory and work force) and an arbitrary "known" pattern of future orders.

The reader who is not interested in the derivation of the decision rules might proceed to Section 4, which presents a self-contained step by step computational procedure.

2. Derivation of the Conditions for Minimum Cost

The first order conditions for minimum cost where future orders are given may be obtained by equating to zero the partial derivatives of cost, C_N, with respect to each independent decision variable. In stating these first order conditions it is convenient to make use of the following notation, which is illustrated with the variable, W, for expressing the differences between the magnitude of a variable in successive time periods:

$$\begin{aligned} \Delta W_t &= W_{t+1} - W_t \\ \Delta^2 W_t &= \Delta W_{t+1} - \Delta W_t = W_{t+2} - 2W_{t+1} + W_t \\ \Delta^3 W_t &= W_{t+3} - 3W_{t+2} + 3W_{t+1} - W_t \\ \Delta^4 W_t &= W_{t+4} - 4W_{t+3} + 6W_{t+2} - 4W_{t+1} + W_t \end{aligned} \tag{2.1}$$

Differentiating C_N, (1.1), with respect to $W_r (r = 1, 2, \cdots, N - 1)$, and noting that

$$\frac{\partial W_{t-1}}{\partial W_r} = \begin{cases} 1 \text{ if } t = r + 1 \\ 0 \text{ otherwise} \end{cases} \tag{2.2}$$

and

$$\frac{\partial W_t}{\partial W_r} = \begin{cases} 1 \text{ if } t = r \\ 0 \text{ otherwise,} \end{cases} \tag{2.3}$$

we obtain:

$$\begin{aligned} \frac{\partial C_N}{\partial W_r} = C_1 - C_6 + 2C_2(\Delta W_{r-1} - C_{11}) - 2C_2(\Delta W_r - C_{11}) \\ - 2C_3C_4(P_r - C_4 W_r) + C_{12} P_r = 0 \\ r = 1, 2, \cdots, N - 1 \end{aligned} \tag{2.4}$$

Solving (2.4) for P_r we obtain:

$$\begin{aligned} P_r &= \frac{C_{10}}{C_{14}} - C_{15}\Delta^2 W_{r-1} + C_{16} W_r \\ &= \frac{C_{10}}{C_{14}} - C_{15} W_{r+1} + C_{23} W_r - C_{15} W_{r-1} \\ & \qquad r = 1, 2, \cdots, N - 1 \end{aligned} \tag{2.5}$$

where we have defined

$$C_{10} \equiv C_1 - C_6$$

$$C_{14} \equiv 2C_3C_4 - C_{12}$$

$$C_{15} \equiv \frac{2C_2}{C_{14}}$$

$$C_{16} \equiv \frac{2C_3C_4^2}{C_{14}}$$

$$C_{23} \equiv C_{16} + 2C_{15}.$$

Thus we find that the production rate of each period is a linear function of the size of the work force in the same and adjacent periods. If we knew the work force decisions, we could readily determine the production decisions.

Since the inventory holding and runout costs depend on the inventory level which, in turn, depends on the cumulative production of *all* previous periods, if we take the partial derivatives of total cost, C_N, with respect to production rates as the second set of decision variables, we obtain a very complicated expression. This may be avoided by considering inventory as the second decision variable instead of production. The production rate for each period would then be uniquely determined through (1.2). Therefore we differentiate the cost function with respect to the inventory in each period and equate to zero in order to compete the first order conditions for minimum cost. Using the production-inventory relation (1.2) we note that:

$$\frac{\partial P_t}{\partial I_r} = \frac{\partial}{\partial I_r}(O_t + I_t - I_{t-1}) = \frac{\partial I_t}{\partial I_r} - \frac{\partial I_{t-1}}{\partial I_r} \tag{2.6}$$
$$= \begin{cases} 1 \text{ if } t = r \\ -1 \text{ if } t = r+1 \\ 0 \text{ otherwise} \end{cases}$$

Hence differentiating C_N, (1.1), with respect to $I_r (r = 1, 2, \cdots, N-1)$ and setting the derivatives equal to zero we obtain

$$\begin{aligned} \frac{\partial C_N}{\partial I_r} &= 2C_3(P_r - C_4W_r) - 2C_3(P_{r+1} - C_4W_{r+1}) + C_5 - C_5 + C_{12}W_r \\ &\quad - C_{12}W_{r+1} + 2C_7(I_r - C_8 - C_9O_r) = 0 \end{aligned} \tag{2.7}$$
$$r = 1, 2, \cdots, N-1$$

Solving for inventory we obtain

$$I_r = \frac{C_3}{C_7}\Delta P_r - \frac{C_{14}}{2C_7}\Delta W_r + C_8 + C_9O_r \qquad r = 1, 2, \cdots, N-1 \tag{2.8}$$

We now use this equation to substitute for $I_r (r = 1, 2, \cdots, N-1)$ in the equations (1.2) and thus eliminate the inventory variable. By this substitution

we obtain equations in the unknowns, production and employment, as shown in (2.9) below. It will be noted that the first period ($r = 1$) must be treated differently from the others for I_0, the initial inventory, is not an unknown decision variable, but a known initial condition.

$$
\begin{aligned}
&P_1 - O_1 = I_1 - I_0 = \frac{C_3}{C_7}\Delta P_1 - \frac{C_{14}}{2C_7}\Delta W_1 + C_8 + C_9 O_1 - I_0 \\
(2.9)\qquad &P_r - O_r = \Delta I_{r-1} \quad = \frac{C_3}{C_7}\Delta^2 P_{r-1} - \frac{C_{14}}{2C_7}\Delta^2 W_{r-1} + C_9 \Delta O_{r-1} \\
&\qquad\qquad\qquad\qquad\qquad r = 2, 3, \cdots, N-1
\end{aligned}
$$

Now using the relation between production and size of work force that has been derived in (2.5), we can eliminate production from the above equations, obtaining:

$$
\begin{aligned}
&\frac{C_{10}}{C_{14}} - C_{15}\Delta^2 W_0 + C_{16} W_1 - O_1 \\
&\qquad = -C_{17}\Delta^3 W_0 + C_{18}\Delta W_1 + C_8 + C_9 O_1 - I_0 \\
(2.10)\qquad &\frac{C_{10}}{C_{14}} - C_{15}\Delta^2 W_{r-1} + C_{16} W_r - O_r \\
&\qquad = -C_{17}\Delta^4 W_{r-2} + C_{18}\Delta^2 W_{r-1} + C_9 \Delta O_{r-1} \\
&\qquad\qquad\qquad\qquad\qquad r = 2, 3, \cdots, N-1
\end{aligned}
$$

where we define

$$
C_{17} \equiv \frac{C_3 C_{15}}{C_7}
$$

$$
C_{18} \equiv \frac{C_3 C_{16}}{C_7} - \frac{C_{14}}{2C_7}.
$$

Equations (2.10) constitute a set of simultaneous linear relations in the unknown employment levels for the various periods. Expanding the differences by using (2.1) and collecting the unknowns on the left, we can rewrite this system of equations as follows:

$$
\begin{aligned}
&C_{19} W_1 - C_{20} W_2 + C_{17} W_3 \\
&\qquad = (1 + C_9)O_1 + (C_{15} + C_{17})W_0 + C_8 - \frac{C_{10}}{C_{14}} - I_0 \\
&-C_{21} W_1 + C_{22} W_2 - C_{21} W_3 + C_{17} W_4 \\
(2.11)\qquad &\qquad = -C_9 O_1 + (1 + C_9)O_2 - C_{17} W_0 - \frac{C_{10}}{C_{14}} \\
&C_{17} W_{r-2} - C_{21} W_{r-1} + C_{22} W_r - C_{21} W_{r+1} + C_{17} W_{r+2} \\
&\qquad = -C_9 O_{r-1} + (1 + C_9)O_r \qquad - \frac{C_{10}}{C_{14}} \\
&\qquad\qquad\qquad\qquad\qquad r = 3, 4, \cdots, N-1
\end{aligned}
$$

where we define

$$
\begin{aligned}
C_{19} &\equiv C_{16} + C_{18} + 2C_{15} + 3C_{17} \\
C_{20} &\equiv C_{15} + 3C_{17} + C_{18} \\
C_{21} &\equiv C_{15} + 4C_{17} + C_{18} \\
C_{22} &\equiv C_{16} + 2C_{18} + 2C_{15} + 6C_{17}
\end{aligned}
$$

This system has two more unknown variables, $N + 1$, than equations, $N - 1$; this deficiency could be remedied by supplying terminal conditions and writing two more equations. Rather than do this, however, we let N approach infinity so that the terminal conditions have a negligible influence on the employment (and production) of the first few periods.

The structure of this infinite set of linear simultaneous equations is most easily seen when written in the matrix form:

(2.12)

$$
\begin{bmatrix}
C_{19} & -C_{20} & C_{17} & & & & & & & \\
-C_{21} & C_{22} & -C_{21} & C_{17} & & & & \bigcirc & & \\
C_{17} & -C_{21} & C_{22} & -C_{21} & C_{17} & & & & & \\
 & C_{17} & -C_{21} & C_{22} & -C_{21} & C_{17} & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & \bigcirc & & & C_{17} & -C_{21} & C_{22} & -C_{21} & C_{17} & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
$$

$$
\times
\begin{bmatrix}
W_1 \\ W_2 \\ W_3 \\ W_4 \\ \cdot \\ \cdot \\ W_r \\ \cdot \\ \cdot
\end{bmatrix}
=
\begin{bmatrix}
(1 + C_9)O_1 + (C_{15} + C_{17})W_0 + C_8 - \dfrac{C_{10}}{C_{14}} - I_0 \\
-C_9 O_1 + (1 + C_9)O_2 - C_{17} W_0 - \dfrac{C_{10}}{C_{14}} \\
-C_9 O_2 + (1 + C_9)O_3 - \dfrac{C_{10}}{C_{14}} \\
-C_9 O_3 + (1 + C_9)O_4 - \dfrac{C_{10}}{C_{14}} \\
\cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \\
-C_9 O_{r-1} + (1 + C_9)O_r - \dfrac{C_{10}}{C_{14}} \\
\cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot
\end{bmatrix}
$$

This system of equations may be solved for the unknown W's in which we are interested.

We can now summarize the results of this section. For the quadratic cost

function in the decision variables, W_r and $P_r (r = 1, 2, \cdots)$ and "known" future orders, we can obtain from the first order conditions for minimum costs a solution in which a) the work force decisions are functions of future orders and initial conditions, i.e., the solution of equations (2.11), and b) the production level decisions are functions of the work force decisions using equations (2.5).

Because the original cost function is quadratic, when we differentiate to obtain the first order conditions for minimum cost, the functions which are obtained are linear. The relative ease with which such linear equation systems may be solved constitutes an important reason for interest in quadratic decision criteria.

In the next section we consider means of obtaining for the first period a solution of the above conditions for minimum cost.

3. Solution of the Recurrence Relations

A number of techniques are available for the solution of (2.11), even though it is an infinite system in the unknowns W_1, W_2, et cetera. We shall employ here the one that appears to be the most simple and direct.[5] A solution for all the P's and W's is not required since actions will be taken on only the first few steps of the plan. We are primarily interested in solving the set of equations for the immediate actions, P_1 and W_1. Expressions for determining their values will then constitute the desired decision rules.

From the system (2.11) we may obtain a single equation by multiplying each equation by the expression λ^{r-1}, where λ is a certain variable (which may take on complex values) and r is the number of the equation. Thus the first equation is multiplied by unity (λ^0), the second is multiplied by λ, the third by λ^2, and so on. Adding the resulting system of equations, we obtain:

$$\begin{aligned}(C_{19} W_1 - C_{20} W_2 + C_{17} W_3) &+ \lambda(-C_{21} W_1 + C_{22} W_2 - C_{21} W_3 + C_{17} W_4) \\ &+ \sum_{r=3}^{\infty} \lambda^{r-1}(C_{17} W_{r-2} - C_{21} W_{r-1} + C_{22} W_r - C_{21} W_{r+1} + C_{17} W_{r+2}) \\ &= (1 + C_9)O_1 + \sum_{r=2}^{\infty} \lambda^{r-1}[-C_9 O_{r-1} + (1 + C_9)O_r] + C_8 - I_0 \\ &\quad + (C_{15} + C_{17})W_0 - \lambda C_{17} W_0 - \frac{C_{10}}{C_{14}} \sum_{r=1}^{\infty} \lambda^{r-1} \qquad (3.1)\end{aligned}$$

By rearranging terms and noting that,

$$\sum_{r=1}^{\infty} \lambda^{r-1} = \frac{1}{1 - \lambda}$$

we have:

[5] Another method, which involves the inversion of the matrix of coefficients in equation (2.12), was devised by Modigliani [3]. His method allows one to find all the entries of the inverse matrix, and is directly applicable if the coefficients are symmetric about the main diagonal, except for the "corners." The method employed here was developed in a somewhat more general form by Muth [4].

$$
\begin{aligned}
&(C_{17}\lambda^{-2} - C_{21}\lambda^{-1} + C_{22} - C_{21}\lambda + C_{17}\lambda^{2})\left(\sum_{r=1}^{\infty}\lambda^{r-1}W_r\right) \\
(3.2)\qquad &+ [(C_{19} - C_{22}) + C_{21}\lambda^{-1} - C_{17}\lambda^{-2}]W_1 - [(C_{20} - C_{21}) + C_{17}\lambda^{-1}]W_2 \\
&= \sum_{r=1}^{\infty}\lambda^{r-1}[1 + C_9(1-\lambda)]O_r + [C_{15} + C_{17}(1-\lambda)]W_0 - I_0 + C_8 - \frac{C_{10}}{C_{14}(1 -\ }
\end{aligned}
$$

This equation holds for *all* values of λ for which its components conver
If $W_1, W_2, \cdots$ are all bounded, it is sufficient for convergence that λ lie insi
the unit circle of the complex plane, excluding the origin. That is,

$$(3.3)\qquad 0 < |\lambda| < 1.$$

In particular, we can choose values of λ satisfying (3.3) such that the fi term of (3.2) vanishes. Since the series, $\sum_{r=1}^{\infty}\lambda^{r-1}W_r$, converges, the first ter will vanish if the polynomial coefficient is equal to zero, i.e. if

$$(3.4)\qquad C_{17}\lambda^{-2} - C_{21}\lambda^{-1} + C_{22} - C_{21}\lambda + C_{17}\lambda^{2} = 0.$$

If λ_1 satisfies (3.4), as the result of symmetry $1/\lambda_1$ does also. Hence, if w can find any solution, not zero or unity, we can find a solution that satisfi (3.3). Using this fact, we will show later that there are two and only two valu of λ (say, λ_1 and λ_2) which satisfy the restrictions (3.3) as well as the auxiliar equation (3.4).

Inasmuch as λ_1 and λ_2 are roots of the auxiliary equation, we have

$$(3.5)\qquad C_{19} - C_{22} + C_{21}\lambda_i^{-1} - C_{17}\lambda_i^{-2} = C_{19} - C_{21}\lambda_i + C_{17}\lambda_i^{2}, \qquad i = 1, 2$$

Substituting each of these roots into (3.2), and using the relation (3.5), we obtai the following two equations in the two unknowns W_1 and W_2:

$$
\begin{aligned}
&(C_{19} - C_{21}\lambda_i + C_{17}\lambda_i^{2})W_1 - [(C_{20} - C_{21}) + C_{17}\lambda_i^{-1}]W_2 \\
(3.6)\qquad &= [1 + C_9(1-\lambda_i)]\left(\sum_{r=1}^{\infty}\lambda_i^{r-1}O_r\right) + [C_{15} + C_{17}(1-\lambda_i)]W_0 - I_0 \\
&\qquad + C_8 - \frac{C_{10}}{C_{14}}\frac{1}{1-\lambda_i} \qquad i = 1, 2
\end{aligned}
$$

Using any of the numerous methods available for solving such small systems of linear equations, we can then obtain the decision rules for W_1 and W_2. One method is illustrated in Section 4.

Having obtained W_1 and W_2 from (3.6) we can use (2.5) to determine the optimal rate of production, P_1. Planned levels of the labor force and rates of production for periods further into the future (i.e., $W_3, \cdots,$ and $P_2, \cdots$) can probably be calculated most efficiently by successive application of the above decision rules for W_1 and P_1 together with the inventory-production relationships (1.2).[6]

[6] Another possibility is that of solving equations (2.11) recursively once W_1 and W_2 are known. Although the procedure requires only a small number of arithmetical operations,

The Roots of the Auxiliary Equation. We will examine next the problem of finding the roots to the auxiliary equation (3.4) that also satisfy the conditions (3.3). Because of the symmetry of the coefficients of this equation, the problem of finding the roots may be broken down into that of determining the roots of two quadratic equations. We first make a change in variables; let:

$$s = \lambda - 2 + 1/\lambda = (1 - \lambda)^2/\lambda. \tag{3.7}$$

Then (3.4) may be reduced to

$$C_{17}s^2 - (C_{15} + C_{18})s + C_{16} = 0 \tag{3.8}$$

since from (2.11), $C_{22} \equiv C_{16} + 2(C_{15} + C_{18}) + 6C_{17}$ and $C_{21} \equiv (C_{15} + C_{18}) + 4C_{17}$.

The roots of (3.8) are

$$s_j = \frac{1}{2C_{17}}\left[(C_{15} + C_{18}) \pm \sqrt{(C_{15} + C_{18})^2 - 4C_{16}C_{17}}\right] \tag{3.9}$$

for $j = 1$ and 2, respectively.

Secondly, we have the quadratic equations for λ from (3.7):

$$\lambda^2 - (2 + s_j)\lambda + 1 = 0 \qquad j = 1, 2 \tag{3.10}$$

whose roots are

$$\lambda_i = \begin{cases} \frac{1}{2}[(2 + s_j) - \sqrt{s_j(4 + s_j)}], & i = j = 1, 2 \qquad (3.11a) \\ \frac{1}{2}[(2 + s_j) + \sqrt{s_j(4 + s_j)}], & i = 2 + j = 3, 4 \qquad (3.11b) \end{cases}$$

If the roots s_j are complex, we can write the radical $\sqrt{s_j(4 + s_j)}$ directly in a form that involves real coefficients. Let x and y be the real and imaginary parts, respectively, of $s_j(4 + s_j)$ and let $r = \sqrt{x^2 + y^2}$. It is well-known (see, for example, [1]) that

$$\sqrt{s_j(4 + s_j)} = \frac{1}{\sqrt{2}}[\sqrt{r + x} \pm i\sqrt{r - x}]. \tag{3.12}$$

We will now list two important properties of the roots to the auxiliary equation (3.4). First, the four roots are either all real or all complex.[7] Second, exactly two of these roots (λ_1 and λ_2) have moduli less than one, while the moduli of the other two (λ_3 and λ_4) exceed one since the parameters, C_{16}, C_{17} and $(C_{15} + C_{18})$ all have the same sign, which in turn follows from the conditions of footnote 1. Furthermore, the roots are distinct except for the "hairline" case $(C_{15} + C_{18})^2 =$

---

it is computationally unstable (i.e., round-off errors eventually grow without bound). Techniques to impose computational stability increase the number of operations, and require some degree of mathematical sophistication. One such technique [3] permits the computation of any desired entries of the inverse of the matrix in (2.12).

[7] This follows immediately from equations (3.9) and (3.11). If either s_j is real (and hence positive), so will be the other; therefore the radicals $\sqrt{s_j(4 + s_j)}$ will both be real. Therefore, if either s_j is real, so will all the λ_i. A similar argument holds if either s_j is complex, and this exhausts the possibilities.

$4C_{16}C_{17}$,[8] which can always be avoided simply by carrying estimates of these cost coefficients to more significant figures.

Consequently we know that there are always two (and only two) roots which satisfy the auxiliary equation as well as the condition $0 < |\lambda| < 1$. Furthermore, these "relevant" roots will always be λ_1 and λ_2, given by equation (3.11a).

The Reduced System of Equations. Having the two allowable roots of the auxiliary equation, we are in a position to solve the reduced system (3.6) for the optimal level of the labor force, W_1, given a forecast of orders and the initial conditions of the system. The procedure outlined above is quite straightforward if the roots of the auxiliary equation are real, since a unique solution exists. If, on the other hand, the roots are complex, the previous results can still be cast into a simple computational form. Under these conditions, the second equation of (3.6) is the complex conjugate of the first. Since an equality implies that the real and imaginary parts of the equation must *each* hold independently of the other, either one of the equations would yield the same system of two linear equations having only real coefficients.

4. Computational Procedure for Obtaining the Decision Rules

We shall now illustrate how the method outlined above may be applied to actual computations. We will take as the first illustration the specific cost function discussed in [2]; in this application the roots of the auxiliary equation (3.4) turn out to be real numbers. Another cost structure, the roots of whose resulting auxiliary equation are complex, will then be briefly examined as a second example.

[8] It can be readily verified from equations (3.11) that $\lambda_1 = 1/\lambda_3$ and $\lambda_2 = 1/\lambda_4$. To show that the assertion holds, we need only show that no roots have a modulus equal to unity. First, if the roots s_j are real, we know that $\sqrt{s_j(4 + s_j)} > s_j > 0$. It immediately follows that $\lambda_i < 1 (i = 1, 2)$ and $\lambda_i > 1$ $(i = 3, 4)$. Second, if the roots s_j are complex (conjugates), assume that the modulus of some λ_i (and hence all) is equal to unity. Write the roots λ_i in trigonometric form as $\cos \phi \pm i \sin \phi$; then $s_j = \lambda_i + 1/\lambda_i - 2 = 2(\cos \phi - 1)$, a real, non-positive quantity. But this is a contradiction. Since C_{16}, C_{17} and $(C_{15} + C_{18})$ all possess the same sign, the s_j have positive real parts. Therefore, none of the roots has a modulus equal to unity. Because the s_j are non-zero, the roots λ_1 and λ_2 are distinct unless $s_1 = s_2$, which situation is possible only for the "hairline" case $(C_{15} + C_{18})^2 = 4C_{16}C_{17}$.

[9] The necessary and sufficient condition for a unique solution is that the determinant of coefficients does not vanish. The value of this determinant is

$$\frac{C_{17}(\lambda_1 - \lambda_2)}{(\lambda_1\lambda_2)^2}[C_{17}(\lambda_1 + \lambda_2 - 1) - (C_{19} - C_{22} + C_{21})\lambda_1\lambda_2].$$

Since the roots have a modulus less than one, we know

$$(1 - \lambda_1)(1 - \lambda_2) > 0.$$

It follows that

$$\lambda_1 + \lambda_2 - 1 < \lambda_1\lambda_2 < (1 + C_7/C_3)\lambda_1\lambda_2 = \frac{C_{19} - C_{22} + C_{21}}{C_{17}}\lambda_1\lambda_2.$$

The determinant then vanishes if and only if $\lambda_1 = \lambda_2$, namely, if $(C_{15} + C_{18})^2 = 4C_{16}C_{17}$.

EXAMPLE 1. REAL ROOTS

The cost data employed in [2] were the following:

Step 1: List of the Cost Data[10]

$$
\begin{array}{llll}
C_1 = 340. & & C_7 = & .0825 \\
C_2 = 64.3 & & C_8 = & 320. \\
C_3 = .20 & & C_9 = & 0. \\
C_4 = 5.67 & & C_{11} = & 0. \\
C_5 = 51.2 & & C_{12} = & 0. \\
C_6 = 281. & & C_{13} = & 0.
\end{array}
$$

Next, we evaluate the derived coefficients (which were introduced in Section 2 to simplify the notation):

Step 2: Calculation of the Derived Coefficients

$$
\begin{array}{lll}
C_{14} & = 2C_3C_4 - C_{12} & = 2.268000 \\
C_{10}/C_{14} & = (C_1 - C_6)/C_{14} & = 26.014109 \\
C_{15} & = 2C_2/C_{14} & = 56.701940 \\
C_{16} & = 2C_3C_4^2/C_{14} & = 5.670000 \\
C_{17} & = C_3C_{15}/C_7 & = 137.459248 \\
C_{18} & = (2C_3C_{16} - C_{14})/2C_7 & = 0. \\
C_{19} & = C_{16} + C_{18} + 2C_{15} + 3C_{17} & = 531.451624 \\
C_{20} & = C_{15} + 3C_{17} + C_{18} & = 469.079684 \\
C_{21} & = C_{15} + 4C_{17} + C_{18} & = 606.538932 \\
C_{22} & = C_{16} + 2C_{18} + 2C_{15} + 6C_{17} & = 943.829368 \\
C_{23} & = C_{16} + 2C_{15} & = 119.073880
\end{array}
$$

It is desirable to carry these and succeeding calculations to a large number of decimal places, in spite of inaccuracies in the original cost data, because there is a tendency for rounding errors to be exaggerated through the subtraction of numbers that are of the same order of magnitude. Upon completing the calculations of the decision rules, the extra decimal places that cannot be justified in terms of the accuracy of the original cost estimates may be dropped.

Step 3: Calculation of the Roots s. The next step is that of finding the roots of the auxiliary equation. When the equation is symmetric, as it is here, we have from (3.9):

$$
s = \frac{1}{2C_{17}}\left[(C_{15} + C_{18}) \pm \sqrt{(C_{15} + C_{18})^2 - 4C_{16}C_{17}}\right],
$$

from which we obtain:

$$
s_1 = .242173, \qquad s_2 = .170327.
$$

Step 4: Calculation of the Roots λ. We can substitute these two values of s into equation (3.11a)

[10] Note that these values of the parameter satisfy the conditions of footnote 1 for an interior minimum of the cost function.

$$\lambda = \tfrac{1}{2}[(2 + s) - \sqrt{s(4 + s)}],$$

yielding

$$\lambda_1 = .614298, \qquad \lambda_2 = .663762.$$

Step 5: Check Substitutions into the Auxiliary Equations. That these roots satisfy the auxiliary equation (3.4) may be verified by direct substitution. We have

$$C_{17} - C_{21}\lambda_1 + C_{22}\lambda_1^2 - C_{21}\lambda_1^3 + C_{17}\lambda_1^4 = 0$$

$$(137.459248) - (606.538932)(.614298) + (943.829368)(.377362)$$
$$- (606.538932)(.231813) + (137.459248)(.142402) = -.000205$$

$$C_{17} - C_{21}\lambda_2 + C_{22}\lambda_2^2 - C_{21}\lambda_2^3 + C_{17}\lambda_2^4 = 0$$

$$(137.459248) - (606.538932)(.663762) + (943.829368)(.440580)$$
$$- (606.538932)(.292440) + (137.459248)(.194111) = .000205$$

Since .000205 is close to zero (and within the range of rounding errors) we can safely proceed to the next step.

Step 6: The Reduced System of Equations. We will next substitute the numerical values of λ determined in Step 4 into equations (3.6), which are:

$$(C_{19} - C_{21}\lambda_i + C_{17}\lambda_i^2)W_1 + C_{17}(1 - \lambda_i^{-1})W_2$$
$$= [1 + C_9(1 - \lambda_i)]\left[\sum_{r=1}^{\infty} \lambda_i^{r-1}O_r\right] + [C_{15} + C_{17}(1 - \lambda_i)]W_0 - I_0$$
$$+ \left[C_8 - \frac{C_{10}}{C_{14}(1 - \lambda_i)}\right], \qquad i = 1, 2$$

Performing the indicated arithmetic, we obtain the following equations in the two unknowns W_1 and W_2—the variables on the right-hand side of the equations, $O_r (r = 1, 2, 3, \cdots)$, W_0, and I_0, are known.

$$210.727868W_1 - 86.307088W_2 = \sum_{r=1}^{\infty} \lambda_1^{r-1}O_r + 109.720247W_0 - I_0 + 252.553865$$

$$189.415925W_1 - 69.631907W_2 = \sum_{r=1}^{\infty} \lambda_2^{r-1}O_r + 102.920963W_0 - I_0 + 242.631859$$

Step 7: The Solution of the Equations for W_1. Several methods are available for solving this relatively simple system of equations. One convenient method is to eliminate the variable W_2 from the system as follows. First multiply the first equation of Step 6 by the factor $-(69.631907/86.307088) = -.806792$; then add the second equation to this new one. Performing these operations and dividing by the resulting coefficient of W_1, we obtain:

$$W_1 = \sum_{r=1}^{\infty} [-.041582\lambda_1^{r-1} + .051540\lambda_2^{r-1}]O_r + .742153W_0 - .009958I_0 + 2.003536$$

This is the employment decision rule, given by Equation 10 of [2].

Step 8: The Solution of the Equations for W_2. The value of W_1, from Step 7, can now be substituted into the first equation of Step 6 to obtain W_2.

$$W_2 = -.011587\left(\sum_{r=1}^{\infty} \lambda_1^{r-1} O_r\right) - 1.271329W_0$$
$$+ .011587I_0 - 2.926342 + 2.441704W_1$$
$$= \sum_{r=1}^{\infty} [-.113118\lambda_1^{r-1} + .125845\lambda_2^{r-1}]O_r + .540789W_0 - .012727I_0 + 1.965700$$

Step 9: Check Substitutions into the Equations. Again, it is advisable to check the work. Substituting the expressions for W_1 and W_2 into the lefthand side of the first equation of Step 6, we obtain the following expression which may then be compared for equality with the right-hand side of the equation.

$$210.727868\left[\sum_{r=1}^{\infty} (-.041582\lambda_1^{r-1} + .051540\lambda_2^{r-1})O_r + .742153W_0 \right.$$
$$\left. -.009958I_0 + 2.003536\right]$$
$$-86.307088\left[\sum_{r=1}^{\infty} (-.113118\lambda_1^{r-1} + .125845\lambda_2^{r-1})O_r + .540789W_0 \right.$$
$$\left. - .012727I_0 + 1.965700\right]$$

Simplifying, we obtain:

$$\sum_{r=1}^{\infty} [1.000399\lambda_1^{r-1} - .000401\lambda_2^{r-1}]O_r + 109.718395W_0 - .999998I_0 + 252.547027$$

Proceeding similarly for the second equation of Step 6, we obtain:

$$\sum_{r=1}^{\infty} [.000329\lambda_1^{r-1} + .999670\lambda_2^{r-1}]O_r + 102.919428W_0 - .999999I_0 + 242.626185$$

Since the coefficients above agree with those of Step 6 (within the range of expected rounding errors), we can proceed to the next step.

Step 10: Solution for P_1. Equation (2.5) relates the optimal rate of production, P_1, to planned levels of the work force. Making the substitutions of the two work force rules, from Steps 2, 7, and 8, we can express the optimal production plan directly in terms of the initial conditions and the forecasts of incoming orders as:

$$P_1 = C_{10}/C_{14} - C_{15}W_2 + C_{23}W_1 - C_{15}W_0$$
$$= 26.014109 - 56.701940W_2 + 119.073880W_1 - 56.701940W_0$$
$$= \sum_{r=1}^{\infty} [1.462680\lambda_1^{r-1} - .998588\lambda_2^{r-1}]O_r + 1.005312W_0$$
$$- .464092I_0 + 153.123911,$$

which is the production decision rule, Equation (11) of [2].

TABLE 1

Worksheet for Calculation of Weights

Step 11, (Real Roots)

Col. 1	Col. 2	Col. 3	Col. 4 Weights for Work Force Rule	Col. 5 Weights for Production Rule
r	λ_1^{r-1}	λ_2^{r-1}	$-.041582\ \lambda_1^{r-1}$ $+.051540\ \lambda_2^{r-1}$	$1.462608\ \lambda_1^{r-1}$ $-.998588\ \lambda_2^{r-1}$
1	1.000000	1.000000	.009958	.464092
2	.614298	.663762	.008666	.235696
3	.377361	.440580	.007016	.112002
4	.231812	.292440	.005433	.047041
5	.142402	.194111	.004083	.014452
6	.087477	.128844	.003004	−.000711
7	.053737	.085522	.002174	−.006801
8	.033010	.056766	.001553	−.008401
9	.020278	.037679	.001099	−.007964
10	.012457	.025010	.000772	−.006754
11	.007652	.016601	.000538	−.005386
12	.004701	.011019	.000373	−.004127
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
Total............	2.592675	2.974084	.045476	.822369

Step 11: Calculation of Forecast Weights. The only remaining step is the calculation of the weights to be applied to forecasts of future orders for the work force rule (from Step 7) and for the production rule (from Step 10). Since these weights are linear combinations of successive powers of the roots λ_i, they may be computed on a relatively simple worksheet (Table 1).

In the first column of Table 1 we have the index representing the number of time periods ahead. In Columns 2 and 3, the successive powers of the roots λ_1 and λ_2, respectively, are computed. The weight of the forecast of orders in the r^{th} period for the labor force rule is given in Column 4 as $-.041582\ \lambda_1^{r-1} + .051540\ \lambda_2^{r-1}$, a weighted sum of the respective entries in the previous two columns (see Step 7). Similarly, the weights for the production rule in Column 5 are $1.462608\ \lambda_1^{r-1} - .998580\ \lambda_2^{r-1}$ (see Step 10).[11]

EXAMPLE 2: COMPLEX ROOTS

The computation of the decision rules from the cost function is somewhat more complicated if the roots of the auxiliary equation turn out at Step 3 to be complex numbers. To illustrate this case, we will change two of the parameters in the

[11] The weights given by Columns 4 and 5 of the worksheet are not identical with those given in [2]. In the previous article a small adjustment in the weights was made in order to shorten to 12 months the infinite forecast horizon which results from the theory which has been derived here.

previous example as follows: let $C_2 = 72.3375$ and $C_3 = .2375$ and carry through the modified computations.

Step 2: Calculation of the Derived Coefficients. The derived parameters are almost all changed; they now become:

$$
\begin{array}{llll}
C_{14} & = 2.693250 & C_{18} = & 0. \\
C_{10}/C_{14} & = 21.906618 & C_{19} = & 577.030198 \\
C_{15} & = 53.717627 & C_{20} = & 517.642571 \\
C_{16} & = 5.670000 & C_{21} = & 672.284219 \\
C_{17} & = 154.641648 & C_{22} = & 1040.955142
\end{array}
$$

$$C_{23} = 113.105254$$

Step 3: Calculation of the Roots s. We determine the roots to the auxiliary equation, by substituting[12] in (3.9):

$$s_1 = .173684 + .080618i$$

$$s_2 = .173684 - .080618i$$

where $i = \sqrt{-1}$.

Upon obtaining the conjugate roots, s, the root with the positive imaginary part is designated s_1 and the root with the negative imaginary part s_2. In the succeeding calculation, the λ root which corresponds to s_1 is designated λ_1, and similarly for s_2 and λ_2. Attention to this notation is necessary to avoid errors of sign.

Step 4A: Calculation of $\sqrt{s(4+s)}$. Equation (3.12) is a standard formula for expressing the square root of a complex number directly in terms of its real and imaginary parts. Since we need the square root of $s(4+s)$ we proceed as follows:

$$s(4+s) = (.173684 \pm .080618i)(4.173684 \pm .080618i) = .718403 \pm .350476i$$

$$\equiv x \pm yi$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(.718403)^2 + (.350476)^2} = .799335$$

$$\sqrt{s(4+s)} = \frac{1}{\sqrt{2}}\left[\sqrt{r+x} \pm i\sqrt{r-x}\right]$$

$$= .707107\ (1.231966 \pm .284486i)$$

$$= .871132 \pm .201162i$$

[12] In the following calculations a knowledge of the routine manipulation of complex numbers is required. The essential operations are outlined below, but for explanation consult a textbook on college algebra or trigonometry, reference [5] for example.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Multiplication: $(a + bi) \times (c + di) = (ac - bd) + (bc + ad)i$

Division: $\dfrac{1}{b + ci} = \dfrac{(b - ci)}{b^2 + c^2}.$

Step 4B: Calculation of the Roots λ. Now we can substitute these values into equation (3.11a) to obtain:

$$\lambda_i = \tfrac{1}{2}[(2 + s_j) - \sqrt{s_j(4 + s_j)}] \qquad i = j = 1, 2$$

$$= \tfrac{1}{2}[(2.173684 \pm .080618i) - (.871131 \pm .201162i)]$$

$$\lambda_1 = .651276 + (-.060272)i \equiv a + bi$$

$$\lambda_2 = .651276 - (-\ .060272)i \equiv a - bi$$

These roots have the radius

$$\rho = \sqrt{a^2 + b^2} = \sqrt{(.651276)^2 + (.060272)^2} = .654059$$

and argument

$$\phi = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{-.060272}{.651276} = -5.2872^\circ \quad \text{or} \quad -5^\circ\, 17.23'$$

so that

$$\lambda_1 = \rho\,(\cos\phi + i\sin\phi) = .65406\,(\cos 5.29^\circ - i\sin 5.29^\circ)$$

$$\lambda_2 = \rho\,(\cos\phi - i\sin\phi) = .65406\,(\cos 5.29^\circ + i\sin 5.29^\circ).$$

The calculation of the check substitutions, Step 5, is left to the reader.

Step 6A: The Reduced System of Equations. Substituting the values of λ given by Step 4 into a modified version of equation (3.6) we obtain the system:

$$(C_{19} - C_{21}\lambda_i + C_{17}\lambda_i^2)W_1 + C_{17}(1 - \lambda_i^{-1})W_2$$
$$= [1 + C_9(1 - \lambda_i)]\left\{\sum_{r=1}^{\infty} \rho^{r-1}[\cos(r-1)\phi \pm i\sin(r-1)\phi]O_r\right\}$$
$$+ [C_{15} + C_{17}(1 - \lambda_i)]W_0 - I_0 + \left[C_8 - \frac{C_{10}}{C_{14}(1 - \lambda_i)}\right].$$

$$(204.218764 \pm 28.379463i)W_1 + (-80.786807 \mp 21.787616i)W_2$$
$$= \sum_{r=1}^{\infty} \rho^{r-1}[\cos(r-1)\phi \pm i\sin(r-1)\phi]O_r$$
$$+ (107.644881 \pm 9.320561i)W_0 - I_0 + (259.002687 \pm 10.542516i)$$

Step 6B: The Equations Involving Only Real Coefficients. Since the real and imaginary parts of the above equations must each be equal, we can equate these two parts separately in order to obtain the following system which involves only real coefficients:

$$204.218764W_1 - 80.786807W_2 = \sum_{r=1}^{\infty} [\rho^{r-1}\cos(r-1)\phi]O_r$$
$$+ 107.644881W_0 - I_0 + 259.002687$$

$$28.379463W_1 - 21.787616W_2 = \sum_{r=1}^{\infty} [\rho^{r-1}\sin(r-1)\phi]O_r$$
$$+ 9.320561W_0 + 10.542516$$

TABLE 2

Worksheet for Calculations of Weights

Step 11, (Complex Roots)

Col. 1	Col. 2	Col. 3	Col. 4	Col. 5 Weights For Work Force Rule	Col. 6 Weights for Production Rule
r	$\cos(r-1)\phi$	$\sin(r-1)\phi$	ρ^{r-1}	$[.010102 \cos(r-1)\phi - .037457 \sin(r-1)\phi] \times \rho^{r-1}$	$[.435773 \cos(r-1)\phi + .849670 \sin(r-1)\phi] \times \rho^{r-1}$
1	1.00000	.00000	1.000000	.010102	.435773
2	.99574	−.09214	.654059	.008837	.232602
3	.98307	−.18353	.427793	.007189	.116554
4	.96192	−.27332	.279802	.005583	.052308
5	.93263	−.36078	.183007	.004197	.018277
6	.89800	−.44001	.119697	.003059	.002090
7	.85060	−.52582	.078289	.002215	−.005958
8	.79854	−.60192	.051206	.001568	−.008370
9	.73967	−.67297	.033492	.001095	−.008355
10	.67450	−.73828	.021906	.000755	−.007303
11	.60361	−.79729	.014328	.000515	−.005938
12	.52753	−.84948	.009371	.000348	−.004609
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
Total.....	—	—	2.867573	.046154	.804475

Step 7: Solution of the Equations for W_1. Eliminating W_2 from the equations above, we obtain the work force rule:

$$W_1 = \sum_{r=1}^{\infty} \rho^{r-1}[.010102 \cos (r - 1)\phi - .037457 \sin (r - 1)\phi]O_r$$

$$+ .738304W_0 - .010102I_0 + 2.221549$$

The next three steps, 8, 9, and 10, are basically the same as those of the preceding case.

Leaving the detail of these steps to the reader, we report the production rule that is obtained:

$$P_1 = \sum_{r=1}^{\infty} \rho^{r-1}[.435773 \cos (r - 1)\phi + .849670 \sin (r - 1)\phi]O_r$$

$$+ 1.110097W_0 - .435773I_0 + 143.729914.$$

Step 11: Calculation of Forecast Weights. The worksheet for calculating these weights now requires more columns than previously (Table 2). The weights for the work force rule and the production rule are given, respectively, in Columns 5 and 6 which are computed from the first three columns.

This completes the computation procedure for obtaining the two decision rules.

5. Conclusion

Once these decision rules have been obtained, computation of the production and employment schedule for any production period requires but a few minutes. These rules may continue in use unchanged as long as there is no significant change in the cost structure.[13]

For the quadratic cost function in the decision variables, W_r, and P_r ($r = 1, 2, \cdots$), and initial conditions and forecasts, we have obtained the first order conditions for a minimum cost. The solution of these conditions then yielded for the next period the optimal level of the work force, W_1, and the optimal rate of production, P_1, as functions of the forecasts of orders and the levels of inventories and employment at the beginning of that period. We finally presented a procedure for computing the solution, and this procedure was illustrated with the cost parameters developed in the previous article.

The certainty equivalence property enables us to use these solutions, which were derived for "known" future order receipts, as decision rules for the usual situation in which forecasts of future order receipts are subject to error. The action that is indicated by the decision rule for the first period is optimal in the sense that it is the best action that can be taken on the basis of information currently available. The tentative plans for future actions based on present information may also be obtained, but, of course, these plans will undoubtedly be modified as new information becomes available before they are put into effect. A forecasting method should be used whose expected error is zero, or more loosely, whose algebraic average error is zero. Of course, the more accurate the forecasts, the better the decisions and the lower the costs.

Although the procedures presented here and in [2] were developed for a particular factory, they should be of immediate usefulness in facilitating production and employment decisions elsewhere.[14] Moreover, the general techniques for obtaining decision rules from quadratic criterion functions is applicable to many other decision problems.

References

[1] H. B. Dwight, *Tables of Integrals and Other Mathematical Data* (Macmillan, 1955), p. 13.
[2] Charles C. Holt, Franco Modigliani, and Herbert A. Simon, "A Linear Decision

[13] Since the coefficients of the decision rules depend only on the ratios of the cost parameters, a proportional change in all costs does not require any change in the rules.

[14] However, some more work needs to be done on problem of making "best" estimates of the parameters in the cost function.

An extension of the above methods may be used to obtain estimates of the overall cost implications for the firm of changes in the cost parameters. These would be useful when a decision is under consideration that will change the cost parameters—a new labor contract for example. Such cost implications can be determined by recomputing the decision rules, applying them to long-term sales forecasts to obtain forecasts of production and employment decisions, and finally computing forecasts of the overall costs. Note that an increase in any cost parameter will cause an increase in forecasted costs but less than would be expected under the original decision rule, since the new decision rule which is obtained for the changed cost structure will to some degree avoid the action that incurs the component of costs that has become more expensive.

Rule for Production and Employment Scheduling," *Management Science*, October, 1955.

[3] FRANCO MODIGLIANI (with the assistance of TAKEHIKO MATSUDA), "An Alternative Method for Inverting the Quadratic Decision Matrix," ONR Research Memorandum No. 22, G.S.I.A., Carnegie Institute of Technology, August, 1954.

[4] JOHN F. MUTH, "Locally Optimal Decision Rules," Unpublished, May, 1955.

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[6] HERBERT A. SIMON, "Dynamic Programming Under Uncertainty with a Quadratic Criterion Function," *Econometrica*, to be published in January, 1956.

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V-16

DYNAMIC VERSION OF THE ECONOMIC LOT SIZE MODEL*†

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A forward algorithm for a solution to the following dynamic version of the economic lot size model is given: allowing the possibility of demands for a single item, inventory holding charges, and setup costs to vary over N periods, we desire a minimum total cost inventory management scheme which satisfies known demand in every period. Disjoint planning horizons are shown to be possible which eliminate the necessity of having data for the full N periods.

1. Introduction

By now the "square root formula" [7] (equation 8 below) for an economic lot size under the assumption of a steady-state demand rate is well known. The calculation is predicated upon a balancing of the costs of holding inventory against the costs of placing an order. When the assumption of a steady-state demand rate is dropped—i.e., when the amounts demanded in each period are known but are different—and furthermore, when inventory costs vary from period to period, the square root formula (applied to the overall average demand and costs) no longer assures a minimum cost solution. We shall present a simple algorithm for solving the dynamic version of the model.[1]

The mathematical model may be viewed as a "one-way temporal feasibility" problem, in that it is feasible to order inventory in period t for demand in period $t + k$ but not vice versa. This suggests that the same model also permits an alternative interpretation as the following "one-way technological feasibility" problem [1]. Suppose a manufacturer produces an item having N possible values for a certain critical dimension; for example, the item may be steel beams of various strengths.[2] He anticipates a known demand schedule for the N types of steel beams, and it is feasible to substitute a beam of strength g_1 for a beam of strength g_2 if and only if $g_1 > g_2$. Producing each kind of a beam requires a setup cost, and using a beam in excess of the required strength incurs a charge in terms of wasted steel. The operator of the steel mill wishes to know how many beams of each type to produce in order to minimize total costs.

2. Mathematical Model

As in the standard lot size formulation, we assume that the buying (or manufacturing) costs and selling price of the item are constant throughout all time

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[1] Elsewhere [6] we have discussed a further generalization in which period sales are a function of price, and costs are not necessarily proportional to output or the amount purchased.

[2] We are indebted to Professor W. Sadowski, Central School of Planning and Statistics, Warsaw University, who suggested this application.

periods, and consequently only the costs of inventory management are of concern. In the t-th period, $t = 1, 2, \cdots, N$, we let

d_t = amount demanded
i_t = interest charge per unit of inventory carried forward to period $t + 1$
s_t = ordering (or setup) cost
x_t = amount ordered (or manufactured).[3]

We assume that all period demands and costs are non-negative. The problem is to find a program $x_t \geqq 0$, $t = 1, 2, \cdots, N$, such that all demands are met at a minimum total cost; any such program, which need not be unique, will be termed *optimal.*

Of course one method of solving the optimization problem is to enumerate 2^{N-1} combinations of either ordering or not ordering in each period (we assume an order is placed in the first period).[4] A more efficient algorithm evolves from a dynamic programming characterization of an optimal policy [2, 3, 4].

Let I denote the inventory entering a period and I_0 initial inventory; for period t

$$I = I_0 + \sum_{j=1}^{t-1} x_j - \sum_{j=1}^{t-1} d_j \geqq 0. \tag{1}$$

We may write the functional equation [2, 4] representing the minimal cost policy for periods t through N, given incoming inventory I, as

$$f_t(I) = \min_{\substack{x_t \geqq 0 \\ I + x_t \geqq d_t}} [i_{t-1} I + \delta(x_t)s_t + f_{t+1}(I + x_t - d_t)] \tag{2}$$

where

$$\delta(x_t) = \begin{cases} 0 \text{ if } x_t = 0 \\ 1 \text{ if } x_t > 0. \end{cases} \tag{3}$$

In period N we have

$$f_N(I) = \min_{\substack{x_N \geqq 0 \\ I + x_N = d_N}} [i_{N-1} I + \delta(x_N)s_N]. \tag{4}$$

Consequently we compute f_t, starting at $t = N$, as a function of I; ultimately we derive f_1, thereby obtaining an optimal solution as I for period 1 is specified. Theorem 2 below establishes that it is permissible to confine consideration to only $N + 2 - t$, $t > 1$, values of I at period t.

By taking cognizance of the special properties of our model, we may formulate an alternative functional equation which has the advantage of potentially requiring less than N periods' data to obtain an optimal program; that is, it may

[3] We confine ourselves, as one does in the static model, to situations in which (nearly) constant lead or delivery time is a workable approximation to reality.

[4] Formally the model may also be posed as a fixed charge linear programming problem; see W. M. Hirsch and G. B. Dantzig, "The Fixed Charge Problem," RAND Corporation RM-1383, December 1954.

be possible without any loss of optimality to narrow our program commitment to a shorter "planning horizon" than N periods on the sole basis of data for this horizon. Just as one may prove that in a linear programming model it suffices to investigate only *basic* sets of variables in search of an optimal solution, we shall demonstrate that in our model an optimal solution exists among a very simple class of policies.

It is necessary to postulate that $d_1 \geqq 0$ is demand in period 1 *net* of starting inventory.[5] Then the fundamental proposition underlying our approach asserts that it is sufficient to consider programs in which at period t one does not *both* place an order and bring in inventory.

Theorem 1. There exists an optimal program such that $Ix_t = 0$ for all t (where I is inventory entering period t).

Proof: Suppose an optimal program suggests both to place an order in period t and to bring in I (i.e., $Ix_t > 0$). Then it is no more costly to reschedule the purchase of I by including the quantity in x_t, for this alteration does not incur any *additional* ordering cost and does save the cost $i_{t-1}I \geqq 0$.

Note that the theorem does not hold if our model includes buying or production costs which are not constant and identical for all periods. In the latter case, economies of scale might very well call for the carrying of inventory into period t even when an order or setup takes place in t [6].

Two corollaries follow from the theorem.

Theorem 2. There exists an optimal program such that for all t

$$x_t = 0 \quad \text{or} \quad \textstyle\sum_{j=t}^{k} d_j \quad \text{for some} \quad k,\ t \leqq k \leqq N.$$

Proof: Since all demands must be met, any other value for x_t implies there exists a period $t^* \geqq t$ such that $Ix_{t^*} > 0$; but Theorem 1 assures that it is sufficient to consider programs in which such a condition does not arise.

The implication of Theorem 2 is that we can limit the values of I in (2) for period t to zero and the cumulative sums of demand for periods t up to N. If initial inventory is zero, then only $N(N + 1)/2$ different values of I in toto over the entire N periods need be examined.

Theorem 3. There exists an optimal program such that if d_{t^*} is satisfied by some $x_{t^{**}}$, $t^{**} < t^*$, then d_t, $t = t^{**} + 1, \cdots, t^* - 1$, is also satisfied by $x_{t^{**}}$.

Proof: In a program not satisfying the theorem, either I for period t^{**} is positive or I for period t^* is brought into some period t', $t^{**} < t' < t^*$, where $x_{t'} > 0$; but again by Theorem 1, it is sufficient to consider programs in which such conditions do not arise.

[5] In other words, we let $I_0 = 0$ by netting out starting inventory from demand in period 1. If the level of starting inventory in fact exceeds the total demand in period 1, then the "forward" algorithm to be suggested may not be correct. In particular, Theorem 1 below may not hold for period 1; in such a case (2) still remains applicable. A sufficient condition for the existence of a forward solution is that s_t is monotonically non-increasing. An optimal solution is found then by using up initial inventory period by period until, at some t, the inventory remaining does not meet the demand; at this point, our suggested algorithm is commenced.

We next investigate a condition under which we may divide our problem into two smaller subproblems.

Theorem 4. Given that $I = 0$ for period t, it is optimal to consider periods 1 through $t - 1$ by themselves.

Proof: By hypothesis, (2) in period $t - 1$ for the N period model is

$$f_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} [i_{t-2}I + \delta(x_{t-1})s_{t-1} + f_t(0)], \tag{5}$$

and for the $t - 1$ period model is correspondingly

$$g_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} [i_{t-2}I + \delta(x_{t-1})s_{t-1}]. \tag{6}$$

But the functional relations (5) and (6) differ only by a constant $f_t(0)$. Consequently what is optimal for (6) is optimal for (5), and by the recursive structure of the model, the latter conclusion continues to hold for all the earlier periods.

We may now offer an alternative formulation to (2). Let $F(t)$ denote the minimal cost program for periods 1 through t. Then

$$F(t) = \min \begin{bmatrix} \min\limits_{1 \leqq j < t} \left[s_j + \sum\limits_{h=j}^{t-1} \sum\limits_{k=h+1}^{t} i_h d_k + F(j - 1) \right] \\ s_t + F(t - 1) \end{bmatrix} \tag{7}$$

where $F(1) = s_1$ and $F(0) = 0$. That is, the minimum cost for the first t periods comprises a setup cost in period j, plus charges for filling demand d_k, $k = j + 1, \cdots, t$, by carrying inventory from period j, plus the cost of adopting an optimal policy in periods 1 through $j - 1$ taken by themselves. Theorems 2, 3, and 4 guarantee that at period t we shall find an optimum program of this type. With the present formulation, (7) is computed, starting at $t = 1$. At any period t, (7) implies that only t policies need to be considered. The minimum in (7) need not be unique, so that there may be alternative optimal solutions. When we derive $F(N)$, we shall have solved the problem for N is the last period to be considered.

Finally we come to what is perhaps the most interesting property of our model.

The Planning Horizon Theorem.[6] If at period t^* the minimum in (7) occurs for $j = t^{**} \leqq t^*$, then in periods $t > t^*$ it is sufficient to consider only $t^{**} \leqq j \leqq t$. In particular, if $t^* = t^{**}$, then it is sufficient to consider programs such that $x_{t^*} > 0$.

Proof: Without loss of optimality we restrict our attention to programs of the form specified in Theorems 1–4. Suppose a program suggests that d_t is satisfied by $x_{t^{***}}$, where $t^{***} < t^{**} \leqq t^* < t$. Then by Theorem 3 d_{t^*} is also satisfied by $x_{t^{***}}$. But by hypothesis we know that costs are not increased by rescheduling the program to let d_{t^*} be satisfied by $x_{t^{**}} > 0$.

[6] The reader may wish to prove the corresponding theorem for (2): Let I^{**} be the value of incoming inventory associated with $\min_I f_{t^*}(I)$; then in period $t < t^*$ it is sufficient to consider only $0 \leqq I \leqq I^{**} + \sum_{j=t}^{t^*-1} d_j$. In particular, if $I^{**} = 0$, then it is sufficient to consider programs such that $I = 0$ at period t^*.

The planning horizon theorem states in part that if it is optimal to incur a setup cost in period t^* when periods 1 through t^* are considered by themselves, then we may let $x_{t^*} > 0$ in the N period model without foregoing optimality. By Theorems 1 and 4 it follows further that we may adopt an optimal program for periods 1 through $t^* - 1$ considered separately.

3. The Algorithm

The algorithm at period t^*, $t^* = 1, 2, \cdots, N$, may be generally stated as

1. Consider the policies of ordering at period t^{**}, $t^{**} = 1, 2, \cdots, t^*$, and filling demands d_t, $t = t^{**}, t^{**} + 1, \cdots, t^*$, by this order.
2. Determine the total cost of these t^* different policies by adding the ordering and holding costs associated with placing an order at period t^{**}, and the cost of acting optimally for periods 1 through $t^{**} - 1$ considered by themselves. The latter cost has been determined previously in the computations for periods $t = 1, 2, \cdots, t^* - 1$.
3. From these t^* alternatives, select the minimum cost policy for periods 1 through t^* considered independently.
4. Proceed to period $t^* + 1$ (or stop if $t^* = N$).

Table 1 portrays the symbolic scheme for the algorithm. The notation $(1, 2, \cdots, t^{**})\ \underline{t^{**} + 1, t^{**} + 2, \cdots, t^*}$ in Table 1 indicates that an order is placed in period $t^{**} + 1$ to cover the demands of d_t, $t = t^{**} + 1, t^{**} + 2, \cdots, t^*$, and the optimal policy is adopted for periods 1 through t^{**} considered separately. At the bottom of the table we record the minimum cost plan for periods 1 through t^*.

In general, it may be necessary to test N policies at the N-th period, implying a table of $N(N + 1)/2$ entries (versus 2^{N-1} for all possibilities). Thus the forward algorithm (7) is at least as efficient as (2). As we shall see, the number of entries usually is much smaller than this number if we make full use of the planning horizon theorem.

TABLE 1

Month t	1	2	3	4	$\cdots$	N
Ordering cost	s_1	s_2	s_3	s_4	$\cdots$	s_N
Demand	d_1	d_2	d_3	d_4	$\cdots$	d_N
$(1, 2, \cdots, t-1)\underline{t}$	$\underline{1}$	$(1)\underline{2}$	$(1, 2)\underline{3}$	$(1, 2, 3)\underline{4}$	$\cdots$	$(1, 2, \cdots, N-1)\underline{N}$
$(1, 2, \cdots, t-2)$ $\underline{t-1, t}$		$\underline{12}$	$(1)\underline{23}$	$(1, 2)\underline{34}$	$\cdots$	$(1, 2, \cdots, N-2)$ $\underline{N-1, N}$
$(1, 2, \cdots, t-3)$ $\underline{t-2, t-1, t}$			$\underline{123}$	$(1)\underline{234}$	$\cdots$	$(1, 2, \cdots, N-3)$ $\underline{N-2, N-1, N}$
$(1, 2, \cdots, t-4)$ $\underline{t-3, \cdots, t}$				$\underline{1243}$	$\cdots$	$(1, 2, \cdots, N-4)$ $\underline{N-3, N-2, N-1, N}$
$\vdots$						$\vdots$
Minimum cost						
Optimal policy $(1, 2, \cdots, t)$	(1)	(1, 2)	(1, 2, 3)	(1, 2, 3, 4)		$(1, 2, \cdots, N)$

4. An Example

Table 2 presents a sample set of data for a 12 month period; to simplify computations we have let $i_t = 1$ for all t; Table 3 contains the specific calculations.

To illustrate, the optimal plan for period 1 alone is to order (entailing an ordering cost of 85), Table 3. Two possibilities must be evaluated for period 2: order in period 2, and use the best policy for period 1 considered alone (at a cost of $102 + 85 = 187$); or order in period 1 for both periods, and carry inventory into period 2 (at a cost of $85 + 29 = 114$). The better policy is the latter one. In period 3 there are three alternatives: order in period 3, and use the best policy for periods 1 and 2 considered alone (at a cost of $102 + 114 = 216$); or order in period 2 for the latter two periods and use the best policy for period 1

TABLE 2

Month t	d_t	s_t	i_t
1	69	85	1
2	29	102	1
3	36	102	1
4	61	101	1
5	61	98	1
6	26	114	1
7	34	105	1
8	67	86	1
9	45	119	1
10	67	110	1
11	79	98	1
12	56	114	1
Average..................	52.5	102.8	1

TABLE 3

Month t 0	1	2	3	4	5	6	7	8	9	10	11	12
Ordering cost	85	102	102	101	98	114	105	86	119	110	98	114
Demand 0	69	29	36	61	61	26	34	67	45	67	79	56
	85	187	216	287	375	462	505	555	674	710	808	903
		114	223	277	348	401	496	572	600	741	789	864
			186			400	469			734		901
							502					
Minimum cost	85	114	186	277	348	400	469	555	600	710	789	864
Optimal policy*	1	12	123	34	45	456	567	8	89	10	10, 11	11, 12

* Only the last order period is shown; 567 indicates that the optimal policy for periods 1 through 7 is to order in period 5 to satisfy d_5, d_6, and d_7, and adopt an optimal policy for periods 1 through 4 considered separately.

considered alone (at a cost of $102 + 36 + 85 = 223$); or order in period 1 for the entire three periods (at a cost of $85 + 29 + 36 + 36 = 186$).

In our example, it is clear that it would never pay to carry inventory from periods 1 or 2 to meet d_4, since the carrying charges would exceed the ordering cost in period 4. A fortiori it would never pay to carry inventory from periods 1 or 2 to meet $d_5, d_6, \cdots, d_N$, because to do so would also imply that inventory was being carried to period 4 (Theorem 3).

Note that periods 1 through 8, and 8 through 10 comprise planning horizons. Whenever a time horizon (or a simplification of the type mentioned in the previous paragraph) arises, the entries in the table can be truncated below the southeast diagonal through the entry for $(1, 2, \cdots, t^* - 1)\underline{t^*}$, as we have done in Table 3.

For our set of data the optimal policy is

1. Order at period 11, $x_{11} = 79 + 56 = 135$, and use the optimal policy for periods 1 through 10, implying
2. Order at period 10, $x_{10} = 67$, and use the optimal policy for periods 1 through 9, implying
3. Order at period 8, $x_8 = 67 + 45 = 112$, and use the optimal policy for periods 1 through 7, implying
4. Order at period 5, $x_5 = 61 + 26 + 34 = 121$, and use the optimal policy for periods 1 through 4, implying
5. Order at period 3, $x_3 = 36 + 61 = 97$, and use the optimal policy for periods 1 through 2, implying
6. Order at period 1, $x_1 = 69 + 29 = 98$.

The total cost of the optimal policy is 864.

By use of the suggested tabular form, it is also relatively easy to make sensitivity analyses of the solution. For example, the ordering cost in period 2 would have to decrease by more than 73 in order to make it less costly to setup in period 2 than carry inventory from period 1; ordering cost in period 11 would have to increase by more than 37 in order to make it less costly to order in period 10 for the last three periods.

5. A Steady State Example

In the case of steady state demand and constant ordering and holding costs, our algorithm yields the same result as the standard "square root formula." Assume that throughout the entire year monthly demand $d = 52.5$, ordering (setup) cost $s = 102.80$, and interest charge $i = 1$. The square root formula for the order quantity gives

$$Q = \sqrt{2\,ds/i} = \sqrt{2 \times 52.5 \times 102.8/1} = 104. \tag{8}$$

Since this is approximately two months demand, we round Q to 105 units for comparison purposes.

Applying our algorithm yields that for the first two and three periods, the optimal policies are $\underline{12}$ and $(1, 2)\underline{3}$, indicating that the first two periods comprise

a planning horizon. In the steady state case, all planning horizons are the same, i.e., orders will be placed every two months. Therefore annual costs are easily obtained as six times the costs for one planning horizon, amounting to 931.80.

Annual total variable costs may be calculated with the standard lot size model as[7]

$$12[(Q - d)i/2 + ds/Q] = 12[(105 - 52.5)1/2 + 52.5(102.80)/105]$$
$$= 931.80.$$

Thus the two models are equally as costly. If the square root formula had not resulted in the ordering of an integral number of months' supply, the costs under the two methods would have been different due to the discrete division of time in our model. However this difference vanishes once the length of our time period is reduced.

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[7] To effect comparability with our model, the inventory carrying charge term has been reduced from $Qi/2$ to $(Q - d)i/2$. This change is required because our model, for the sake of simplicity, applies carrying charges to ending inventory only. As the reader may convince himself, adding carrying charges to that inventory used within a period would change total costs by a fixed amount and would not affect the solution of an optimal policy.

V–17

PROGRAMMING OF ECONOMIC LOT SIZES*[1]

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This paper studies the planning problem faced by a machine shop required to produce many different items so as to meet a rigid delivery schedule, remain within capacity limitations, and at the same time minimize the use of premium-cost overtime labor. It differs from alternative approaches to this well-known problem by allowing for setup cost indivisibilities.

As an approximation, the following linear programming model is suggested: Let an activity be defined as a sequence of the inputs required to satisfy the delivery requirements for a single item over time. The input coefficients for each such activity may then be constructed so as to allow for all setup costs incurred when the activity is operated at the level of unity or at zero. It is then shown that in any solution to this problem, all activity levels will turn out to be either unity or zero, except for those related to a group of items which, in number, must be equal to or less than the original number of capacity constraints. This result means that the linear programming solution should provide a good approximation whenever the number of items being manufactured is large in comparison with the number of capacity constraints.

1. Background

It is common knowledge that the presence of "setup costs" in a manufacturing process raises questions of indivisibilities [4], and that such indivisibilities constitute a formidable obstacle to any attempt to phrase economic lot size problems in terms of linear programming. In economists' language, this amounts to saying that the presence of economies of scale contradicts the assumptions of marginal analysis, along with such economic theories as linear programming, which are so deeply rooted in marginalism.

This paper reports upon the successful use of linear programming in a special instance involving setup costs and economic lot sizes.[3] Unlike a number of the

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[2] The author is deeply indebted to E. Greenwood, general manager, and also to C. Correll, A. Pastick, and A. Goldman, all of Norden-Ketay, Inc., Milford and Stamford, Connecticut.

[3] If the total costs of producing x units of a particular item in a single lot are given by: $a\delta + bx$

$$\text{where} \quad x\begin{pmatrix} >0 \\ \\ =0 \end{pmatrix} \quad \text{implies} \quad \delta\begin{pmatrix} =1 \\ \\ =0 \end{pmatrix}$$

then the constant a is said to represent the "setup cost" for that item and the constant

more recent proposals [e.g., 1], the particular model is a non-stochastic one, and in this sense is of less general applicability. The distinctive feature of the approach outlined here is that capacity limitations—and hence interdependence between individual items—are treated as an explicit part of the economic lot size decision. Since the background of the individual problem makes it possible to justify a number of simplifications in the mathematical model, it seems worthwhile to summarize the leading features of that background:

The plant in question sells most of its output to the armed services of the United States. The "end items" of this plant are in turn an input for other manufacturing activities, and the timing of deliveries takes on even greater importance here than in the routine production of consumers' goods. Indeed, in his competitive bid for any particular product, the manufacturer stipulates not only the price at which he will undertake to produce the item, but also the dates at which individual units will be delivered to the Government. Because the actual timing of deliveries affects the manufacturer's reputation—hence his ability to obtain future contract awards—one assumption that underlies all production planning is that the manufacturer will adhere to the promised delivery dates.

Since the plant does not produce large quantities of any individual end item, the productive process is of the batch type rather than continuous. As is typical of many metal-working establishments, the first step is to produce individual parts in the plant's own machine shop and to procure certain parts from other manufacturers. Once all the parts for a particular finished unit are on hand, these are assembled, tested, and the item is turned over to the Government. Note that if the Government contract calls for delivery of 25% of the finished units over each of four successive months, only 25% of the total requirement for each individual part needs to be available at the time actual assembly is initiated.[4] This means that one of the significant choices to be made is that of splitting production lots for individual parts so as to meet the initial delivery requirements, but still defer a portion of the machining work until the latter part of the delivery cycle. Lot-splitting does, of course, bring about an increase in setup costs, and so an optimum lot size decision entails an economic balance between the advantages of reducing setup costs, and the advantages of smoothing out the production program over time.

Although there is some overlap between planning for the machine shop and for the final assembly and testing operations, this paper is primarily concerned with the machine shop itself, and only to a secondary extent with the problems created by this overlap. Actual planning of the machine shop's activities takes place at two distinct echelons of the plant's management, and correspondingly at two different levels of abstraction. Short-range scheduling is concerned solely

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b the "incremental unit cost." Evidently setup costs are minimized by concentrating an entire production requirement into a single lot, rather than by splitting up that requirement among several lots.

[4] In practice, "buffer stock" considerations may indicate that more than 25% of the total for each part ought to be available before final assembly is initiated.

with such details as which parts are to be manufactured, and which individuals and machines are to be used. Long-range scheduling (up to eighteen months ahead) is concerned with the general problem of whether the machine shop's existing resources will be able to meet the company's future delivery commitments, and if not, what policies should be adopted to supplement the existing resources: overtime work, recruiting and training of additional personnel, and outside procurement of certain parts.[5] This paper is concerned almost entirely with the long-range problem, as distinct from the day-to-day operation.

Within the company, the traditional procedure for long-range production planning has emphasized calculations made upon the assumption that each of the parts was to be run off without splitting any of the lots—despite the fact that lot-splitting is far from a rare occurrence. Once the simplifying assumption is made, it is then largely a matter of arithmetic to take the end item delivery schedules, pool this information with the "operation sheet" machining time estimates, and arrive at the man-hour requirements for machining during each of the months in which parts are to be produced for a particular end item. Given these estimates of requirements, in turn it is possible to subtract off the straight-time man-hours available from the existing work force, and come up with a figure for the deficit or surplus of manpower over requirements. In case of an impending deficit, it is up to the long-range planning group to recommend whether to order overtime work, to attempt outside procurement for certain of the parts that would normally have been made in the company's own plant, or to alter the initially stipulated schedule of parts deliveries to the final assembly operation. In practice, of course, a tight scheduling problem will force the planning group to depart from the assumption of no split lots, and thereby to depart from minimizing setup costs.

In the linear programming calculations—just as with current methods of scheduling—two simplifying features of this particular manufacturing operation are exploited. Neither is essential to the use of linear programming, but both are highly convenient for expository purposes: (1) Limitations on the availability of specific machines have been disregarded. It will ordinarily be true that if a particular production plan stays within the limitations of the manpower available with a particular time period, the plan will also be within the capabilities of the plant's machine tool equipment. (2) Inventory-holding costs have been neglected. Physical storage costs are quite low, and the contractual arrangement of Government "progress payments" makes the interest cost element a minor one.

2. A linear programming formulation

The linear programming model of the machine shop's operations is intended to provide numerical answers to the following general problem: Given a large

[5] Decisions on the purchase of new equipment constitute an additional degree of freedom, but since the payout period for such equipment normally extends over several years, the company's policies on equipment purchase may be regarded as fixed—at least as far as an eighteen-month production schedule is concerned.

number of individual parts to be machined, and given delivery requirements for each of these parts over a series of time periods, determine how many of each of the parts should be machined in each time period—taking account of the fact that there are limits upon the amount of straight-time and of overtime productive capacity available during the individual periods, and also that lot-splitting increases the total amount of setup time required.

In determining an output schedule, the objective is assumed to be the minimization of overtime labor requirements. This criterion for choice among alternative production plans implies: (a) that the straight-time services from the projected work force represent a fixed commitment on the company's part, and that nothing can be saved by failing to use up these services; (b) that the total labor requirements fall within the man-hours available from straight-time plus overtime work so that the question of outside procurement does not arise;[6] and (c) that the only remaining variable costs are those that increase with the total number of overtime man-hours.

Underlying this linear programming formulation is the definition of an activity as a *sequence* of inputs over time that satisfies the delivery requirements for a particular part. (Individual parts are distinguished from one another by the subscript i, and the alternative sequences for the i-th part by the subscript ij.) Since, in general, there will be more than one sequence that is feasible from the viewpoint of delivery requirements for the i-th part, the linear programming variables x_{ij} refer to the fraction of the total requirement for the i-th part that is supplied by the j-th sequence of inputs for that part. Although no physical meaning can be attached to a fractional value of x_{ij}[7] (e.g., a solution specifying that half the requirements for a given part are to be met by a one-lot sequence of output and half by a split-lot sequence), there is no guarantee that such proper fractions will be absent from a linear programming solution. All that can be guaranteed is that if there are T time periods distinguished within the model, there will be at most T parts for which the x_{ij} fractions turn out to be intermediate between zero and one. (A proof of this assertion is given in section 5.)

[6] From the viewpoint of model formulation, the question of outside procurement is an inessential complication. Ordering from an outside supplier differs from internal production only in that it costs money and imposes no drain upon the internal availability of labor.

From this same viewpoint, the question of recruiting and training new personnel is also an inessential complication. An activity of this sort could be incorporated directly within the model—provided that the training cost per man was known.

[7] For the special but nonetheless interesting case in which delivery requirements recur at a steady rate over the indefinite future, there is a meaningful interpretation that can be attached to fractional values of x_{ij}—i.e., that the actual lot size be intermediate between the quantities specified in the initial definition of the x_{ij} alternatives. With this interpretation, the non-linear inventory problem with storage and capacity restrictions described by Rifas in the Churchman, Ackoff, and Arnoff volume [2, ch. 10] can be transformed into a straightforward exercise in linear programming.

Unknowns, coefficients, and constants for the programming model are defined in the following way:

(a) *unknowns*

x_{ij} = fraction of the total requirement for the i-th part to be supplied by the j-th alternative sequence of inputs. ($i = 1, \cdots, I; j = 1, \cdots, J.$)

l_t = number of hours of overtime labor required during time period $t (t = 1, \cdots, T.)$

s_t = "slack" variable for straight-time labor during time period t. (all t)

v_t = "slack" variable for overtime labor during time period t. (all t)

(b) *coefficients*

β_{ijt} = labor input required during period t in order to carry out the j-th alternative production sequence for part i. (all i, j, and t)

(c) *constants*

S_t = maximum availability of straight-time labor man-hours during the t-th time period. (all t)

V_t = maximum availability of overtime labor man-hours during the t-th time period. (all t)

With these definitions, the linear programming model becomes:

$$\text{Minimize} \sum_t l_t \tag{2.1}$$

subject to:

$$\sum_j x_{ij} = 1 \qquad (i = 1, \cdots, I) \tag{2.2}$$

$$\sum_i \sum_j \beta_{ijt} x_{ij} - l_t + s_t = S_t \qquad (t = 1, \cdots, T) \tag{2.3}$$

$$l_t + v_t = V_t \qquad (t = 1, \cdots, T) \tag{2.4}$$

$$x_{ij}, l_t, s_t, v_t \geqq 0 \qquad (\text{all } i, j, t) \tag{2.5}$$

Expression (2.1) indicates the minimand—the sum of overtime labor requirements—and conditions (2.2)–(2.5) list the constraints that must be satisfied by the unknowns x_{ij}, l_t, s_t, and v_t. Equations (2.2) say that the total requirement for the i-th part must be met by a combination of one or more sequences of production for that part. (2.3) ensures that within each time period the total number of man-hours required to satisfy the individual output programs will not exceed the amount available of straight-time labor plus the overtime to be ordered for that period. Equations (2.4) place upper bounds upon the use of overtime labor. And finally, conditions (2.5) impose the usual non-negativity requirements upon all unknowns.

To define more precisely what is meant by the x_{ij} variables and the β_{ijt} coefficients, it is easiest to refer to a three-period numerical example. ($T = 3$.) Let the specific part under discussion be part 1, ($i = 1$), and let the deliveries scheduled, the setup time, and the incremental labor requirements for that part be:

a_1 = 10 man-hours = setup time for part 1.

$b_1 \;= .9$ man-hours/part $=$ incremental labor required per unit of output of part 1.

$R_{11} = 30$ units $=$ delivery requirements for part 1 at end of period 1

$R_{12} = 30$ units $=$ delivery requirements for part 1 at end of period 2

$R_{13} = 40$ units $=$ delivery requirements for part 1 at end of period 3

With these numerical values, there are exactly four alternative sequences for labor input and parts output that need to be considered explicitly within a linear programming model.[8] These four are distinguished from one another by the index j.

j index	1	2	3	4
Number of separate lots	1	2	2	3
Delivery requirement to be produced in period 1.	$R_{11} + R_{12} + R_{13} = 100$ units	$R_{11} = 30$ units	$R_{11} + R_{12} =$ 60 units	$R_{11} = 30$ units
Delivery requirement to be produced in period 2.	0	$R_{12} + R_{13} =$ 70 units	0	$R_{12} = 30$ units
Delivery requirement to be produced in period 3.	0	0	$R_{13} = 40$ units	$R_{13} = 40$ units

It can be seen that each of the four output sequences just listed corresponds to one of the four possible combinations of periods in which a production setup occurs. $(2^{T-1} = 4)$ Once a particular combination is specified, the j-th plan is uniquely determined by the rule that each delivery requirement is to be satisfied out of production during the *nearest* preceding period in which setup costs for that part are being incurred. It is not at all self-evident that the only production sequences deserving consideration are those indicated by this rule. At a later point, however, we shall prove that this is indeed the case, and that no reduction in overall costs can be achieved by substituting other output sequences in place of these. (see Appendix.) Hence, for the three-period model, the four output plans are said to "dominate" all others. Corresponding to these alternatives, the period-by-period inputs of labor required to satisfy the delivery requirements for part 1—that is, the β_{1jt} coefficients are:

j index	1	2	3	4
x_{ij} unknown	x_{11}	x_{12}	x_{13}	x_{14}
β_{1j1}, period 1 input coefficients	$a_1 + 100\, b_1 =$ 100 man-hours	$a_1 + 30\, b_1 =$ 37 man-hours	$a_1 + 60\, b_1 =$ 64 man-hours	$a_1 + 30\, b_1 =$ 37 man-hours
β_{1j2}, period 2 input coefficients	0	$a_1 + 70\, b_1 =$ 73 man-hours	0	$a_1 + 30\, b_1 =$ 37 man-hours
β_{1j3}, period 3 input coefficients	0	0	$a_1 + 40\, b_1 =$ 46 man-hours	$a_1 + 40\, b_1 =$ 46 man-hours

[8] If the programming model distinguishes between T time periods, there will be at most $2^T - 1$ distinct combination of periods within which some production could occur—hence

3. Aggregation of individual items

It has already been emphasized that in order for the model described by (2.1)–(2.5) to be a useful one, the number of individual parts I must be quite large in relation to the number of time periods, T. (In the particular machine shop, this qualification creates few difficulties. The number of parts required for any one finished item would seldom amount to less than 100 distinct pieces.) But since the number of equations in the system equals $(2T + I)$, this also means that any conventional simplex computations of (2.1)–(2.5) would involve substantial costs. In general, there are two ways around a difficulty of this kind: One might be to construct a computing routine expressly designed to exploit the special structure of this linear programming matrix.[9] The other would be to aggregate the original model in some suitable way, obtain an optimal linear programming solution to the aggregative model, and then translate this solution back into a detailed production plan for each part. The second course is the one that will be followed here.

The aggregation principle that seems most natural for this problem is to say that two parts belong to the same production category if they have a similar ratio of setup labor to total single-lot labor time, and if they also have a similar pattern of delivery requirements. In other words:

let R_{it} = delivery requirements for part i at end of period t

a_i = setup time for part i

b_i = incremental labor required per unit of output of part i

Then two parts ($i = 1$ and 2, respectively) are said to be in the same production category k if and only if there are two factors of proportionality α and λ such that:

$$\frac{R_{2t}}{R_{1t}} = \lambda \qquad \text{(all } t\text{)} \tag{3.1}$$

---

at most $2^T - 1$ "activities" for each parts category i. Furthermore, if the first period's delivery requirements are greater than zero ($R_{i1} > 0$), this upper bound becomes 2^{T-1}. Even for $T = 6$, $2^{T-1} = 32$, an easily manageable number of activities. Although strict logic compels the enumeration of all such lot-splitting possibilities, in practice it should not be difficult to reduce the number substantially by common-sense inspection.

[9] If he examines the detached coefficients matrix associated with equations (2.2)–(2.4), the reader will observe that every basis of rank $(2T + I)$ that can be formed from this matrix may be partitioned in the following way:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where A is an *identity* matrix of rank $(T + I)$, and D is a square matrix of rank T. The matrices B and C are rectangular—the former with $(T + I)$ rows and T columns, the later with T rows and $(T + I)$ columns. The numerical difficulties connected with solving a system of such equations are much closer to the order of magnitude of T, rather than of $(2T + I)$.

and

$$\frac{a_1}{a_1 + b_1 \sum_t R_{1t}} = \frac{a_2}{a_2 + b_2 \sum_t R_{2t}} = \alpha \tag{3.2}$$

If conditions (3.1) and (3.2) hold, then the labor input coefficients for the two parts will be related to one another by a single factor of proportionality—a factor equal to the ratio of the two setup cost coefficients:

$$\frac{\beta_{2jt}}{\beta_{1jt}} = \frac{a_2}{a_1} = \frac{a_2 + b_2 \sum_t R_{2t}}{a_1 + b_1 \sum_t R_{1t}} \qquad (\text{all } j, t) \tag{3.3}$$

In other words, if conditions (3.1) and (3.2) apply, and if the j-th setup sequence is an optimal one for part 1, it will also be an optimal one for part 2. Hence there is no reason to distinguish between the two parts within a linear programming model. All that needs to be done is to adopt one of them (e.g., part 1) as a standard unit of measurement, and then to express the aggregate requirement for that class of parts in equation (2.2) as:

$$\text{aggregate requirement} = 1 + \frac{a_2}{a_1} = 1 + \frac{a_2 + b_2 \sum_t R_{2t}}{a_1 + b_1 \sum_t R_{1t}}. \tag{3.4}$$

By following this principle of aggregation, it will ordinarily be possible to make a substantial reduction in the number of equations listed in (2.2), and so reduce the burden of computations without lessening the inherent accuracy of the linear programming model.

In practice, the aggregation conditions (3.1) and (3.2) do not seem unduly stringent. Conditions (3.1) say, e.g., that if at the end of period t, 25% of the total requirement for part 1 is to become available for final assembly, then 25% of the total for part 2 must also become available at that time. When both parts are required for the same end item, the timing of delivery requirements will usually be identical, and so there should be no difficulty in constructing a small number of groups such that each part within a given class will satisfy conditions (3.1). Once this kind of preliminary grouping has been effected, it should be easy to define production categories k that also satisfy conditions (3.2)—at least to whatever degree of approximation is warranted by the goodness of the original estimates of a_i, b_i, and R_{it}. Table 1 illustrates this point for the case of one typical end item actually produced by our manufacturer—an end item requiring 110 distinct parts, each with the same pattern of delivery requirements. Here the quality of the raw data was such that the six-category classification scheme shown for these parts in Table 1 appeared entirely satisfactory for purposes of long-range production planning.

In following through the aggregation procedure just described, it seems convenient to define the unit of measurement—i.e., the "standard" part in each

production category k—as one for which the total of setup time plus single-lot running time equals one hour. The requirement for all parts in category k may then be expressed in terms of this standard as follows:

$$
\begin{aligned}
Q_k &= \text{aggregate number of ``standard'' hours' worth of parts in category } k \\
&= \sum_{i \epsilon k} (a_i + b_i \sum_t R_{it})
\end{aligned}
\tag{3.5}
$$

To summarize: Given the labor input coefficients a_i and b_i and also the delivery requirements R_{it} for each of many distinct parts i, a small number of production categories will ordinarily suffice for the purpose of an aggregative linear programming model. Furthermore, once an optimal solution has been calculated, there should be no difficulty in translating the aggregative results back into a detailed plan for the production of each distinct part.

What makes such a translation possible? E.g., what if the linear programming solution called for 1,000 hours' worth of parts in a given category to be produced in a single lot, and 1,000 hours' worth by a split-lot plan? It is perfectly true that no sense could be made of a detailed plan that called for producing half of *every* distinct part with a single-lot program and half with split lots. But it would make perfectly good sense to translate the aggregative solution into a detailed plan that called for producing one distinct group of parts according to the single-lot plan and another group according to the split-lot plan—provided that the total "standard" time for parts in each of these two groups came to 1,000 hours apiece. The whole trick consists of observing that when the number of distinct parts is large, and that when one is dealing with groups of such parts, the alternative production programs are not mutually exclusive, and that under these conditions, one can always spell out a meaningful detailed plan for any convex combination of the stated alternatives.

Whenever the number of distinct parts in a production category exceeds more than a handful, there should be no serious difficulty in translating the aggre-

TABLE 1

A system of aggregation for 110 distinct parts, as classified by setup labor ratio α_i

Production category k	Class interval for the setup labor ratio $\alpha_i = \frac{a_i}{a_i + b_i \Sigma_t R_{it}}$ (all $i \epsilon k$)	Number of distinct parts within category k	Maximum number of "standard" hours required for any single part in category k	Total "standard" hours required for all parts in category $k = Q_k$
1	$0 \leqq \alpha_i < .10$	9 parts	1,046 man-hours	4,064 man-hours
2	$.10 \leqq \alpha_i < .20$	33 parts	567 man-hours	4,774 man-hours
3	$.10 \leqq \alpha_i < .30$	32 parts	176 man-hours	2,097 man-hours
4	$.30 \leqq \alpha_i < .40$	21 parts	90 man-hours	654 man-hours
5	$.40 \leqq \alpha_i < .50$	11 parts	66 man-hours	286 man-hours
6	$.50 \leqq \alpha_i < 1.00$	4 parts	34 man-hours	98 man-hours
Total.....		110 parts		11,973 man-hours

gative solution back into a detailed program for the output of each part.[10] The formulation of the model ensures that not only will the total parts requirement be satisfied in terms of "standard" units, but also that the production of each of these parts can be time-phased in such a way as to satisfy the initially stipulated delivery requirements.

4. A numerical example

This illustrative example will refer to a case involving three time periods and five production categories. In following through the calculations, the first step is to obtain numerical values for the setup time ratios α_k and the percentage delivery requirements $R_{kt} \div \sum_t R_{kt}$ within each of the five production categories k. These parameters, along with the constants Q_k, S_t, and V_t, are all listed in Table 2. With this information available, it is then a straightforward matter to construct the β_{kjt} labor input coefficients, and then the matrix of detached coefficients (Table 3) for the linear programming model indicated abstractly by conditions (2.1)–(2.5).[11] The only change introduced by the aggregation procedure is the replacement of the index i by the index k ranging in value from 1 to K. That is, instead of x_{ij} variables which represent the fraction of the requirement for the i-th part supplied by the j-th alternative sequence, we now have x_{kj} variables which represent the total number of "standard" hours' worth of parts in category k that are to be produced by the j-th sequence. Along with this change, it is, of course, necessary to replace the constants of unity in equations (2.2) with the Q_k, the total number of "standard" hours' worth of parts required in category k.

Altogether this system involves 25 unknowns and 11 equations. Of the unknowns, 16 are of the x_{kj} type, and there are three each of the l_t, s_t, and v_t type.[12] Since the matrix shown in Table 3 indicates non-zero coefficients only, the first row of that matrix (numbered 0) contains just three entries—the cost

[10] If the reader insists upon some precision in the definition of a "handful," and if he is willing to recognize that the a_i, b_i, and R_{it} parameters are each a bit fuzzy, I would venture the guess that no real translation difficulties will occur if the number of distinct parts within a given category k exceeds 10, and if the maximum time required for any single part in a given category is less than 20% of the total. On this score, see Table 1.

[11] Except for the position of the decimal point, the β_{1jt} that appear in Table 3 are identical with those calculated on p. 120 above. All other β_{kjt} were obtained by a similar process.

[12] The reader may wonder why only two alternative programs ($j = 1$ and 2) are listed for parts categories 4 and 5. In strict logic, even though no delivery requirements for these parts exist during period 1, one should still consider the possibility of producing them during that period as well as during 2 and 3. But period 1 production of these items would only be profitable if, in an optimal solution, the "shadow price" associated with labor in period 1, turned out to be lower than that associated with labor in period 2. Since the *a priori* considerations were against this outcome, all activities corresponding to positive amounts of period 1 output were omitted from the linear programming tableau shown in Table 3. As things worked out, the optimal solution substantiated these conjectures, and so nothing was lost by discarding the possibility of period 1 output for parts categories 4 and 5.

TABLE 2

Parameters and constants for the numerical example

	Parts category k				
	1	2	3	4	5
α_k	.1	.2	.3	.2	.3
$R_{k1} \div \Sigma_t R_{kt}$	.3	.3	.3	0	0
$R_{k2} \div \Sigma_t R_{kt}$	.3	.3	.3	.4	.4
$R_{k3} \div \Sigma_t R_{kt}$	.4	.4	.4	.6	.6
Q_k	3,500	4,100	2,900	4,800	3,200
Time period, t		S_t	V_t		
1		6,000	1,500		
2		6,000	1,500		
3		6,000	1,500		

coefficients associated with the three overtime labor variables l_t in the minimand, expression (2.1). Following the minimand, the next five rows correspond to equations (2.2), the requirements for output in each of the five parts categories. Then come the three rows numbered 6, 7, and 8—one for each time period—constraining the input of labor to fall within the number of man-hours available. Rows 6, 7 and 8 correspond therefore to equation group (2.3). And finally, the last three rows (numbers 9, 10, and 11) coincide with equations (2.4)—the upper bound conditions upon the use of overtime work in any one time period.

Using the simplex method of calculation, and taking advantage of the special structure of the matrix shown in Table 3, it proved to be an easy matter to calculate an optimal solution to this model, and to determine that the optimum was unique. The optimal solution, along with the corresponding "shadow prices" or "dual variables," is shown in Table 4. According to this solution, it pays to use a one-lot production plan for the output of every part in categories 3 and 5. (Since these two categories are the ones for which the setup cost parameter α_k is largest, this outcome is an entirely reasonable one.) All parts in category 2 are to be produced in two lots—60% of the output in period 1, the remainder in period 3, and none at all in period 2. And in the case of both categories 1 and 4, it pays to combine two lot-splitting plans. That is, 1,915 "standard" hours' worth of parts in the first category are to be produced by splitting production between time periods 1 and 3, and the remaining parts in that category by splitting production among all three time periods. Similarly, 1,479 hours' worth of parts in category 4 are to be turned out in a single lot during period 2, and the remaining output of 3,321 is to be obtained by splitting production between periods 2 and 3.

The dual variables for each equation (u_1, u_2, $\cdots$, u_{11}) measure the potential change in the minimand (overtime labor cost) per unit change in the constant associated with that equation. An extra hour's worth of straight-time labor available in period 2, for example, would make it possible to reduce the total amount

TABLE 3

Matrix of detached coefficients. A numerical example of the linear programming model (2.1)–(2.5)

Equations \ Unknowns	x_{kj}																l_t			s_t			v_t			Constants Q_k, S_t, V_t
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{21}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{51}	x_{52}	l_1	l_2	l_3	s_1	s_2	s_3	v_1	v_2	v_3	
0) Cost coefficients																	1.00	1.00	1.00							— (0
1) Output of parts, category 1	1.00	1.00	1.00	1.00																						3,500 (1
2) Output of parts, category 2					1.00	1.00	1.00	1.00																		4,100 (2
3) Output of parts, category 3									1.00	1.00	1.00	1.00														2,900 (3
4) Output of parts, category 4													1.00	1.00												4,800 (4
5) Output of parts, category 5															1.00	1.00										3,200 (5
6) Labor input, period 1	1 00	.37	.64	.37	1.00	.44	.68	.44	1.00	.51	.72	.51					−1.00			1.00						6,000 (6
7) Labor input, period 2		73		.37		.76		.44		.79		.51	1.00	.52	1.00	.58		−1.00			1.00					6,000 (7
8) Labor input, period 3			.46	.46			.52	.52			.58	.58		.68		.72			−1.00			1.00				6,000 (8
9) Overtime, period 1																	1.00						1.00			1,500 (9
10) Overtime, period 2																		1.00						1.00		1,500 (10
11) Overtime, period 3																			1.00						1.00	1,500 (11

TABLE 4

Values of dual variables and of non-zero primal variables in the optimal solution

a) *Non-zero primal variables*

Parts category i	Initial parameters and constants		Values of l_{ki} variables, in "standard" hours		
	α_k	Q_k	One-lot plans	Two-lot plans	three-lot plans
1	.1	3,500 hours	—	$x_{13} = 1{,}915$	$x_{14} = 1{,}585$
2	.2	4,100 hours	—	$x_{23} = 4{,}100$	—
3	.3	2,900 hours	$x_{31} = 2{,}900$	—	—
4	.2	4,800 hours	$x_{41} = 1{,}479$	$x_{42} = 3{,}321$	—
5	.3	3,200 hours	$x_{51} = 3{,}200$	—	

Time period t	man-hours		
	l_t (overtime)	s_t (slack)	v_t (slack)
1	1,500	—	—
2	992	—	508
3	—	—	1,500
Minimand = Σl_t =	2,492		

b) *Dual variables*, change in minimand per unit change in value of the constant associated with the particular equation.

Output requirement equations (2.2)

u_1 = 1.202 overtime hours/"standard" hour's worth of parts, category 1
u_2 = 1.299 overtime hours/"standard" hour's worth of parts, category 2
u_3 = 1.370 overtime hours/"standard" hour's worth of parts, category 3
u_4 = 1.000 overtime hours/"standard" hour's worth of parts, category 4
u_5 = 1.000 overtime hours/"standard" hour's worth of parts, category 5

Labor availability equations (2.3)

u_6 = −1.370 overtime hours/hour's worth of straight-time labor in time period 1
u_7 = −1.000 overtime hours/hour's worth of straight-time labor in time period 2
u_8 = −.706 overtime hours/hour's worth of straight-time labor in time period 3

Overtime limitation equations (2.4)

u_9 = −.370 overtime hours/hour's worth of overtime labor in time period 1
u_{10} = 0 overtime hours/hour's worth of overtime labor in time period 2
u_{11} = 0 overtime hours/hour's worth of overtime labor in time period 3

of overtime by exactly one hour. Hence $u_7 = -1$. But an extra hour available in period 1 could be employed so as to avoid a substantial amount of lot-splitting, and for this reason $u_6 = -1.370$.[13] Such values are immediately suggestive of

[13] Although it will not necessarily always be true that $u_6 \leqq u_7 \leqq u_8 \leqq 0$, this ranking will hold whenever: (a) inventory costs are negligible, and (b) the output sequences are defined so that production of each item is permitted in any of the time periods prior to delivery.

"break-even" points for the worth of additional labor in the machine shop beyond the amounts already assumed available. Similarly the dual variables u_1, $\cdots$, u_5—those associated with the output requirement equations (1)–(5)—are indicative of the incremental worth of any external supply of parts in each of these five categories.

One theorem about the properties of the first five dual variables will be asserted without proof: If all parts within two categories satisfy condition (3.1) but not (3.2), and if the first category's setup time parameter is lower than that of the second, then the "implicit cost" of meeting an additional hour's worth of output requirements in the first category will be no higher than that of an hour's worth in the second category. Hence $0 \leqq u_1 \leqq u_2 \leqq u_3$. Also $0 \leqq u_4 \leqq u_5$.

If one were concerned purely with the formal aspects of this economic lot size problem, the discussion of the numerical example could end at this point. Given the optimizing criterion and the constraints listed in (2.1)–(2.5), an optimal solution has been produced for the aggregative scheduling problem, and in principle it has been shown how this could be translated back into a detailed time-phased plan for the output of each distinct part. But if one's interest is with the actual managerial problem that is represented by this model, something more needs to be said. In our idealization of the machine shop's activities, all interactions have been neglected between the machine shop and the final assembly area. In particular, we have ignored the possibility that by splitting parts production within the machine shop, we may disrupt the smooth flow of final assembly work on any one series of end items. To the extent that this intermittent pattern of final assembly costs more than a continuous flow of work, the "suboptimization" calculated for the machine shop is a misleading one. This does not mean that the linear programming analysis is useless—only that the results of this analysis have to be integrated with what is also known about the final assembly operation.

Here, for example, one of the men actually responsible for production planning suggested that it might be possible to transfer skilled final assembly machinists from their usual jobs, and to bring them temporarily into the machine shop to help meet the initial period's peak demand there. Since this proposal would make it possible to avoid all lot-splitting, it contains several attractive features—not only the obvious reduction in setup costs,[14] but also the very real benefits to be derived in the final assembly area by having 100% of every part available at the time that final assembly is initiated for any one series of end items. Against both of these prospective benefits, it is, of course, also necessary to evaluate the immediate cost of disrupting final assembly activities by such a temporary transfer. The linear programming analysis of the machine shop cannot by itself indicate that such transfers would be in the best interests of the plant as a whole, but it can at least indicate the order of magnitude of the direct

[14] If manpower were available early enough to make single-lot production possible for all parts, the actual time of 20,492 man-hours $(\Sigma S_t + \Sigma l_t)$ could be reduced to the "standard" time of 18,500 hours $(Q_1 + Q_2 + Q_3 + Q_4 + Q_5)$. The excess labor requirement for split-lot production amounts therefore to 1,992 man-hours.

labor savings inherent in single-lot production of all parts. Surely this calculation is not the only thing relevant to the question of whether workers ought to be transferred temporarily, but it does represent one of the pieces of information needed in order to arrive at a sound decision.

5. A theorem on the occurrence of fractional values for the x_{ij} variables

At an earlier point, it was convenient to assert without proof that the applicability of the linear programming proposal did not depend upon the possibility of aggregating distinct parts into the output categories defined by (3.1) and (3.2), but only upon the existence of a large number of distinct parts i—each of them with a small labor input requirement by comparison with the total availability of labor. The precise form of this assertion is as follows: Consider the model described by (2.1)–(2.5). Then if there are I parts and T time periods, in every basic feasible solution there will be at least $(I - T)$ parts for which exactly one x_{ij} variable is operated at a positive level. Thus, except for at most T parts, the linear programming solution will immediately indicate a detailed feasible time-phased plan for the output of each item in the machine shop. For each of these T parts, there is indeed the possibility that the linear programming solution will require half the lot to be produced according to a one-lot plan and half according to a split-lot plan. The physical absurdity of such a solution is obvious, but if T is sufficiently small in relation to I, the few parts that will be affected should cause no difficulty from the viewpoint of long-range planning. Another way to state this result is to say that when the number of parts to be scheduled far exceeds the number of individual time periods (a reasonable enough assumption when one end item alone may contain 110 distinct components), the very multiplicity of parts acts in such a way as to smooth out the "lumpiness" associated with setup costs.

The preceding theorem may be restated as follows:

(5.1) If, in a basic feasible linear programming solution to the model indicated by (2.1)–(2.5), there are m parts for which exactly one x_{ij} variable appears at a positive intensity in that solution, then $m \geqq I - T$.

Proof. Let n represent the number of the variables l_t and v_t operated at positive levels in the particular basic feasible solution. (In order for equations (2.4) to be satisfied, $n \geqq T$.) The expression $(I - m)$ represents the number of parts for which two or more x_{ij} variables are operated at a positive intensity in the particular solution. Now since there are altogether $(2T + I)$ restraint equations listed in this model, at most $(2T + I)$ variables will appear at a positive level in the solution. I.e.:

$$2T + I \geqq n + m + 2(I - m) \tag{5.2}$$

and since $n \geqq T$

$$T + I \geqq m + 2(I - m) \tag{5.3}$$

$$\therefore m \geqq I - T, \tag{5.4}$$

which was to be proved.

6. Summary

This paper may be recapitulated as follows: Starting with a production scheduling problem that involves indivisibilities in the form of setup costs, a linear programming model has been constructed that is not identical with the original problem, but which provides an excellent approximation when the number of distinct parts is large in comparison with the number of time periods, T. In this approximation to the original problem, the variables do not refer to the size of each production lot within each time period, but rather to the fraction of the total requirement for any given part that is satisfied by a particular sequence of production for that part. The linear programming formulation ensures that, except for at most T individual parts, these fractions will all turn out to be either zero or one. With this exception, therefore any "basic feasible" solution will automatically avoid the possibility of meeting one portion of the requirement for a given part by a one-lot program of output and another portion of the requirements with a split-lot program. Although this physically absurd option is built into the model, a theorem ensures that the option will be exercised only rarely.

How serious a distortion of reality is implied by a linear programming solution that calls for the production of a few parts in this physically absurd manner? From a purely abstract standpoint, such a solution is completely infeasible, and it is easy to construct numerical examples for which the linear programming solution could not be "patched up" without a large increase in the total system costs. Despite this perfectly valid formal objection, it may seriously be doubted that this difficulty really detracts from the usefulness of the model. The detailed optimal solution to such a model is hardly intended as a literal forecast of production activities up to eighteen months in the future, but only as a guide to making a number of immediate decisions that will affect the future—overtime, recruiting and training of new personnel, and outside procurement of certain parts. For the purpose of choosing among these broad alternatives—although not for the detailed short-run scheduling problem—the few apparent infeasibilities should be of minor significance.

This same line of reasoning should do much to dispel another kind of objection that may be raised against the model presented here. The usefulness of this proposal depends upon the magnitude of the number of distinct time periods, T. Since the number of alternate production activities to be enumerated for a single part category is of the order of 2^T, the capacity of current computing equipment would not be taxed by a model with $T \leqq 8$, but would clearly be swamped for $T \geqq 15$.[15] Certainly there is no guarantee that it will always be satisfactory to plan production over an 18-month period in time units as large as one to three months. Indeed a determined critic would be within his rights in pointing out that it might be necessary to plan a single

[15] Even for large values of T, it would still be possible to enumerate just a small number of the more interesting alternative production programs for any one part. A solution based upon such an incomplete enumeration would still be a feasible one, and should be near-optimal—even though no *a priori* guarantees can be made as to its optimality.

year's operation ahead in terms of 365 individual time units—each one day in length. The answer to such a critic can only come from a study of the empirical problem to which the model is to be applied. Assuming that the purpose of the model is to aid in answering certain broad questions dealing with overtime, outside procurement, etc., it should not be a serious limitation upon the problem formulator for him to keep the value of T well within the limits of present-day computing feasibility.

7. Significance of the results

The production scheduling example discussed in this paper is by no means an isolated instance in which, starting with a problem that entailed indivisibilities in terms of one set of variables, it was nevertheless possible to redefine the variables so as to transform the original problem into a new one that could be studied from the computational viewpoint of linear programming. This same approach has already been illustrated in the newsprint trim problem [5], in the coat-and-pants problem [3], in Salveson's machine loading problem [6, pp. 234–245], and doubtless in others. There appears to be an entire class of optimization problems that involve indivisibilities in terms of one set of variables, but which can nevertheless be translated into the linear programming format. Some precise characterization of this class of problems seems to be needed, but is lacking at present.

Although the economist's primary interest is not in numerical analysis, but rather in the possibility of market analogue solutions to welfare maximization problems, the indivisibility of setup costs places him in an awkward position. As long as he regards the individual "activity" as one of determining the lot size for a given part in a particular time period, there need be no set of intra-firm shadow prices that is compatible with a cost-minimizing equilibrium, and hence no possibility of a market analogue solution. The curious aspect of the production problem outlined here is that it is possible to redefine activities and commodities so as to end up with a linear programming system—i.e., one for which, in principle, a market analogue solution is possible. From the viewpoint of the theory of market decentralization, the chief feature of this alternative version is that the individual activities represent a greater degree of vertical integration than is assumed in the initial statement of the problem. Paradoxically enough, successful decentralization requires that the manager of each activity have a longer "span of control" than the size of the individual lot in a particular time period. It is necessary for each such manager to be familiar with the entire program of labor inputs that is implied by his particular sequence of output for the individual part.

Appendix

"Dominance" properties of the set of alternative production programs for a given item

The machine shop is engaged in producing a number of items with a resource input that is homogeneous except for date. If an item is produced in the t-th

time period ($t = 1, 2, \cdots T$), resource inputs are required from the total available in that period, but from no other. The amount of resources used in the t-th period by producing x_t units of a particular item is given by:

$$a\delta_t + bx_t \tag{1}$$

where

$$x_t \begin{pmatrix} >0 \\ \\ =0 \end{pmatrix} \quad \text{implies} \quad \delta_t \begin{pmatrix} =1 \\ \\ =0 \end{pmatrix}$$

The non-negative constant a is said to represent the "setup cost" for that item and the non-negative constant b the "incremental unit cost." Since $\delta_t = 0, 1$, there are altogether 2^T column vectors of the following form:

$$\Delta_j = \begin{bmatrix} \delta_{j1} \\ \delta_{j2} \\ \vdots \\ \delta_{jt} \\ \vdots \\ \delta_{jT} \end{bmatrix} \tag{2}$$

Now suppose that the firm is to deliver R_t units of the item in the t-th period. Corresponding to each of the 2^T vectors Δ_j, the time phased production vector X_j may be written, where:

$$X_j = \begin{bmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{jt} \\ \vdots \\ x_{jT} \end{bmatrix} \tag{3}$$

and where the output levels x_{jt} are determined according to either (4), (5), or (6). These conditions are equivalent to the rule that each delivery requirement be satisfied out of production during the nearest preceding period in which setup costs are being incurred:

(4) if $\delta_{jt} = 0$, then $x_{jt} = 0$

(5) if $\delta_{jt} = \delta_{j,t+1} = 1$, then $x_{jt} = R_t$

(6) if $\delta_{jt} = 1$, $\delta_{j,t+1} = 0$, and if $\bar{\gamma}$ is the largest integer such that $\delta_{j,t+\gamma} = 0$ for $\gamma = 1, \cdots, \bar{\gamma}$, then $x_{jt} = \sum_{\gamma=0}^{\bar{\gamma}} R_{t+\gamma}$.

The setup plan Δ_j and the corresponding output plan X_j are said to be "feasible" from the viewpoint of delivery requirements if the components of X_j also satisfy:

$$\sum_{\tau=1}^{t} x_{j\tau} \geqq \sum_{\tau=1}^{t} R_\tau \qquad (t = 1, 2, \cdots, T-1) \tag{7a}$$

and

(7b)
$$\sum_{\tau=1}^{T} x_{j\tau} = \sum_{\tau=1}^{T} R_\tau$$

For each of the "feasible" Δ_j and X_j vectors, the resource input column vector β_j may be defined as follows:

(8)
$$\beta_j = a\Delta_j + bX_j \qquad (j = 1, \cdots, J)$$

The $T \times J$ matrix B is composed of the vectors β_j :

(9)
$$B = (\beta_1, \beta_2, \cdots, \beta_j, \cdots, \beta_J)$$

Now let the "implicit value" or "shadow price" of any resources used in the t-th time period be represented by $u_t \cdot (u_t \leqq 0;\ u_t \gtrless u_{t-1})$. The column vector formed from these components is termed U:

(10)
$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_t \\ \vdots \\ u_T \end{bmatrix}$$

Suppose that a setup plan Δ_r and a production plan X_r satisfy conditions (11)–(13):

(11)
$$x_{rt} \begin{pmatrix} >0 \\ \\ =0 \end{pmatrix} \text{ if and only if } \delta_{rt} \begin{pmatrix} =1 \\ \\ =0 \end{pmatrix}$$

(12a)
$$\sum_{\tau=1}^{t} x_{r\tau} \geqq \sum_{\tau=1}^{t} R_\tau \qquad (t = 1, 2, \cdots, T-1)$$

(12b)
$$\sum_{\tau=1}^{T} x_{r\tau} = \sum_{\tau=1}^{T} R_\tau$$

and

(13)
$$x_{rt} \geq 0 \qquad (\text{all } t)$$

Denote by β_r the vector of resource inputs that is required in order to carry out this production plan:

(14)
$$\beta_r = a\Delta_r + bX_r$$

"Dominance" theorem: If the vector $U \leqq 0$, there is no pair of vectors Δ_r and X_r satisfying conditions (11)–(13) for which it is also true that:

(15)
$$U'\beta_j < U'\beta_r \leqq 0 \qquad (\text{all } \beta_j \in B)$$

In words, this theorem says that if the production program X_r is feasible from the viewpoint of delivery requirements, then there will always be at least one program X_j within the previously enumerated set that has an implicit cost at

least as low as that for program X_r. This is the sense in which the set of resource input vectors B is said to "dominate" all others.

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[R. E. Levitan pointed out ("A Note on Professor Manne's 'Dominance' Theorem," *Management Science*, Vol. 5, No. 3 (April 1959), 332) that the proof of the theorem given in the original paper was in error. For this reason the original proof is omitted. Levitan supplied a proof of the theorem, and in addition noted that the theorem is a slightly generalized version of theorems 1, 2, 3 in Wagner and Whitin's paper in Chapter 16. The generalization is to the case where the unit costs are constant but not identical in each period. Wagner and Whitin's theorems 1–3 and their proofs apply in this case if the second sentence in the proof of theorem 1 is replaced by the following: "Let s be the largest integer less than t for which $x_s > 0$. If the sum of the unit purchase cost in period s and unit inventory carrying costs in periods $s, \cdots, t-1$ equals or exceeds (is less than) the unit purchase cost in period t, then it is no more costly to increase (decrease) purchases by ϵ in period t and decrease (increase) them by ϵ in period s where $\epsilon = \min(I, x_s, x_t)$. Repetition of the construction produces an alternative optimal program satisfying the assertion of the theorem."
—The Editor.]

ON A DYNAMIC PROGRAMMING APPROACH TO THE CATERER PROBLEM—I*

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Summary

In this paper, it is shown that the "caterer" problem, a problem in mathematical economics and logistics which has been discussed by Jacobs, Gaddum, Hoffman and Sokolowsky, and Prager, can be reduced to the problem of determining the maximum of the linear form $L_n = \sum_{i=1}^{n} v_i$, subject to a series of constraints of the form $v_1 \leqq b_1$, $v_1 + v_2 \leqq b_2$, $v_1 + v_2 + v_3 \leqq b_3, \cdots$, $v_1 + v_2 + \cdots + v_k \leqq b_k$, $v_2 + v_3 + \cdots + v_{k+1} \leqq b_{k+1}, \cdots, v_{n-k+1} + \cdots + v_n \leqq b_n$, $0 \leqq v_i \leqq r_i$, $i = 1, 2, \cdots, n$, under an assumption concerning the non-accumulation of dirty laundry.

This maximization problem is solved explicitly, using the functional equation technique of dynamic programming.

1. Introduction

The purpose of this paper is to show how the functional equation method of dynamic programming may be used to obtain an explicit solution of the problem of determining the maximum of the linear form

$$L_n(v) = v_1 + v_2 + \cdots + v_n, \tag{1}$$

over all v_i subject to the constraints

$$\text{(2)}\quad \text{(a)}\qquad r_i \geqq v_i \geqq 0$$

$$\text{(b)}\qquad \begin{aligned} v_1 &\leqq b_1 \\ v_1 + v_2 &\leqq b_2 \\ &\vdots \\ v_1 + v_2 + \cdots + v_k &\leqq b_k \\ v_2 + v_3 + \cdots + v_{k+1} &\leqq b_{k+1} \\ &\vdots \\ v_{n-k+1} + \cdots + v_n &\leqq b_n. \end{aligned}$$

The origin of this problem lies in the "caterer" problem, a problem of some interest in recent years in connection with economic, industrial and military scheduling problems.

2. Discussion

A large number of mathematical models of economic activities culminate in the problem of determining the maximum or minimum of a linear function sub-

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ject to a set of linear constraints. The importance of having available computational algorithms for the numerical resolution of these problems can hardly be over-estimated, both as far as application of the results are concerned and as far as further theoretical study is concerned. Foremost of these algorithms is the "simplex" method of Dantzig, together with its modifications by Charnes, Lemke, Beale, and others.

In the study of universal methods, insufficient attention has been paid to the underlying structure of the processes generating the minimization and maximization problems. Ideally what is desired is a systematic fitting to each process of a computational algorithm specifically designed for the process. There has been barely a start made in the mathematical theory of computational algorithms; cf. the discussion in [1]. In particular, little effort has been devoted to the question of analytic solution of these minimization and maximization problems.

In this paper we wish to consider the interesting minimization problem posed above using the functional equation approach of dynamic programming, [2]. The problem from which it is derived, the "caterer" problem has been discussed by a number of mathematicians over the last few years, see Jacobs, [3], Gaddum, Hoffman and Sokolowsky, [4], and Prager, [5].

Our interest in the possibility of an explicit solution of the type we present here was aroused by the solution obtained by O. Gross in the case where $k = 2$.

3. The Caterer Problem

Let us now state the caterer problem in the following form: (cf. Jacobs, [3], Prager, [5])

"A caterer knows that in connection with the meals he has arranged to serve during the next n days, he will need r_j fresh napkins on the jth day, $j = 1, 2, \cdots, n$. There are two types of laundry service available. One type requires p days and costs b cents per napkin; a faster service requires q days, $q < p$, but costs c cents per napkin, $c > b$. Beginning with no usable napkins on hand or in the laundry, the caterer meets the demands by purchasing napkins at a cents per napkin. How does the caterer purchase and launder napkins so as to minimize the total cost for n days?"

As is known from the above references, and also J. W. Caddum, A. J. Hoffman and D. Sokolowsky, [4], this problem can be resolved by linear programming techniques in some cases.

In this paper we shall approach the problem using the approach of dynamic programming.

4. Dynamic Programming Approach—I

The first approach to the problem by means of dynamic programming proceeds as follows. The state of the process at any time may be specified by the stage, i.e. day, and by the number of napkins due back from the laundry in 1, 2, up to p days hence. On the basis of that information, we must make a decision as to how many napkins to purchase, and how to launder the accumulated dirty napkins.

It is not difficult to formulate the problem in this way, using the functional equation approach. Unfortunately, if p is large, we founder on the shoals of dimensionality.

As we shall see, the proper dimensionality of the problem is $p - q$, when formulated in a different manner.

5. Formulation of Problem

In place of this approach, let us proceed with the equations defining the process in the usual way until an appropriate point at which we shall reintroduce the dynamic programming approach.

It is first of all clear from the above formulation of the problem that we may just as well purchase all the napkins at one time at the start of the process. Let us then begin by solving the simpler problem of determining the laundering process to employ given an initial stock of S napkins. Clearly

(1) $$S \geqq \max_k r_k .$$

Let us now make a simplifying assumption that all the dirty napkins returned at the end of each day are sent out to the laundry, either to the fast service or to the slow service. There are many justifications for this assumption as far as applications are concerned, which we shall not enter into at the moment.

The process then continues as follows. At the end of the k^{th} day, the caterer divides r_k, the quantity of dirty napkins on hand, into two parts, $r_k = u_k + v_k$, with u_k sent to the q-day laundry and v_k sent to the p-day laundry.

Continuing in this way, we see that the quantity, x_k, of clean napkins available at the beginning of the k^{th} day is determined by the following recurrence relation,

(2) $$x_1 = S,$$
$$x_k = (x_{k-1} - r_{k-1}) + u_{k-q} + v_{k-p}$$

where $u_k = v_k = 0$ for $k \leqq 0$.

The cost incurred on the k^{th} day is

(3) $$bv_k + cu_k \qquad k = 1, 2, \cdots, N - 1$$

Hence the total cost is

(4) $$C_N = b \sum_{k=1}^{N-1} v_k + c \sum_{k=1}^{N-1} u_k .$$

The problem is to minimize C_N subject to the constraints on the u_k

(5) (a) $$0 \leqq u_k \leqq r_k$$

(b) $$x_k \geqq r_k , \qquad k = 1, 2, \cdots, N.$$

In order to illustrate the method, we shall consider two particular cases.

(6) a. $$q = 1, \qquad p = 2$$

b. $$q = 1, \qquad p = 3$$

The general case will be discussed following this.

6. The case $q = 1, p = 2$

The equations in (5.2) assume the form

$$
\begin{aligned}
x_1 &= S \\
x_2 &= (x_1 - r_1) + u_1 \\
x_3 &= (x_2 - r_2) + u_2 + v_1 \\
&\vdots \\
x_{n-1} &= (x_{n-2} - r_{n-2}) + u_{n-2} + v_{n-3}, \\
x_n &= (x_{n-1} - r_{n-1}) + u_{n-1} + v_{n-2}
\end{aligned} \tag{1}
$$

Let us now solve for the x_k in terms of the u_k and v_k. Namely

$$
\begin{aligned}
x_1 &= S \\
x_2 &= (S - r_1) + u_1 \\
x_3 &= (S - r_1 - r_2) + (u_1 + u_2) + v_1 \\
x_4 &= (S - r_1 - r_2 - r_3) + (u_1 + u_2 + u_3) + v_1 + v_2, \\
&\vdots \\
x_{n-1} &= (S - r_1 - r_2 - \cdots - r_{n-2}) + (u_1 + u_2 + u_3 + \cdots + u_{n-2}) \\
&\qquad + (v_1 + v_2 + \cdots + v_{n-3}) \\
x_n &= (S - r_1 - r_2 - \cdots - r_{n-1}) + (u_1 + u_2 + u_3 + \cdots + u_{n-1}) \\
&\qquad + (v_1 + v_2 + \cdots + v_{n-2}).
\end{aligned} \tag{2}
$$

Since $r_k = u_k + v_k$, this may be written

$$x_k = S - v_{k-1}, \qquad (v_0 = 0), \qquad k = 1, 2, \cdots, n. \tag{3}$$

Turning to (5.4), we wish to minimize

$$C_N = c \sum_{k=1}^{N-1} r_k + (b - c) \sum_{k=1}^{N-1} v_k, \tag{4}$$

over all v_k subject to the constraints

(5) (a) $$0 \leqq v_k \leqq r_k$$

(b) $$S - v_{k-1} \geqq r_k \quad \text{or} \quad S - r_k \geqq v_{k-1}.$$

Since $(c - b) > 0$, we wish to choose v_k as large as possible. Hence

$$v_k = \min (r_k, S - r_{k+1}), \qquad k = 1, 2, \cdots, N - 1. \tag{6}$$

This determines the structure of the optimal policy. Using this explicit form of the solution it is not difficult to determine the minimizing value of S.

7. The case $q = K, p = K + 1$

It is readily seen upon writing down the equations that the case $q = K$, $p = K + 1$ leads to a system of equations of the same type as given above for $q = 1, p = 2$. This illustrates the fact that it is only the difference $p - q$ which determines the level of difficulty of the problem.

8. The Case $q = 1, p = 3$

In order to illustrate the method which is applicable to the general case, let us consider the case $q = 1, p = 3$.

The equations in (4.2) assume the form

$$
\begin{aligned}
x_1 &= S, \\
x_2 &= x_1 - r_1 + u_1, \\
x_3 &= x_2 - r_2 + u_2, \\
x_4 &= x_3 - r_3 + u_3 + v_1 \\
&\vdots \\
x_n &= x_{n-1} - r_{n-1} + u_{n-1} + v_{n-3}
\end{aligned} \tag{1}
$$

Thus

$$
\begin{aligned}
x_1 &= S, \\
x_2 &= S - r_1 + u_1, \\
x_3 &= (S - r_1 - r_2) + u_1 + u_2, \\
x_4 &= (S - r_1 - r_2 - r_3) + u_1 + u_2 + u_3 + v_1 \\
&\vdots \\
x_n &= (S - r_1 - r_2 - r_3 - \cdots - r_{n-1}) + u_1 + u_2 + u_3 + \cdots + u_{n-1} \\
&\qquad + v_1 + v_2 + \cdots + v_{n-3}.
\end{aligned} \tag{2}
$$

Hence

$$
\begin{aligned}
x_1 &= S \\
x_2 &= S - v_1 \\
x_3 &= S - v_1 - v_2 \\
x_4 &= S - v_2 - v_3 \\
&\vdots \\
x_n &= S - v_{n-2} - v_{n-1}.
\end{aligned} \tag{3}
$$

We wish to maximize $\sum_{k=1}^{N-1} v_k$ subject to the constraints

$$
\begin{aligned}
S - v_1 &\geqq r_1, & & & S - r_1 &\geqq v_1 \\
S - v_1 - v_2 &\geqq r_2 & &\text{or} & S - r_2 &\geqq v_1 + v_2 \\
&\vdots & & & &\vdots \\
S - v_{n-2} - v_{n-1} &\geqq r_{n-1} & & & S - r_{n-1} &\geqq v_{n-2} + v_{n-1}
\end{aligned} \tag{4}
$$

and

(5) $$0 \leqq v_i \leqq r_i$$

9. Dynamic Programming Formulation—II

Our problem reduces to that of maximizing the linear form

(1) $$L_N = \sum_{k=1}^{N} v_k,$$

subject to a set of constraints of the form

(2) $$\text{(a)} \quad \begin{aligned} b_1 &\geqq v_1, \\ b_2 &\geqq v_1 + v_2, \\ &\vdots \\ b_N &\geqq v_N + v_{N-1} \end{aligned} \qquad \text{(b)} \quad r_k \geqq v_k \geqq 0.$$

Having chosen v_1, it is clear that we have a problem of precisely the same type remaining for the other variables $v_2, v_3, \cdots, v_N$. Let us then define the sequence of functions $\{f_k(x)\}$, $k = 1, 2, \cdots, N - 1$, as follows:

(3) $$f_k(x) = \max_{R_k} \sum_{l=k}^{N} v_l,$$

where R_k is the region defined by

(4) $$\text{(a)} \quad \begin{aligned} x &\geqq v_k \geqq 0, \\ b_{k+1} &\geqq v_k + v_{k+1}, \\ &\vdots \\ b_N &\geqq v_{N-1} + v_N, \end{aligned} \qquad \text{(b)} \quad \begin{aligned} r_{k+1} &\geqq v_{k+1} \geqq 0 \\ &\vdots \\ r_N &\geqq v_N \geqq 0. \end{aligned}$$

We have

(5) $$f_{N-1}(x) = \text{Max}\,[v_{N-1} + v_N]$$

where

(6) $$\begin{aligned} &x \geqq v_{N-1} \geqq 0, \\ &b_N \geqq v_{N-1} + v_N, \qquad r_N \geqq v_N \geqq 0. \end{aligned}$$

Hence

(7) $$f_{N-1}(x) = \text{Min}\,[b_N, x + r_N].$$

Employing the principle of optimality, [2], we see that

(8) $$f_k(x) = \underset{0 \leqq v_k \leqq v_k^*}{\text{Max}} [v_k + f_{k+1}(\text{Min}\,(r_{k+1}, b_{k+1} - v_k))]$$

where

$$v_k^* = \text{Min}\,[x, b_{k+1}], \qquad \text{for } k = 1, 2, \cdots, N - 1.$$

10. Explicit Solution

Let us assume that each $f_k(x)$ has the form

(1) $$f_k(x) = \mathrm{Min}\,[P_k, x + Q_k],$$

for $k = 1, 2, \cdots, N-1$. This is true for $k = N - 1$, upon referring to (9.7), and we shall establish it inductively for general k.

Assuming the relation true for $k+1$, substitute in (9.8), obtaining

$$\begin{aligned}
f_k(x) &= \underset{0 \leqq v_k \leqq v_k^*}{\mathrm{Max}} \{v_k + \mathrm{Min}\,[P_{k+1}, \mathrm{Min}\,(r_{k+1}, b_{k+1} - v_k) + Q_{k+1}]\} \\
&= \underset{0 \leqq v_k \leqq v_k^*}{\mathrm{Max}} \{\mathrm{Min}\,[P_{k+1} + v_k, \mathrm{Min}\,(r_{k+1}, b_{k+1} - v_k) + Q_{k+1} + v_k]\} \\
&= \underset{0 \leqq v_k \leqq v_k^*}{\mathrm{Max}} \{\mathrm{Min}\,[P_{k+1} + v_k, \mathrm{Min}\,(r_{k+1} + Q_{k+1} + v_k, b_{k+1} + Q_{k+1})]\} \\
&= \mathrm{Min}\,[P_{k+1} + v_k^*, r_{k+1} + Q_{k+1} + v_k^*, b_{k+1} + Q_{k+1}] \qquad (2)\\
&= \mathrm{Min}\,[P_{k+1} + \mathrm{Min}\,(x, b_{k+1}), r_{k+1} + Q_{k+1} + \mathrm{Min}\,(x, b_{k+1}), b_{k+1} + Q_{k+1}] \\
&= \mathrm{Min}\,[x + P_{k+1}, P_{k+1} + b_{k+1}, x + r_{k+1} + Q_{k+1}, b_{k+1} + Q_{k+1}] \\
&= \mathrm{Min}\,[x + \mathrm{Min}\,(P_{k+1}, r_{k+1} + Q_{k+1}), \mathrm{Min}\,(P_{k+1} + b_{k+1}, Q_{k+1} + b_{k+1})].
\end{aligned}$$

Hence we have the recurrence relation

(3) $$\begin{aligned}
P_k &= \mathrm{Min}\,(P_{k+1} + b_{k+1}, Q_{k+1} + b_{k+1}), & P_{N-1} &= b_N, \\
Q_k &= \mathrm{Min}\,(P_{k+1}, r_{k+1} + Q_{k+1}), & Q_{N-1} &= r_N,
\end{aligned}$$

for $k = 1, 2, \cdots, N-1$.

These recurrence relations determine $f_k(x)$. Furthermore, the optimal policy is determined by the relation

(4) $$v_k = \mathrm{Min}\,[x, b_{k+1}],$$

at each stage.

11. Explicit Solution for the General System

Let us now show that the same method may be used to solve the general maximization problem stated in §1.

Define

(1) $$f_k(x_1, x_2, \cdots, x_{k-1}) = \underset{R_k}{\mathrm{Max}}\,[v_k + v_{k+1} + \cdots + v_n],$$

where R_k is defined, for $k = 1, 2, \cdots, n - K$, by the inequalities

(2) (a) $$\begin{aligned}
x_1 &\geqq v_k \\
x_2 &\geqq v_k + v_{k+1} \\
&\vdots \\
x_{k-1} &\geqq v_k + v_{k+1} + \cdots + v_{k+K-2} \\
b_{k+K-1} &\geqq v_k + v_{k+1} + \cdots + v_{k+K-1} \\
b_{k+K} &\geqq v_{k+1} + v_{k+2} + \cdots + v_{k+K} \\
&\vdots \\
b_n &\geqq v_{n-K+1} + \cdots + v_n
\end{aligned}$$

(b) $$r_i \geqq v_i \geqq 0, \quad i = k+1, \cdots, n.$$

We assume that $n \geqq K$.

Let us first compute $f_{n-K}(x_1, x_2, \cdots, x_{K-1})$. This is the maximum of

$$v_{n-K+1} + \cdots + v_n \tag{3}$$

subject to the constraints

$$(4) \qquad \text{(a)} \quad \begin{aligned} x_1 &\geqq v_{n-K+1} \\ x_2 &\geqq v_{n-K+1} + v_{n-K+2} \\ x_{K-1} &\geqq v_{n-K+1} + \cdots + v_{n-1} \\ b_n &\geqq v_{n-K+1} + \cdots + v_{n-1} + v_n \end{aligned} \qquad \text{(b)} \quad \begin{aligned} & r_i \geqq v_i \geqq 0, \\ & i = n - K + 2, \cdots, n. \end{aligned}$$

Thus

$$(5) \quad f_{n-K}(x_1, x_2, \cdots, x_{K-1})$$
$$\text{Min}\,(b_n, x_{K-1} + r_n, x_{K-2} + r_n + r_{n-1}, \cdots, x_1 + r_{n-K+2} + \cdots + r_n)\,.$$

The recurrence relation for the sequence is

$$(6) \quad f_{k-1}(x_1, x_2, \cdots, x_{K-1})$$
$$= \underset{0 \leqq v_{k-1} \leqq v^*_{k-1}}{\text{Max}} \{v_{k-1} + f_k[\text{Min}(x_2 - v_k, r_k), x_3 - v_k, \cdots, x_{K-1} - v_k, b_{k+K-1} - v_{k-1}]\},$$

where

$$v^*_{k-1} = \text{Min}\,[x_1, x_2, \cdots, x_{K-1}, b_{k+K-2}] \tag{7}$$

Let us now assume that f_k has the form

$$(8) \quad f_k(x_1, x_2, \cdots, x_{K-1}) = \text{Min}[P_{0k}, x_1 + P_{1,k}, x_2 + P_{2,k}, \cdots, x_{K-1} + P_{K-1,k}].$$

Then, substituting in (6),

$$(9) \quad f_{k-1}(x_1, x_2, \cdots, x_{K-1})$$
$$= \underset{0 \leqq v_{k-1} \leqq v^*_{k-1}}{\text{Max}} \{v_{k-1} + \text{Min}[P_{0k}, \text{Min}(x_2 - v_{k-1}, r_{k+1}) + P_{1k},$$
$$x_3 - v_{k-1} + P_{2,k}, \cdots, b_{k+K-2} - v_{k-1} + P_{K-1,k}]\}$$
$$= \underset{0 < v_{k-1} < v^*_{k-1}}{\text{Max}} [\text{Min}\,[P_{0k} + v_{k-1}, x_2 + P_{1k}, v_{k-1} + r_{k+1} + P_{1k},$$
$$x_3 + P_{2,k}, \cdots, b_{k+K-2} + P_{K-1,k}]].$$

The maximum is clearly assumed at $v_{k-1} = v^*_{k-1}$.

Hence we have

$$(10) \quad f_{k-1}(x_1, x_2, \cdots, x_{K-1})$$
$$= \text{Min}[P_{0,k} + v^*_{k-1}, x_2 + P_{1,k}, v^*_{k-1} + r_{k+1}$$
$$+ P_{1,k}, x_3 + P_{2,k}, \cdots, b_{k+K-1} + P_{K-1,k}]$$
$$= \text{Min}\{\text{Min}[P_{0,k}, P_{1,k} + r_k] + \text{Min}[x_1, x_2, \cdots, x_{K-1}, b_{k+K-2}],$$
$$x_2 + P_{1,k}, x_3 + P_{2,k}, \cdots, b_{k+K-2} + P_{K-1,k}\}$$
$$= \text{Min}\{x_1 + w_k, x_2 + w_k, \cdots, x_{K-1} + w_k, b_{k+K-2} + w_k, x_2 + P_{1,k},$$
$$x_3 + P_{2,k}, \cdots, b_{k+K-2} + P_{K-1,k}\},$$

where

$$w_k = \mathrm{Min}[P_{0,k}, P_{1k} + r_k]. \tag{11}$$

Hence

$$f_{k-1}(x_1, x_2, \cdots, x_{K-1}) = \mathrm{Min}\{x_1 + w_k, x_2 + \mathrm{Min}[w_k, P_{1,k}], x_3 + \mathrm{Min}\,[w_k, P_{2,k}], \cdots \mathrm{Min}[b_{k+K-1} + w_k, b_{k+K-2} + P_{K-1,k}]\}. \tag{12}$$

From this equation we can read off the recurrence relations connecting the $P_{i,k}$ and the $P_{i,k-1}$.

12. Discussion

The solution presented in the preceding section yields the optimal policy at each stage, as well as the value of the minimum cost.

There are a number of related problems which can be treated by similar methods. A particularly interesting one is the case where the demand is periodic. In this case, the problem reduces to maximizing

$$L_n = \sum_{i=1}^{n} v_i, \tag{1}$$

subject to a series of constraints

$$\text{(a)}\quad \begin{aligned} &v_1 + v_2 + \cdots + v_K \leqq b_1 \\ &v_2 + v_3 + \cdots + v_{K+1} \leqq b_2 \\ &\vdots \\ &v_{n-K+1} + \cdots + v_n \leqq b_{n-K-1} \\ &v_{n-K+2} + \cdots + v_1 \leqq b_{n-K} \\ &\vdots \\ &v_n + v_1 + \cdots + v_{K-1} \leqq b_n . \end{aligned} \qquad \text{(b)}\quad 0 \leqq v_i \leqq r_i . \tag{2}$$

Furthermore, there are the interesting problems in which there is a storage cost for each excess item, and in which there are more than two types of laundry service.

It is also easy to see that several more general classes of maximization problems subject to linear constraints may be treated by means of the same technique. We shall discuss these topics, together with the question of actual computational solution, in a further paper.

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V–19

ON THE CATERER PROBLEM*

WILLIAM PRAGER

Brown University

1. Introduction

The Caterer Problem was formulated by Jacobs [1][1] as a paraphrase of a practical problem concerning the number of spare engines required to assure given operational levels of a fleet of airplanes. Jacobs stated the problem as follows.

A caterer knows that in connection with the meals he has arranged to serve during the next n days, he will need $r_j(>0)$ fresh napkins on the jth day ($j = 1, 2, \cdots, n$). Laundering normally takes p days; that is, a soiled napkin sent for laundering immediately after use on the jth day is returned in time to be used again on the $(j + p)$th day. However, the laundry also has a higher-cost service which returns the napkins in $q < p$ days (p and q being integers). Having no usable napkins on hand or in the laundry, the caterer will meet his early requirements by purchasing napkins at a cents each. Laundering costs b and c cents a napkin for the normal and high-cost services, respectively, where $b < c < a$. How does the caterer arrange matters to meet his needs and minimize his outlay for the n days?

Formulating this as a problem in linear programming and considering the case $q = p - 1$, Jacobs showed how the problem could be simplified by several transformations to such a degree that an explicit solution could be given. To this writer, Jacobs' analysis appears as a mathematical *tour de force*, which fails, however, to shed light on the features of the problem that make the explicit solution possible. A similar feeling was recently expressed by Hoffman [2].

In the present paper, the Caterer Problem is shown to be equivalent to a Hitchcock Distribution Problem [3] with a very special cost matrix. For the case $q = p - 1$, a simple procedure taking advantage of this fact is developed and shown to yield Jacobs' solution. The possible extension of the procedure to the case $p - q > 1$ is illustrated by a numerical example.

2. The Caterer Problem as a Special Distribution Problem

It is readily seen that the Caterer Problem can be formulated as a Distribution Problem in which the store and each day's hamper of soiled napkins are the origins and each day's requirement of fresh napkins and the final inventory of soiled napkins are the destinations. The cost of "shipping" a napkin from the store to any one of the n days is a. The cost of shipping a napkin from the jth day's hamper of soiled napkins to the kth day is b (when $k - j \geqq p$) or c (when $q \leqq k - j < p$); when $k - j < q$, this cost must be considered as infinite to

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where

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Hence

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$$x_3 + \mathrm{Min}\,[w_k\,, P_{2,k}], \cdots \mathrm{Min}[b_{k+K-1} + w_k\,, b_{k+K-2} + P_{K-1,k}]\}.$$

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TABLE I

Truncated Cost Matrix

	Day	Destination										
		q + 1	q + 2	q + 3	...	p	p + 1	p + 2	p + 3	...	n	Inv.
ORIGIN	1	c	c	c	...	c	b	b	b	...	b	0
	2	∞	c	c	...	c	c	b	b	...	b	0
	3	∞	∞	c	...	c	c	c	b	...	b	0
	...	...	...	...	...	...	...	...	...	...	...	...
	n−q	∞	∞	∞	...	∞	∞	∞	∞	∞	c	0
	Store	a	a	a	...	a	a	a	a	a	a	∞

exclude such impossible shipments. Similarly, the cost of shipping a napkin from the store to the final inventory must be considered as infinite to exclude such wasteful shipments. Finally, the cost of shipping a napkin from any day's hamper of soiled napkins to the final inventory of napkins is zero.

As usual, we present these costs in the form of a cost matrix with the rows corresponding to the origins and the columns to the destinations. Since the napkins for the first q days must be bought at the store and the soiled napkins of the last q days necessarily go to the final inventory, the first q columns and the last q rows may be omitted from the cost matrix. Table I shows the structure of the *truncated cost matrix*. The important feature of this matrix is its nearly triangular character: except for the last line, all entries below the main diagonal are infinite.

Aside from the special structure of its cost matrix, our distribution problem has only one unusual feature: the total number of napkins that will be bought from the store is not known beforehand. A lower bound for this number is obtained as follows. Let R_j denote the total number of napkins used during the first j days,

$$R_j = \sum_{i=1}^{j} r_i. \tag{1}$$

The greatest number of napkins that could be made available for use on the jth day through laundering of napkins used on earlier days is R_{j-q}, where the definition (1) must be supplemented by the convention that $R_j = 0$ for $j \leqq 0$. Thus, if the difference $R_j - R_{j-q}$ is positive, it represents a deficiency of napkins for use on the jth day, which must anyhow be made up by the purchase of new napkins. The largest positive difference $R_j - R_{j-q}$ for $j = 1, \cdots, n$ therefore is a lower bound for the number of napkins that will be bought. It may, of course, be advantageous to purchase in excess of this lower bound if the expenses for express laundry service can thereby be reduced sufficiently.

3. The case $q = p - 1$: Numerical example

To solve the Caterer Problem when $q = p - 1$, we assume at first that the number of napkins bought is given by the lower bound derived in the preceding

TABLE II
Numerical Example

j	1	2	3	4	5	6	7	8	9	10
r_j	50	60	80	70	50	60	90	80	50	100
R_j	50	110	190	260	310	370	460	540	590	690
R_j-R_{j-2}	50	110	140	150	120	110	150	*170*	130	150

TABLE III
Feasible Program ($q = 2$)

j	3	4	5	6	7	8	9	10	I′	r_j
1	20	30								50
2		40	20*		0*					60
3			30*	50*						80
4				10*	60*					70
5					30**	20**				50
6						60**	**			60
7							50**	40	0**	90
8								60	20	80
S′	60				0**					60
r_j	80	70	50	60	90	80	50	100	20	600

section. Having worked out an optimal program under this assumption, we finally check whether this can be improved by the purchase of additional napkins. To familiarize the reader with the various steps of the suggested procedure, these will first be illustrated by a numerical example. It will be shown in Section 4 that this procedure always leads to an optimal program.

For our numerical example, $p = 3$, $q = 2$, and the requirements r_j are given in Table II, which also lists the accumulated requirements R_j and the differences $R_j - R_{j-2}$. It is seen from the last row in Table II that $S = 170$ is a lower bound for the number of napkins that have to be bought. Accepting, for the present, this lower bound as the actual number of napkins bought, we work out the feasible program shown in Table III. Since this table is truncated in the same manner as Table I, the purchases appearing in the table are $S' = S - r_1 - r_2 = 60$ and the final inventory appearing in the table is $I' = S - r_9 - r_{10} = 20$.

Beginning at the right end of the eighth row and proceeding towards the left, we distribute $r_8 = 80$ into the eighth row cells so as to exhaust either the capacities of these cells noted in the bottom row or the amount r_8 available for distribution in the eighth row. We then proceed to distribute $r_7 = 90$ in a similar manner. The feasible program shown in Table III is obtained by continuing in this manner and programming purchases only if the column totals cannot be met otherwise.

This program cannot be improved as long as purchasing remains restricted to $S' = 60$. Indeed, the only way in which the program could be modified while preserving the row and column totals is as follows. Select an even number of

cells with finite cost in such a manner that no row or column contains an odd number of these cells. In Table III such a choice has been indicated by putting an asterisk in each selected cell. Join these cells by a closed path of alternating vertical and horizontal steps. Proceeding along this path, alternatingly increase and decrease the entries in the cells by the same amount. This obviously preserves the row and column totals, but yields a feasible program only if all previously empty cells of the path receive positive entries. For the path indicated in Table III, this condition implies that the amounts in the cells selected on the main diagonal have to be increased. The considered modification of the program therefore leads to an increased use of express laundry service, i.e. to increased expenditure. It is readily seen that the arrangement of the entries in the cells of Table III is such that it is impossible to choose a path of cells yielding a decrease in cost.

So far, S' and hence also I' were considered as fixed at $S' = I' = 60$. Let us now check whether the cost of the program can be reduced by the purchase of additional napkins. An increase in purchase and final inventory involves changes in cells of an open path that begins in some cell of the row S' and proceeds by alternating vertical and horizontal steps to end in some cell of the column I'. In Table III such a path is indicated by double asterisks. This particular path touches three cells on the main diagonal, two cells off this diagonal, and one cell each in the purchase row and the inventory column. If the entries in the cells of this path are alternatingly increased and decreased by the same amount δ, the entries in the cells on the main diagonal are all decreased and those in the cells off this diagonal are all increased. The change in cost is therefore given by

$$\delta(a - 3c + 2b) = \delta(a - b)\left(1 - 3\,\frac{c - b}{a - b}\right).$$

If the considered modification of the program is to result in a saving, we must have

$$\frac{a - b}{c - b} < 3,$$

where the right-hand side is the number of cells in the path that lie on the main diagonal.

Obviously, the path indicated by the double asterisks in Table III is far from being the best of this kind. The best one begins in the first cell of the purchase row, ascends to the first cell of the first row and then descends in a stairlike fashion along the main diagonal to end in the eighth row of the inventory column. Since this best path involves 8 cells on the main diagonal, the program of Table III cannot be improved at all if $(a - b)/(c - b) > 8$.

To be specific, let us assume that $4 < (a - b)/(c - b) < 5$. To lead to an improvement, a path must therefore have at least five cells on the principal diagonal. For the eight-cell path considered above, the amount δ is limited by the smallest entry (i.e. 10) on the main diagonal, because the modification must not lead to negative entries. The modified program is shown in Table IV.

TABLE IV
Modified Program ($p = 3,\ q = 2$)

j	3	4	5	6	7	8	9	10	Inv.	r_j
1	10*	40*								50
2		30*	30*							60
3			20*	60	0*					80
4					70					70
5					20*	30*				50
6						50*	10*			60
7							40*	50*		90
8								50*	30*	80
Purch.	70*									70
r_j	80	70	50	60	90	80	50	100	30	610

TABLE V
Optimal Program ($p = 3,\ q = 2$)

j	3	4	5	6	7	8	9	10	Inv.	r_j
1		50								50
2		10	50							60
3				60	20					80
4					70					70
5						50				50
6						30	30			60
7							20	70		90
8								30	50	80
Purch.	80	10								90
r_j	80	70	50	60	90	80	50	100	50	630

The asterisks in Table IV indicate another path of this kind that includes 7 cells on the main diagonal and hence leads to a further improvement of the program. The amount δ for this path is given by the entry in the first cell of the first row in Table IV. After this second modification there remain six cells with positive entries on the main diagonal, so that the technique can be applied once more. Table V shows the program after this third modification. Since there are now only four cells with positive entries on the main diagonal no further improvement is possible, as $(a - b)/(c - d)$ has been assumed to exceed 4.

4. The case $q = p - 1$: General considerations

We now investigate whether the procedure outlined in the preceding section could fail. In the course of this investigation an analytical description of the procedure will be obtained that shows the final program to be identical with Jacobs'.

The lower bound for the number of napkins that have to be bought is

$$S = \max\,(R_j - R_{j-q}), \qquad (j = 1, 2, \cdots, n) \quad (2)$$

where

$$R_k = \begin{cases} 0 & \text{for } k \leqq 0, \\ \sum_{i=1}^{k} r_k & \text{for } k > 0. \end{cases} \tag{3}$$

For the truncated program (e.g. Table III), we need

$$S' = S - R_q \tag{4}$$

and

$$I' = S - R_n + R_{n-q}\,. \tag{5}$$

In working out the first program, starting from the lower right, could we encounter the following difficulty? As we distribute r_k into the cells of the kth row ($k = 1, 2, \cdots, n - q$), proceeding from right to left and putting into each cell as much as is possible, can we reach the left-most usable cell of this row and still have more left to put into this cell than it will take? It is easily shown that this cannot occur. Indeed, the amount available for the left-most usable cell of the kth row is

$$d_k = \left(\sum_{i=k}^{n-q} r_i - \sum_{i=k+q+1}^{n} r_i - I'\right)^+ = (R_{n-q} - R_{k-1} - R_n + R_{k+q} - I')^+, \tag{6}$$

where $(\cdots)^+$ denotes $\frac{1}{2}\{(\cdots) + |(\cdots)|\}$. Substituting I' from (5), we obtain

$$d_k = (R_{k+q} - R_{k-1} - S)^+ = (r_{k+q} + R_{k+q-1} - R_{k-1} - S)^+ \leqq r_{k+q}\,, \tag{7}$$

because $S \geqq R_{k+q-1} - R_{k-1}$ by (2). Since r_{k+q} is the total for the column that contains the considered cell, and since d_k is the first entry to be made anywhere in this column, the anticipated difficulty cannot arise. Our procedure thus leads to a feasible program.

Next, we show that this program cannot be improved short of purchasing additional napkins. To this end, we observe that our procedure implies certain spatial relations between the left-most entry in one row and the right-most entry in the next higher row.

Let the left-most entry in row k occur in column l. The fact that something was put into column l in row k indicates that the requirements for column $l + 1$ have been met. The first entry to be put into row $k - 1$ can therefore not be to the right of column l; it will be in column l if the requirements of this column have not yet been met; otherwise it will be in column $l - 1$ since no entry has as yet been made in this column. Thus, the right-most entry in a row is either in the same column as the left-most entry in the next lower row or immediately to the left of this column.

From this pattern there follows an important property of any path that starts from a cell in the main diagonal and proceeds by alternating vertical and horizontal steps to return to the starting point without touching any cell twice or

descending below the main diagonal: the top right corner of such a path falls necessarily into an empty cell. To be acceptable, any program change effected by alternatingly decreasing and increasing the entries in the cells of this closed path by a fixed quantity δ, must increase the amount in this previously empty cell and hence also the amount in the starting cell on the main diagonal. Such a program change cannot, therefore, constitute an improvement because it substitutes high-cost laundry service for normal service.

We must now check whether our tentative program can be improved by the purchase of additional napkins. It follows from the discussion of this point in the preceding section that this is possible, whenever the number of non-vanishing entries d_k on the main diagonal exceeds $(a - b)/(c - b)$. Let us order these entries d_k by descending magnitude repeating, if necessary, each value according to its multiplicity. If m is the least integer greater than $(a - b)/(c - b)$ and d_k^* the mth of the ordered values d_k, it is clear that repeated application of the process discussed in reducing the program of Table III to that of Table IV will eventually reduce to zero the entries in all main diagonal cells that contained values d_k of a higher order than d_k^*. At this stage, the number of non-vanishing entries on the main diagonal has fallen below m and further improvement of the program has become impossible. The final entries on the main diagonal are therefore given by

$$z_k = (d_k - d_k^*)^+. \tag{8}$$

This agrees with the first Eq. (2.7) in Jacobs' paper, since our d_k corresponds to Jacobs' H_{k+q}. Once the number of napkins that is sent to the high-cost laundry service is known for each day, the number that must be bought on each day can be worked out by a simple book-keeping method similar to the one that furnished Eq. (2). This yields the second Eq. (2.7), of Jacobs' paper and the third Eq. (2.7) follows from this. Actually, the stepwise improvement procedure suggested here furnishes the high cost service and purchase requirements about as fast as they could be computed from Jacobs' formulas.

5. The case $q = p - 2$: Numerical example

The present synthetic approach has a practical advantage over Jacobs' analytical treatment of the problem: it is readily generalized to the case where $p - q$, though small, exceeds unity. To show this, let us modify the example of Table II by assuming that $q = 2$, as before, but $p = 4$. The program in Table III is still feasible, but in trying to improve it, we must keep in mind that now not only the cells on the main diagonal but also the cells immediately to the right of this diagonal are high-cost cells. As a consequence of this, it is no longer true that we cannot improve this feasible program short of buying additional napkins. Table VI shows the program obtained after improving the first program as much as is possible without increased purchase of napkins. Each improvement step adds an amount δ to a cell on the main diagonal and to the cell immediately to the upper right of it, while removing the amount δ from each of the two remaining cells of a square of four cells.

TABLE VI

Improved Program ($p = 4$, $q = 2$)

j	3	4	5	6	7	8	9	10	Inv.	r_i
1	20*	10	20*							50
2		60								60
3			30*		50*					80
4				60		10				70
5					40*	10	0*			50
6						60				60
7							50*	20	20*	90
8								80		80
Purch.	60*									60
r_j	80	70	50	60	90	80	50	100	30	600

TABLE VII

Optimal Program ($p = 4$, $q = 2$)

j	3	4	5	6	7	8	9	10	Inv.	r_i
1		10	40							50
2				60						60
3			10		70					80
4						70				70
5					20	10	20			50
6								60		60
7							30	20	40	90
8								20	60	80
Purch.	80	60								140
r_j	80	70	50	60	90	80	50	100	100	740

We next check whether further improvements are possible by additional purchases of napkins. The technique is essentially the same as before, except that the stair-like path down the main diagonal now must have horizontal or vertical steps of the minimum length 2, if it is to avoid the high-cost cells off the main diagonal. Such a path is indicated by asterisks in Table VI. Since it has 4 cells on the main diagonal, it will lead to a program of lower cost only if $(a - b)/(c - b) < 4$. Let us assume that this is the case, but that $(a - b)/(c - b) > 3$. The optimal program shown in Table VII is then obtained by applying this improvement procedure twice, once for the path marked in Table VI and then for a similar path whose steps are one cell to the right and below those of the first path. While an analytical description of the optimal program for $p - q > 1$ could be worked out, it appears highly doubtful that it would yield this program faster than the synthetic procedure followed here.[2]

[2] Since this paper was written, Dr. George B. Dantzig has informed the author that the equivalence of the Caterer Problem to a Transportation Problem has been recognized for some time by the members of his research group at the Rand Corporation. More recently, this fact seems even to have been stated in print in a report by S. Hoch (USAF-AMC). Since the author has not been able to obtain a copy of this report, he cannot comment on the relation between the present approach and that of Mr. Hoch.

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LETTERS TO THE EDITOR

Dear Sir:

There are two points arising out of Professor W. Prager's paper "On the Caterer Problem", *Management Science 3* (1956), pp. 15–23, which seem of interest.

One is that the whole Caterer Problem can be regarded as a Transportation Problem, even if the number of napkins to be bought is not regarded as known, if one introduces a large number of napkins originally (i.e. at least as many as the largest sum of p successive daily requirements) but regards the cost of transporting a new napkin to the final inventory as zero—this meaning that the napkin is not really bought at all.

The other point is that if the total number of napkins is given, and there are only two types of laundry involved, the solution can be written down straight away as follows:

Consider the requirement at each "destination" in turn, and satisfy it according to the following rules.

1. Always use new napkins until the stock of new napkins is exhausted.

2. If there are not enough new napkins available, arrange as far as possible to supply napkins last used p or more days earlier through the slow laundry.

3. Use the fast laundry to satisfy any additional demand that cannot be met by these two means. This is Prager's solution when $p - q = 1$. But there is an important proviso when $p - q > 1$. When using the fast laundry, napkins that were last used most recently should be selected. Thus one should prefer to have napkins last used q days ago, rather than napkins last used $q + 1$ days ago, and these in turn should be preferred to napkins last used $q + 2$ days ago (as long as $q + 2 < p$), and so on. The point of this proviso is that if napkin A was last used q days ago, it could not be recovered from the slow laundry for another $p - q$ days; while napkin B which was last used $q + 1$ days ago could be made available from the slow laundry after another $p - q - 1$ days. If there happens to be a heavy demand for napkins $p - q - 1$ days later, then if napkin A is used now, napkin B can be recovered in time from the slow laundry. But if napkin B is used now, napkin A cannot be recovered in time from the slow laundry.

It is a straightforward matter to prove that the above scheme necessarily produces an optimal solution if one exists, though there may well be other equally cheap solutions.

Yours faithfully,

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INVENTORY DEPLETION MANAGEMENT*[1]

CYRUS DERMAN AND MORTON KLEIN

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Consideration is given to problems of choosing the order of issue of items from a stockpile of material whose utility characteristics are changing with time. Conditions are given under which either LIFO (last in, first out) or FIFO (first in, first out) is an optimal issue policy.

1. Introduction

During the past decade there has been an intensified interest in inventory problems. Studies in this area have concentrated on the development and evaluation of stock ordering policies. Two basic papers on this subject are [1] by Arrow, Harris, and Marshack and [2] by Dvoretzky, Kiefer, and Wolfowitz. For an elementary exposition, [5] the paper by Laderman, Littauer and Weiss is available.

Recently Greenwood [3] and Heit [4] have drawn attention to the problem of development and evaluation of stock issuing policies. In this paper three problems in this area will be considered:

Problem 1. A stockpile consists of n items[2]. Associated with the i-th item is an age (length of time in the stockpile) $S_i(i = 1, \cdots, n)$. The field life of an item is a function, $L(S)$, of the age of the item upon issue to the field. When an item's usefulness or life in the field is ended, a new one is issued from the stockpile. Items are to be issued successively until the stockpile is depleted.

The problem of interest, here, is that of finding the order of item issue which maximizes the total field life obtained from the stockpile.

Issue policies which permit the replacement of an item in the field before its usefulness is ended will not be considered here. This case is discussed in Section 7.

Problem 2. An inventory consists of n items of ages $S_1, \cdots, S_n$. Let $X(S)$ be a random variable, and $U(S)$ its expectation, which denotes the utility to management of an item of age S when issued. A withdrawal schedule is given which specifies the times at which items will be required. It is assumed that the schedule exhausts the stock.

It is required to find that order of stock issue which maximizes the total expected utility obtainable from the n items while meeting the given demand schedule.

* Received January 1958.

[1] Research under contract with the U. S. Army Chemical Corps Engineering Command. Originally issued as Technical Report No. 2, Oct. 15, 1957, of the Statistical Engineering Group, Columbia University.

[2] It may be noted that there are situations in which "lot" instead of "item" is appropriate in the statement of all of the listed problems.

It should be noted that *Problems 1* and *2* differ in two respects:

(i) For purposes of mathematical simplicity, it is assumed that a deterministic relationship ($L(S)$) exists between the age of an item and its field life, i.e., $L(S)$ is not the expectation of a random variable, and

(ii) the schedule of usage in *Problem 2* is independent of the function $U(S)$, whereas the usage schedule in *Problem 1* is a function of $L(S)$.

The next problem is a more dynamic version of *Problem 2*.

Problem 3. A stockpile contains n items of different ages. At k different times a new item is added to stock. A withdrawal schedule is given.

It is, again, of interest to determine the order of item issue which maximizes the total expected utility obtainable from the use of all of the items.

If complete knowledge of the functions $L(S)$ and $U(S)$ is available, then optimal policies for any given situation can be obtained by a consideration of all $n!$ different orderings and a selection of the best. In the case of *Problem 2*, computational procedures which are available for the solution of the "optimal assignment problem" may be utilized.[3] However, there are circumstances of even greater interest. In most real cases only limited knowledge of the functions $L(S)$ or $U(S)$ and the ages (S_i) is available; consequently, complete enumeration or the utilization of a computational procedure is not possible.

Greenwood [3] and Heit [4] have reported that the most frequently used depletion policies are LIFO (last in, first out) and FIFO (first in, first out). A LIFO (FIFO) policy would be that of always using the youngest (oldest) item on hand first. Greenwood was primarily interested in a comparison between these two stock issue policies for the case in which the field life function is linear. Heit also compared these policies and several others, which, however, require for their implementation complete knowledge of both the deterioration function and the age of each item.

A LIFO (FIFO) policy is of interest to us since its utilization requires information concerning only the relative ages of the items. The purpose of this paper is to give sufficient conditions on $L(S)$ or $U(S)$ under which LIFO will be optimum over all possible policies. The case where FIFO is optimum will be briefly discussed.

2. The Main Results

Theorem 1. If

(i) $L(S)$ is a non-negative, non-increasing convex function, and

(ii) LIFO is an optimal policy for *Problem 1* when $n = 2$, then LIFO is an optimal policy for *Problem 1* for $n = 3, 4, \cdots$.

Theorem 2. If $U(S)$ is a convex function, then LIFO is an optimal policy for *Problem 2*.

Theorem 3. If $U(S)$ is convex, then LIFO is an optimal policy for *Problem 3*.

[3] See, for example, Kuhn [6] or Munkres [7].

A function f is convex if for every pair of values x_1, x_2 and all values a_1, a_2 such that $0 \leqq a_1, a_2 \leqq 1$, $a_1 + a_2 = 1$,

$$f(a_1x_1 + a_2x_2) \leqq a_1f(x_1) + a_2f(x_2).$$

That is, if a line connects any two points of the graph of a convex function then, all intermediary points of the graph are never above the line. For a concave function, the direction of the inequality is reversed.

Although complete knowledge of $L(S)$ and $U(S)$ may be lacking, it is possible to determine, sometimes from a priori considerations, sometimes empirically, that $U(S)$ is convex. From this point of view, *Theorems 2* and *3* represent satisfactory results. However, *Theorem 1* falls short in that it does not provide such a simple characterization. We shall give below some functions which satisfy (i) and (ii).

3. Proof of Theorem 1

Suppose that LIFO is optimal for $n = k \geqq 2$. Let $n = k + 1$ and $0 < S_1 < S_2 < \cdots < S_{k+1}$ be an arbitrary set of initial ages. For any issue policy, let S^* denote the age of the last item issued. First, observe that none of the $k!$ policies having $S^* = S_1$ can be optimal, for in any such policy, if S_i denotes the initial age of the item issued next to last, we have $S_i > S_1$, and by hypothesis this policy could be improved by interchanging the order of issue of these last two items.

Now for $S^* \neq S_1$, let x denote the total field life obtained from the issue of the k preceding items and x^* denote the largest possible value of x, so that by the inductive assumption, $x^* \geqq L(S_1)$. Let $Q(x) = x + L(x + S^*)$ denote the total field life of all $k + 1$ items. Since L is non-increasing and $S_1 < S^*$, it follows that $L(S_1) \geqq L(S^*)$ and

$$Q(x^*) = x^* + L(x^* + S^*) \geqq x^* \geqq L(S_1) \geqq L(S^*) = Q(0).$$

Thus, since Q is convex, Q is maximized for fixed S^* by $x = x^*$, which is obtained by using a LIFO order on the first k items.

By letting S^* vary over $S_2, S_3, \cdots, S_{k+1}$, we obtain k policies. Suppose the optimal among these is not the one with $S^* = S_{k+1}$, but some other. Then since x^* is a result of LIFO order, the item of age S_{k+1} was issued next to last, and by hypothesis the policy could be improved by interchanging the order of issue of these last two items. Hence, the optimal policy must be the one that has $S^* = S_{k+1}$, which is precisely the LIFO policy with $n = k + 1$. The theorem follows by induction.

As a matter of mathematical interest the question arises as to whether condition (ii) implies condition (i), i.e., does LIFO as an optimal policy for the case $n = 2$ require L to be convex. As a counterexample, consider the case of the non-convex function defined as follows:

$$L(S) = 2 \qquad 0 \leqq S < 1$$
$$= 1 \qquad 1 \leqq S.$$

It is easy to verify that (ii) is satisfied.

4. Proof of Theorem 2

Suppose $n = 2$ and the ages are S_1 and S_2 with $S_1 < S_2$. Let a denote the scheduling interval. It must be shown that

$$U(S_1) + U(S_2 + a) \geqq U(S_2) + U(S_1 + a). \tag{1}$$

$U(S)$ is a convex function, hence,

$$\frac{U(S_2) - U(S_1)}{S_2 - S_1} \leqq \frac{U(S_2 + a) - U(S_1 + a)}{(S_2 + a) - (S_1 + a)}.$$

Equivalently

$$U(S_2) - U(S_1) \leqq U(S_2 + a) - U(S_1 + a)$$

which yields (1).

Now let n be any integer greater than 2. Any policy other than LIFO will have the property that there will be two successive items issued in such a way that the first item issued will be older than the second. However, using (1), it follows that such a policy could be improved by interchanging the order of issue of these two items. This process may be continued until no further interchanges are advantageous and the LIFO order is reached. Thus, since no policy provides a greater expected utility than LIFO, *Theorem 2* is proved.

5. Proof of Theorem 3

Consider first the case in which there are n original items in the stockpile and just one item is to be added at some time. For any stock issue policy, the total expected utility is composed of two parts: the expected utility from the original item and the expected utility from the new item. It follows from *Theorem 2* that, no matter where the new item is placed in the issue order, the expected utility from the *original* items is maximized by the use of a LIFO policy for the originals. Hence, the optimal policy must be of the form in which the original items are used in the LIFO order.

Now consider the total expected utility as being composed of the expected utility evolving from the items used prior to the entrance of the new item and the expected utility from the remainder of the stockpile (including the new item). Since a LIFO ordering is used on the original items, the expected utility is maximized by maximizing the second part. However, by *Theorem 2* this is accomplished by using a LIFO ordering for these items.

The case in which k new items are added at different times to the stockpile is easily proved by induction.

6. Special Field Life Functions

In this section we show that certain classes of functions satisfy conditions (i) and (ii) of *Theorem 1*. At present, we have no satisfactory characterization of such a class. We shall say that a function belongs to the class $\mathcal{C}$ if it satisfies (i) and (ii).

Theorem 4. If $L(S)$ is of the form $(a/b + S)(a > 0,\ b \geqq 0)$ then L belongs to $\mathcal{C}$.

Proof: (i) is clearly satisfied. Let S_1, S_2 be any non-negative real numbers with $S_1 < S_2$. To prove (ii) we must show that

$$L(S_1) + L(S_2 + L(S_1)) - L(S_2) - L(S_1 + L(S_2)) > 0. \tag{2}$$

On substitution we get

$$\frac{a}{b + S_1} + \frac{a}{b + \left(S_2 + \dfrac{a}{b + S_1}\right)} - \frac{a}{b + S_2} - \frac{a}{b + \left(S_1 + \dfrac{a}{b + S_2}\right)}$$

$$= \frac{\dfrac{a}{b + S_1} - \dfrac{a}{b + S_2}}{b^2 + b(S_1 + S_2) + S_1 S_2 + a} > 0$$

since

$$\frac{a}{b + S_1} > \frac{a}{b + S_2}.$$

Hence (2) is established.

Theorem 5. If $L(S)$ is of the form $ce^{-kS}(c, k > 0)$ then L belongs to $\mathcal{C}$.

Proof: (i) is again clearly satisfied. In order to prove (2) and hence (ii) we consider the function of S for fixed S_1

$$F(S_1, S) = L(S_1) + L(S + L(S_1)) - L(S) - L(S_1 + L(S)).$$

In the case under study we have

$$F(S_1, S) = ce^{-kS_1} + ce^{-k(S+ce^{-kS_1})} - ce^{-kS} - ce^{-k(S_1+ce^{-kS})}.$$

Clearly we have $F(S_1, S_1) = \lim_{S\to\infty} F(S_1, S) = 0$. Also

$$\frac{dF(S_1, S)}{dS} = c\{-ke^{-k(S+ce^{-kS_1})} + ke^{-kS} - ck^2e^{-kS-k(S_1+ce^{-kS})}\}$$

$$= ck[e^{-kS}(1 - cke^{-k(S_1+ce^{-kS})}) - e^{-k(S+ce^{-kS_1})}]$$

$$= cke^{-kS}[1 - cke^{-k(S_1+ce^{-kS})} - e^{-ce^{-kS_1}}].$$

The first factor of the above is always positive. The second factor, since its derivative is negative is either always negative or at first positive and then always negative. The first contingency is ruled out because of the values of

$F(S_1, S)$ at S_1 and ∞. Thus the second is the case and therefore $F(S_1, S) > 0$ for all $S > S_1$.

7. Partial Use of Items in Problem 1

It was previously remarked that, in the determination of the optimum issue policy for *Problem 1*, only those policies in which an item is completely used were considered. It is conceivable that a policy calling for only partial use of some of the items might yield a greater total field life than the optimum obtained over the class of policies which were considered.

Consider the case in which $n = 2$ and the function $L(S)$ satisfies conditions (i) and (ii) of *Theorem 1*. A partial usage policy would be to use one of the items just for T units of time, where $0 \leqq T \leqq L(S_i)$, $i = 1$ or 2 according to whether the younger or older of the two items is used first, and then use (completely) the remaining item.

The problem is to choose i and T in order to maximize the function

$$Q(i, T) = T + L(S_{3-i} + T).$$

Since T and L are convex $Q(i, T)$ achieves its maximum either at $T = 0$ or at the largest possible value of T. Using this fact and conditions (i) and (ii) of *Theorem 1*, we have

$$\max_{i,T} Q(i, T) = \max\,[Q(1, 0), Q(2, 0), Q(1, L(S_1)), Q(2, L(S_2))] = \max\,[Q(1, L(S_1)), Q(2, L(S_2))] = Q(1, L(S_1)).$$

Hence, LIFO (using items until failure) is still the optimum issue policy. The method of induction, as used in the proof of *Theorem 1* can be used again to show that LIFO is optimal for $n > 2$.

8. FIFO as an Optimal Policy

If $L(S)$ is linear and decreasing it can be seen easily that FIFO is optimal for *Problem* 1. This rules out any such result as convexity of $L(S)$ being a sufficient condition for the optimality of LIFO for *Problem 1*.

If $U(S)$ is concave, then a reversal of the argument shows that FIFO is optimal for *Problems 2* and *3*.

Note that these remarks are somewhat less than rigorous, since $L(S)$ linear (or $U(S)$ concave) and decreasing would lead to negative values of U and L for large enough S. Thus, it is tacitly assumed that the ages are within a range such that U, L are positive.

9. Remarks

The characterization of the expected utility functions for which LIFO and FIFO inventory depletion policies are optimal, even for such simple problems as were considered, is fairly useful:

(a) These policies are easily understood and for the most part easily implemented in practice.

(b) For *Problems 2* and *3* neither the demand schedule nor the replacement schedule need be known or regular; the precise function need not be known either. Only information as to the convexity or concavity (within limits) of the expected utility function is required for an optimal policy to be chosen. Since the optimality of LIFO holds for any given schedule, it is easy to see that it is also optimal for a random schedule, provided that the scheduling is independent of the issue policy. This information may be used to approximate a solution to the following more general version of *Problem 1*:

Suppose there are, instead of one, many sources of demand. This changes the situation so that instead of the issue of the i-th item awaiting the failure of the $(i - 1)$-th item, the demand schedule takes on the character of a random demand schedule. Since LIFO is optimal for *Problem 2*, it is reasonable to utilize it as an approximately optimal policy for this problem also.

(c) Reasonable approximations to many other functional forms can be obtained via convex (concave) functions. Hence, an even wider range of potential approximate application for *Theorems 2* and *3*.

It should be noted that even the narrow scope of the problems posed here is not fully explored. Other kinds of objective functions are clearly of interest as are the policy requirements imposed by other kinds of field life and expected utility functions.

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[P. Zehna pointed out in "Optimal Depletion Policies," Chapter 15 in K. Arrow, S. Karlin, and H. Scarf (eds.), *Studies in Applied Probability and Management Science*, Stanford: Stanford University Press, 1962, 265–267, that certain modifications of the original statement and proof of *Theorem 1* were necessary. His corrected proof is incorporated in this reprinted version of the original paper. In addition, a minor error in the original argument in section 7 has been corrected. —The Editor.]

LIFO VS FIFO IN INVENTORY DEPLETION MANAGEMENT*

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1. Introduction and Purpose

In a recent paper (1), Derman and Klein consider the following problem: A stockpile consists of n items. Associated with the ith item is an age (length of time in the stockpile) S_i ($i = 1, 2, \cdots, n$). The field life of an item is a function, $L(S)$ ($\geqq 0$), of the age of the item upon issue to the field. When an item's usefulness or life in the field is ended, a replacement is issued from the stockpile. Items are to be issued successively until the stockpile is depleted. The problem of interest is to find the order of item issue which maximizes the total field life obtained from the stockpile.

In (1), it is pointed out that if a complete knowledge of the function $L(S)$ is available, then optimal policies for any given situation can be obtained either by a consideration of all $n!$ different orderings and the consequent selection of the best, or by using an algorithm which will lead to the solution. For large n (and even moderately sized n) heavy numerical calculations are involved. Furthermore, in the usual circumstances there is only limited knowledge available about the function $L(S)$ and the ages (S_i), e.g., general shape of $L(S)$ and possibly the ranking of the S_i, making it impossible to use these techniques to find an optimal policy.

In their paper, Derman and Klein present sufficient conditions on $L(S)$ under which a LIFO (last in, first out) policy is optimal. This corresponds to issuing the youngest item on hand first, and its utilization requires information concerning the relative ages of the items, rather than their absolute ages. In particular, they show that if

(i) $L(S)$ is a non-increasing convex function, and

(ii) LIFO is an optimal policy when $n = 2$, then LIFO is an optimal policy for $n = 3, 4, \cdots$.

The purpose of this paper is to present an alternate set of sufficient conditions on $L(S)$ under which a LIFO policy is optimal. In addition, this paper will present two sets of sufficient conditions on $L(S)$ under which a FIFO (first in, first out) policy is optimal. This corresponds to issuing the oldest item on hand first, and its utilization, like LIFO, requires information concerning only the relative ages of the items. Moreover, the second set of conditions will not involve verifying the results for the case of two items ($n = 2$).

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2. A Set of Sufficient Conditions under which LIFO Is Optimal

The following theorem gives a set of sufficient conditions under which a LIFO policy is optimal.[1]

Theorem 1:

(i) If $\frac{dL}{dS} = L'(S) \geqq -1$ and

(ii) LIFO is an optimal policy when $n = 2$, then LIFO is an optimal policy for $n = 3, 4, \cdots$.

Proof: The proof is similar to that used by Derman and Klein, and will be by induction. The function to be maximized by choosing an optimum issue policy may be written in the form $Q(x) = x + L(x + S^*)$ where x denotes the field life resulting from the use of the first $n - 1$ items issued and S^* is the initial age of the nth item issued. $L(x + S^*)$ is then the field life resulting from the nth item issued. From condition (i) it follows that $Q(x)$ is a non-decreasing function since $Q'(x) = 1 + L'(S)$ is always non-negative. Hence $Q(x)$ is maximized by making x as large as possible. Derman and Klein use their condition (i) only to show that $Q(x)$ is maximized by making x as large as possible. Hence, the remainder of the Derman-Klein proof is applicable since their condition (ii) is the same as that given above. The proof will be repeated here for the sake of completeness. By the induction assumption x is maximized by using a LIFO policy in the first $n - 1$ items issued. Now, if the issuing policy is such that the oldest item is not issued last, it follows that in maximizing x it will be issued next to last. If so, it follows from condition (ii) that the issuing policy could be improved by interchanging the order of issue of the last two items. Thus, the optimal policy must be of the form where the oldest item is issued last. This being established, the theorem follows from the application, again, of the induction assumption.

It should be pointed out that condition (i), $L'(S) \geqq -1$, does not imply convexity of $L(S)$ nor does convexity of $L(S)$ imply that $L'(S) \geqq -1$. Neither is a stronger condition than the other.

3. A Set of Sufficient Conditions under which A FIFO Policy Is Optimal

The following theorem gives a set of sufficient conditions under which a FIFO policy is optimal.

Theorem 2:

(i) If $L'(S) \geqq -1$, and

(ii) FIFO is an optimal policy when $n = 2$, then FIFO is an optimal policy for $n = 3, 4, \cdots$.

Proof: The proof will be by induction. The function to be maximized by choosing an optimum issue policy may be written in the form $Q(x) = x + L(x + S^*)$ where x denotes the field life resulting from the use of the first $n - 1$ items issued and S^* is the initial age of the nth item issued. $L(x + S^*)$ is then the field life resulting from the nth item issued. From condition (i), $Q(x)$ is maxi-

[1] This result was obtained jointly with Professor A. Dvoretzky.

mized by making x as large as possible. By the induction assumption x is maximized by using a FIFO policy on the first $n - 1$ items issued. Now if the issuing policy is such that the youngest item is not issued last, it follows that in maximizing x it will be issued next to last. At this point there are only two items left. Hence, from condition (ii) the issuing policy could be improved by interchanging the order of issue of the remaining two items. Thus, the optimal policy must be of the form where the youngest item is issued last. This being established, the theorem follows from the application, again, of the induction assumption.

Thus, it is interesting to note that the problem of determining whether LIFO or FIFO is an optimal policy resolves itself into determining whether LIFO or FIFO is optimal for the case $n = 2$.

4. A Set of Sufficient Conditions under which FIFO Is Optimal for $n = 2$

The following theorem gives a set of sufficient conditions under which a FIFO policy is optimal.

Theorem 3:

(i) If $L'(S) \geqq -1$ and

(ii) $L(S)$ is a non-increasing or non-decreasing concave function, then FIFO is an optimal policy.

Proof: Condition (i) implies that if the theorem is true for $n = 2$, it is true for all n. Hence, it is sufficient to show that condition (ii) implies that the theorem is true for $n = 2$.

A) Assume that $L(S)$ is a non-increasing function.

If $S_2 > S_1$, it is necessary to show that

$$* \qquad L(S_2) + L[S_1 + L(S_2)] \geqq L(S_1) + L[S_2 + L(S_1)].$$

Since $L(S)$ is non-increasing, it follows that

$$L[S_2 + L(S_1)] \leqq L[S_2 + L(S_2)].$$

Using this inequality, it follows that * holds whenever

$$L(S_2) + L[S_1 + L(S_2)] \geqq L(S_1) + L[S_2 + L(S_2)].$$

From condition (ii), $L(S)$ is concave. Hence,

$$\frac{L(S_2) - L(S_1)}{S_2 - S_1} \geqq \frac{L[S_2 + L(S_2)] - L[S_1 + L(S_2)]}{[S_2 + L(S_2)] - [S_1 + L(S_2)]}.$$

Equivalently,

$$L(S_2) + L[S_1 + L(S_2)] \geqq L(S_1) + L[S_2 + L(S_2)],$$

and the result is obtained. It has been tacitly assumed that $L(S)$ is concave only for S such that L is positive. In the above proof the range of the argument never goes outside of this region provided S_2 lies within since $[S_2 + L(S_2)]$ is the maximum value of S considered, and this must always be in the region where the function is concave when $L'(S) \geqq -1$. If S_2 is such that $L(S_2)$ is already 0, * holds trivially for all values of $S_1 < S_2$.

B) Assume that $L(S)$ is a non-decreasing function.

Again it is necessary to show that $*$ holds. Since $L(S)$ is a non-decreasing function

$$L[S_2 + L(S_1)] \leqq L[S_2 + L(S_2)],$$

and the same proof as in (A) goes through.

5. Remarks

In the statement of the original problem, only those policies in which an item is completely used were considered. It is conceivable that a policy calling for only partial use of some of the items might yield a greater total field life than the optimum obtained over the class of policies which were considered. Derman and Klein show that under their conditions LIFO (using items until failure) is still the optimum policy. Essentially the same proof will go through for Theorems 1, 2, and 3 in this paper, with, of course, the optimal policy being FIFO (using items until failure) for the last two theorems.

For Theorem 3, the precise function $L(S)$ need not be known. Only information as to its concavity (within limits) and its derivative never being less than minus one is required for an optimal policy of FIFO to be chosen.

Finally, a result similar to Theorem 3 for LIFO is desirable since Theorems 1 and 2 require that $L(S)$ be known exactly in order to verify the results for $n = 2$.

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[Minor changes have been made in this article to conform with corrections made in the reprinted version of the paper by Derman and Klein, Chapter 21. —The Editor.]

DISCUSSION: SEQUENCING n JOBS ON TWO MACHINES WITH ARBITRARY TIME LAGS*

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This note presents an alternate proof of a result of L. G. Mitten, solving the problem of sequencing n jobs through two machines with arbitrary time lags when the job sequences are the same for both machines. The more difficult general problem is also discussed and partially solved.

Introduction

In [2], L. G. Mitten solved the problem of sequencing n jobs on two machines with arbitrary time lags while minimizing total elapsed time, when it is assumed that the job sequences on both machines are the same. He mentioned that the general problem would sometimes involve different job sequences on the two machines and hence would be quite difficult.

In the present note an alternate proof of Mitten's result is shown to follow as a corollary of a special three-stage problem solved in [1] when properly interpreted. Also Mitten's general case is discussed and partially solved.

An Alternate Derivation of Mitten's Result

In [2], L. G. Mitten solved the following problem. Each of n jobs must be run first on machine I and then on machine II. Running times (including any set-up or tear-down times) for the i-th job are A_i on I and B_i on II. Let $D_i \geq 0$ be the arbitrary time lag associated with item i, such that job i is not started on II sooner than D_i time units after it was started on I, nor finished on II sooner than D_i time units after it was finished on I. A rule is given for determining the sequence in which the jobs are to be run on the machines, using the same sequence for both machines, in order to minimize the time between the start of the first job on machine I and the completion of the last job on machine II.

Mitten's solution to this problem is as follows. Sequence those jobs whose $A_i < B_i$ in order of increasing value of D_i, all such jobs coming before jobs whose $A_i \geq B_i$, which in turn are sequenced in order of decreasing value of D_i.

We proceed as follows.

Write D_i, the arbitrary time lag for the i-th job lot, as

$$D_i = M_i + \min(A_i, B_i) \tag{1}$$

where $M_i \geq -\min(A_i, B_i)$.

If $M_i \geq 0$, we can interpret M_i as the processing time of the i-th job lot on all intermediate stages or non-bottleneck machines.

If $M_i < 0$, then we can interpret this case (following Mitten) as one involv-

* Received Nov. 1958.

ing lap-phasing, starting an item on the next stage before its entire job lot is available.

Then Mitten's problem can be interpreted as a restricted three-stage problem, which was treated in [1].

The general three-stage time matrix discussed in [1], in terms of present notation, is

A_1	A_2				A_n
M_1	M_2				M_n
B_1	B_2				B_n

For this sequence $S_0 = (1, 2, \cdots, n)$ the total time to process all the jobs is

$$(2) \qquad T(S_0) = \max_{1 \le u \le v \le n} \left(\sum_{i=1}^{u} A_i + \sum_{i=u}^{v} M_i + \sum_{i=v}^{n} B_i \right).$$

We wish to permute the columns of the time matrix to find the minimum $T(S)$.

This three-stage problem was solved in [1] only for a special set of assumptions leading to the conclusion that $u = v$ in (2). In the present problem the same restriction $u = v$ holds since the total time for a given sequence, $S = (1, 2, 3, \cdots, n)$, is

$$(3) \qquad T(S) = \max_{1 \le u \le n} \left\{ \sum_{i=1}^{u} A_i + M_u + \sum_{i=u}^{n} B_i \right\},$$

that is, there is no bottleneck on the intermediate stage. Then the solution in [1] was shown to reduce to an equivalent two-stage problem with an optimal solution given by the transitive rule: Item i precedes item j if

$$(4) \qquad \min (A_i + M_i, B_j + M_j) < \min (A_j + M_j, B_i + M_i)$$

with ties ordered either way. In [1] it was shown that this led to an easy method of scheduling:

Find the smallest number of the set of $2n$ numbers $(A_i + M_i, B_i + M_i)$, $i = 1, 2, \cdots, n$. If it is an $A_i + M_i$, place that item i first; if it is a $B_i + M_i$, place that item last. Then repeat on the reduced set of items until all are ordered.

Mitten's rule gives an alternate interpretation of (4).

If $A_i < B_i$, $B_j < A_j$, item i precedes item j from (4).

If $A_i < B_i$, $A_j < B_j$, then (4) implies item i precedes item j if $A_i + M_i < A_j + M_j$, that is, if $D_i < D_j$.

Similarly, if $B_i < A_i$, $B_j < A_j$, item i precedes item j if $D_j < D_i$.

The two rules are equivalent (except possibly for conventions concerning ties). In any case, both give the same total time.

Note that Mitten's rule also applies to the standard two-stage problem in [1] and is perhaps easier to remember than the rule given in [1].

The General Case Allowing Different Job Sequences

In the problem not treated by Mitten, where different job sequences are allowed for the two bottleneck machines, the following remarks will reduce the problem in most cases down to a relatively small list of sequences whose total times can be compared and the optimal sequence found.

One can easily show that for every sequence S_{I} on machine I there is an optimal sequence on machine II given by processing items in order of their availability times on II. For suppose the job sequence on II is not in the same order as the sequence $\{t_i\}$ where t_i is the availability time for item i on II for a given fixed sequence S_{I} on I. If items i and j of S_{II} are not in the same order as t_i and t_j, then they both must have started on II at a time after max (t_i, t_j). Thus we can interchange *consecutive* items i and j on II *without loss of time.* By successive interchanges, starting from the left, of consecutive pairs of those items which are out of order we can reorder S_{II} to match up with the sequence $\{t_i\}$ without loss of time. Then start each item on machine II as soon as possible. This reduces the problem from $(n!)^2$ cases to $n!$ cases.

But then use symmetrical arguments (reversing the time scale) to find an optimal sequence on I for a given sequence on II. Repeating this process we eventually find a pair of mutually optimal sequences $(S_{\mathrm{I}}, S_{\mathrm{II}})$. However, and this is the real difficulty, there may be many such pairs of sequences which satisfy this necessary condition for over-all optimality.

Nevertheless, the above technique leads to a proof of the following useful result.

Theorem

A necessary condition for a reversal of order of consecutive items i, j on I to j, i on II in a pair of mutually optimal sequences $(S_{\mathrm{I}}, S_{\mathrm{II}})$ is that

$$M_i > M_j + \max(A_j, B_j). \tag{5}$$

This is also a sufficient condition provided item i is not reversed with its preceding item on I and item j is not reversed with its following item on II.

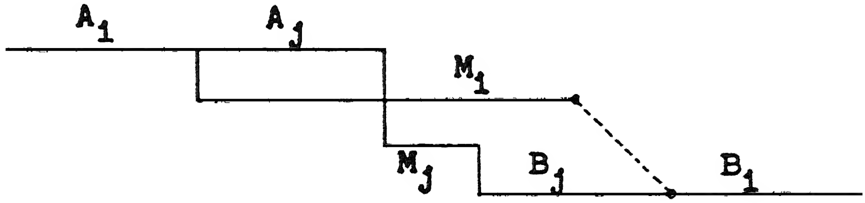
Since $t_i > t_j$, $M_i > A_j + M_j$. Symmetrically, $t'_i > t'_j$ or $M_i > B_j + M_j$, giving (5).

Now for simplicity assume there is only one M_i satisfying (5) for several items j, k, l, say. The question of which item should be interchanged with item i seems to be too hard to answer by any simple decision rule. We propose to try each case and compare. Then if we try reversing i with j where

$$M_j + A_j + B_j > M_i > M_j + \max(A_j, B_j), \tag{5$'$}$$

the two consecutive columns for items i and j in the time matrix

A_i	A_j
M_i	M_j
B_i	B_j



GANTT CHART 1

can be replaced by a single column

$A_i + A_j$
M_j
$B_j + B_i$

corresponding to single fictitious items, as far as computing total time is concerned. To see this consider Gantt Chart 1. Here M_j is the analogous time lag between the finish of the job pair (i, j) on I and the start of the job pair (j, i) on II.

Similarly, if

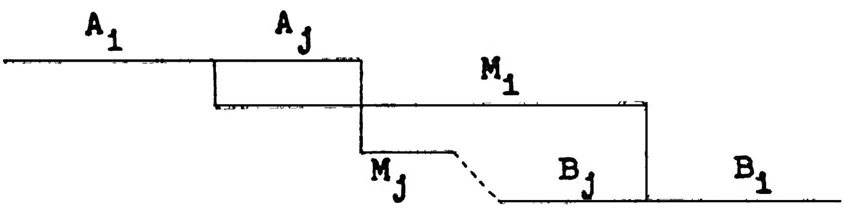
$$M_i > M_j + A_j + B_j, \tag{5''}$$

the new column is

$A_i + A_j$
$M_i - A_j - B_j$
$B_j + B_i$

since from Gantt Chart 2 the analogous delay time is $M_i - A_j - B_j$.

Thus we have replaced the reversed pair of items by a single fictitious item and now are left with a problem of the type solved by Mitten. All that is required is to insert this "item" into its proper place in the sequence given by Mitten's rule.



GANTT CHART 2

Next we try each possible reversal with item i, and compute the total time for such optimally ordered sequences and compare. Note it may still be best to have no reversal for item i even though (5) is satisfied for item i and some item j since they may not be adjacent in the optimal sequencing.

In general, there may be several pairs of items satisfying (5) or even combinations of 3 or more items calling for permutations of order from S_{I} to S_{II}. If there are not too many possible cases, each of these can be worked out and compared.

Further analysis yields some dominance rules concerning which items should be interchanged but the results are too special to be of very much value.

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VI–24

VARIETY IN RETAILING*

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Many marketing problems which promise to be amenable to the techniques of operations research have apparently not been subjected to systematic analysis. This article is a first attempt at an analysis of one such area—the number of items stocked by a retailer and its relation to his sales, his costs, and his profits.

The analysis of the relations among these variables permits the development of criteria for an optimal variety of items; that is, of expressions which can indicate to the retailer whether an increase or a reduction in the number of commodities, styles and brands which he offers for sale will enhance his profits. The discussion also throws some light on a number of well-known retailing phenomena like the growth of suburban shopping centers and supermarkets. By and large, these results are reassuring rather than startling.

The tentative nature of the model cannot be overemphasized. Its structure has purposely been greatly simplified. In particular, linearity has been assumed wherever it does not seem to conflict directly with the properties which the expressions are intended to describe. It is, therefore, noteworthy how often non-linearities have imposed themselves on the model or have arisen out of the mathematical manipulations.

I. Equilibrium of the Consumer

1. *The gains from increased variety*

A shopper does not know in advance (with certainty) whether he will obtain what he wants by entering a particular shop, i.e. whether it does or does not carry some of the items he desires. Generally, there will be one or several alternative sets of items, the availability at acceptable prices of any one of which will make the shopping trip successful in the consumer's view. The greater the number of items carried by the store he enters, the greater, ordinarily, is the consumer's reason for expecting that the shopping trip will in this sense be successful. Of course, this is only true so long as any additional items carried are not known to exclude all commodities desired by the consumer. For example, a known addition to a store's line of paints will not help attract necktie shoppers.

This can readily be translated into probabilistic terms. Let N be the number of different items, i.e., the number of varieties, sold by the retailer. Then we

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may write

$$p(N) \tag{1}$$

for the probability that the consumer will find some set of items in the store which will make his trip successful. On the usual convention, we have $0 \leqq p(N) \leqq 1$ where, for example, $p(N) = 1$ means certain foreknowledge of success. In the absence of specific customer information about the nature of the items carried in the store, $p(N)$ will be close to unity only if the customer is easily satisfied or if N is very large.

Since an increase in the number of items stocked is taken to increase the probability of success in shopping, we also have $dp/dN \geqq 0$.

It must be emphasized that since we are here not primarily interested in the influence of prices and advertising, they are both assumed to remain unchanged throughout. In particular, they are taken to be unaffected by the number of items stocked by the retailer. Of course, this is not likely to occur in practice. The influence of both these variables is clear. When there is a decrease in prices or an increase in informative advertising, there will be an increase in the probability of successful shopping trips, for consumers are then more likely to know which store carries the items they want and are more likely to find them offered for sale at acceptable prices.

2. *The costs of shopping*

In going to some particular store the customer incurs some costs. Some of these represent the cost and trouble of transportation. If the distance of the consumer from the store is D, we assume that for him these costs are strictly proportionate to D and are given by $c_d D$ where c_d is a constant.

Moreover, the difficulty of shopping increases with the number of items stocked by the store—the more items stocked the further we must walk to get to the spot where some items are kept. Roughly speaking, the average distance walked to an item may be expected to increase as the square root of the number of items carried by the store if it is all located on one story since area increases as the square of the radius of a circle or the length of the sides of a rectangle. For similar reasons, if the store operates with a multi-story building we might expect these costs to vary as the cube root of the number of items offered for sale. For our purposes, we shall assume that these costs are directly proportionate with the square root of the number of items stocked, and are given by $c_n\sqrt{N}$.

Finally, there are costs which do not vary with the number of items sold or the consumers' distance from the store. Simply taking the initiative to shop involves time and effort as well as opportunity costs, including other shopping opportunities foregone. For example, a shopper knows that by spending the day shopping, she may be giving up a chance to catch up with her darning or to spend a quiet evening at home. For some who enjoy shopping this cost, c_i, (and perhaps c_n) may be negative. It should be emphasized that c_i is defined as a total, not an average cost, and includes the opportunity cost of foregoing other alternative shopping trips.

Thus the costs of shopping to the consumer are assumed to be given by the sum of these three classes of cost, i.e., by

$$c_dD + c_n\sqrt{N} + c_i\,. \tag{2}$$

3. *The demand function*

Presumably the decision to shop or not to shop at a given retail outlet will result from a weighing of the probability of success as given by (1) against the costs of shopping (2). We assume that the typical consumer does this simply by assigning unconsciously subjective weights w and v (both of which are taken to be positive) to the two components and then seeing which is the larger. The constants v and w are viewed as being invariant over a collection of stores offering similar types of assortments. Here the relevant probability function $p(N)$ is presumably subjective and its relation to the objective probability function is a matter for empirical investigation.

Thus the consumer will not shop at this store unless for him

$$f(N, D) = wp(N) - v(c_dD + c_n\sqrt{N} + c_i) \tag{3}$$

is positive.[1] The function $f(N, D)$ is a measure of the consumer's expected net benefit from entering the store in question and shows how this will vary with D, his distance from the store, and N, the number of items offered for sale there.

We can now examine the effect of variation in the number of items carried. An increase in the variety of items handled by the store will involve an increase in the probability of success but it will also increase shopping costs. We must see how $f(N, D)$ will vary with N. Direct observation yields the following results:

a) When the number of items stocked is small the function will be negative. Specifically, $f(0, D) < 0$ since, when nothing is stocked by the store, the probability of success $p(0) = 0$; and hence the term of which w is the coefficient, and which is ordinarily the only non-negative term in (3), becomes zero. This point amounts to the trivial observation that it does not pay to shop in an empty store.

b) For very large values of N, f will also be negative since the first term in (3) can never[2] exceed w while $vc_n\sqrt{N}$ grows indefinitely large.

c) For intermediate values of N and small values of D, f will be positive if w is sufficiently large relative to v, i.e. if the probability of finding what he wants is weighted sufficiently highly by the customer relative to shopping cost.

d) In this case if the expression is assumed to be continuous throughout, i.e., if $p(N)$ is continuous, f must attain at least one maximum in the intermediate range.

e) After f has passed its maximum, the term $-vc_n\sqrt{N}$ will ultimately domi-

[1] Since c_i takes account of foregone opportunities to shop elsewhere (i.e., $f(N, D)$ is a measure of *net* benefits), the value of f will be negative for any store which does not offer the consumer maximum expected gross benefit.

[2] Of course, this is really only an artifact resulting from the linearity of the expression and the constancy of the model's coefficients but it seems also to be rather reasonable, especially in view of what follows.

nate the expression. Since this term decreases at a decreasing rate (positive second partial derivative with respect to N), this must eventually also be characteristic of f.

f) Marginal and average values of f will be negative for low values of N. They will subsequently become positive and finally decline to a negative value again after reaching a maximum.

4. *Economic implications*

These conclusions have several rather common sense economic implications:

a) Increased variety is an advantage to a consumer only up to a point. Ultimately a store may stock so large a variety of items that shopping costs become prohibitive. This suggests why Sears Roebuck might find it profitable to catalogue many lines which Macy's will not carry. By issuing separate catalogues for different lines some mail order houses have been able to reduce c_n further and thereby have made an even larger N feasible.

b) The minimum number of items necessary to induce a consumer to shop at a given store will increase with D, his distance from that store. This is simply the plausible assertion that the high shopping costs of a distant consumer can only be overcome by a high probability of a successful shopping trip.

c) The optimum variety from the consumer's point of view, i.e., that value of N for which $f(N, D)$ is a maximum, is independent of his distance from the retailer. This was assumed directly in the form postulated for the function f. For the term in $f(N, D)$ which contains D does not contain N. This term will therefore drop out when we solve for the optimum N by setting $\delta f/\delta N = 0$.

d) For every value of N, there will be a maximum consumer distance from the store beyond which it will not pay this consumer to purchase from this shop. The net benefit is a function also of the place of residence of the consumer. Thus about each store and for a particular net benefit, i.e., for a particular value of $f(N, D)$, we may conceive of a contour line or indifference curve associated with a locus of residence about the store. The maximum shopping distance is given by the equation of the indifference curve which offers the consumer zero net benefit from shopping at this store. This maximum distance is obtained by setting $f(N, D) = 0$ (zero net benefit) and solving for D to yield

$$D_m = \frac{w}{vc_d}\,p(N) - \frac{1}{c_d}\,(c_n\sqrt{N} + c_i). \tag{4}$$

More economic implications of our model will be indicated in the next section.

II. The Retailer's Demand Situation

1. *The aggregate demand function*

From the point of view of the retailer, a function very much like $f(N, D)$ may be taken to determine the proportion of the population which shops at his establishment. This may be related directly to sales in the following manner: Suppose, once a customer decides to shop at this particular store, the number

of items he buys is independent of the number of items available. This assumption is clearly false and we shall modify it later. This premise implies that the volume of a store's sales will depend directly on the number of individuals who can be induced to shop there, i.e., on a relationship like (3). At any distance from the retailer, sales will, in the simplest circumstances, vary directly with the proportion of the population which decides to shop at this outlet. This is strictly in accord with our decision to employ linear assumptions wherever possible. In this case, the proportion of the population residing at a distance D from the store which will decide to shop at this store is given by a function similar in form to $f(N, D)$. Let us take capital letters to indicate the parameters in the new function analogous to those represented by lower case letters in f. The new function may then be written

$$(5) \quad F(N, D) = WP(N) - V(C_d D + C_n\sqrt{N} + C_i) \equiv A(N) - VC_d D,$$

where $A(N) = WP(N) - V(C_n\sqrt{N} + C_i)$. It should be noted that since $VC_d D$ does not vary with N both $F(N, D)$ and $A(N)$ will be similarly affected by changes in the value of N. It should also be observed that while f and F are similar in form the latter should be derived independently from the data rather than from some process of aggregation of the f's of different shoppers.

To determine the volume of Sales, we must also know something about the density of population in the area surrounding the store; that is, the distribution of population within the area whose boundaries are given by the relationship gotten by substituting the parameters of F for those of f in (4) and which lie within a distance D_m from the store. We discuss only two very simple possibilities in line with our determination to simplify the model to the utmost:

case i. population per square mile is everywhere given by the constant K so that Population within an area of Radius D_a is $K\pi D_a^2$.

case ii. the store is located at the point of greatest population concentration, and population density, K/D varies inversely with the distance from the retailer. The area lying within a distance D_a from the store is πD_a^2. The population within the circular area of radius D_a is given by

$$\int_0^{\pi D_a^2} \frac{K}{D}\, d\text{ Area} = \int_0^{D_a} \frac{K}{D}\, 2\pi D\, dD = 2\pi K D_a.$$

First consider case i. Here

$$\begin{aligned}
\text{Sales} &= \int_0^{D_m} F(N, D) d \text{ Population} \\
&= \int_0^{D_m} [A(N) - VC_d D] 2\pi K D\, dD \\
&= 2\pi K \left[A(N)\frac{D^2}{2} - VC_d \frac{D^3}{3} \right]_0^{D_m} \\
&= 2\pi K D_m^2 \left(\frac{A(N)}{2} - VC_d \frac{D_m}{3} \right).
\end{aligned}$$

Now from (4) and (5)

$$D_m \doteq \frac{A(N)}{VC_d}.$$

Substituting this in our results yields[3]

$$\text{(6)} \qquad \text{Sales} = \frac{2\pi K}{V^2C_d^2} A(N)^3(\tfrac{1}{2} - \tfrac{1}{3}) = \frac{1}{3}\frac{\pi K}{V^2C_d^2} A(N)^3 = \tfrac{1}{3}VC_d\pi KD_m^3.$$

Turning now to case ii,

$$\text{Sales} = \int_0^{D_m} [A(N) - VC_dD]2\pi K\, dD$$

$$= 2\pi K\left[A(N)D - \frac{VC_dD^2}{2}\right]_0^{D_m}.$$

Thus[4]

$$\text{(7)} \qquad \text{Sales} = 2\pi KD_m\left[A(N) - \frac{VC_dD_m}{2}\right] = \frac{\pi K}{VC_d} A(N)^2 = VC_d\pi KD_m^2.$$

In both cases sales will vary directly with the maximum customer distance, in one case as the cube of that distance and in the other as the square of that distance.

2. *Economic implications*

These results also enable us to discuss the relationship between the expected sales of the retailer and the number of lines he offers for sale directly in terms of $A(N)$. This, as can be seen from (5) will vary with N as does $F(N, D)$ or, by analogy, $f(N, D)$ whose shape we have already analyzed. We are thereby led to a number of economic conclusions.

a) An increased number of items will at first yield increasing average returns, then decreasing marginal and average returns. Finally, it will yield negative marginal returns.

b) This means that, even neglecting considerations of retailer costs, it will

[3] This result can be made intuitively plausible as follows: $A(N)$ is an index of any one customer's inducement to purchase when he is located near the store. It also determines the maximum shopping distance—i.e. the radius D_m is proportionate to $A(N)$—specifically $D_m = A(N)/VC_d$ by (4). Because population is taken to be uniformly distributed, the number of persons in the area will be proportionate to πD_m^2. Since total sales equals sales per person times the number of persons, they will be given by a constant multiplied by $A(N)\pi D_m^2 = A(N)^3\pi/(VC_d)^2$ which is essentially our result.

[4] This implies that sales would be positive for very large or small values of N when $A(N)$ is negative, which is clearly nonsense. This peculiarity arises because $A(N)$, sales per customer would supposedly have negative values at these values of N whereas in fact sales can never be less than zero. It would be more appropriate (though the complication is not worth it) to employ instead of $A(N)$ the function with discontinuous derivative given by $A(N)$ for intermediate values of N but which is zero elsewhere.

not pay a store to proliferate limitlessly the variety of items it carries. There will be some maximum value of sales which can be found by setting the derivatives of (6) and (7) with respect to N equal to zero.

c) The existence of a range of rising average sales deserves attention. This conclusion asserts that not only will total sales be increased up to a point by an increase in the variety stocked but the sales *per item carried* will also rise. Increased variety will then be attracting a disproportionate number of additional customers.

d) Our results are obviously in line with the common sense explanation of the reason why large retailers tend to locate at metropolitan centers since high population density means a large value of K.

e) The results are consistent with the recent increase in emphasis on decentralized retailing and large suburban shopping centers. In terms of our model this can be accounted for in two ways—the increased movement of population toward the suburbs which involves a rise in the K pertaining to the suburban relative to the metropolitan dealers, and the increasing difficulty of driving into and parking in cities which in our model involves increases in C_d and C_i for the metropolitan dealers. This suggests that C_d and C_i are themselves functions of population density.

f) The analysis also fits in with the supermarket phenomenon in grocery retailing. The size of these giant stores can partly be explained by their relatively low C_n—the relative ease with which a consumer can get at additional items which results from the layout permitted by their spaciousness and from their self-service arrangements. Parking lots offer a relatively low C_i. The supermarkets' methods of handling and prepackaging also reduces their inventory and handling costs, and this, as shall be seen in the next section, tends to make for a high value of N, the number of items stocked.

Supermarkets are still increasing the number of items they handle and going into the sale of toiletries, housewares, clothing and appliances. However, it is our impression that no very great further increase in the number of items carried is to be expected in the absence of a marked autonomous or induced change in the value of the coefficients.

3. *Purchases per customer and the number of items stocked*

So far, we have retained the false assumption that purchases per consumer are independent of the number of items stocked by a retailer. Yet up to a point the more items stocked the more likely is a consumer to run into things he had not been planning to buy on this trip but which on being observed become irresistible. This may well serve as a partial offset to the ultimately diminishing returns to an increased number of items carried. But there would appear to be limits to this offset. The customer may not be able to look over more items in a very large store than in a moderately large store simply because of time limitations. After some point, further increases in N may then yield no further increases in sales per customer, though it is conceivable that this value of N is well beyond the relevant range.

III. The Profit Maximizing Variety

1. *The retailers' costs*

To determine an optimum variety from the point of view of the retailers' profits we must include in our model a discussion of the effects of changes in variety on his costs. These effects will primarily involve inventory and handling costs.

In inventory theory, it is customary, as first approximation, to deal with inventory costs as follows:[5] Let E be the mean expected sales volume of all commodities per period, r be the handling, clerical and other related costs of each reordering, T be the warehousing costs per item per period and I the quantity ordered for inventory each time stocks are replaced. Suppose, moreover, that inventory is replaced when, and only when, stocks on hand fall to level R. Then inventory costs per commodity are, on these simplest assumptions, given by

$$\frac{E}{I}r + \left(\frac{I}{2} + R\right)T. \tag{8}$$

The first term represents the cost of keeping the inventory replenished, for E/I is the number of times during a period that inventory will be depleted if sales go on at a steady rate. Since r is the cost of reordering once, then total reordering cost will be r multiplied by E/I, the number of times reordering will take place. The second term in (8) represents warehousing cost since the quantity of the commodity held in stock will vary between $I + R$ and R so that the average level of inventory will be approximately $(I/2) + R$.

Costs can be minimized by picking an appropriate level of I. Setting the first derivative of (8) equal to zero yields the well known result $I = \sqrt{2Er/T}$. Substituting this in (8) gives minimum

$$\text{cost per item} = \frac{Er}{\sqrt{\dfrac{2Er}{T}}} + \frac{1}{2}T\sqrt{\frac{2Er}{T}} + RT$$
$$= \sqrt{2rTE} + RT.$$

In the simplest circumstances, the total cost of carrying N items will be equal to fixed costs, Q plus N times the cost of handling one item plus the additional costs resulting from the increased complexity of handling a variety of items. We may then take this cost to be given by

$$Q + N\sqrt{2rTE} + NRT + a\sqrt{N} \tag{9}$$

where the last term is again given a square root form on the argument that the average distance to any one item will increase as the square root of the number of items so long as all handling takes place in a one-story building.

We can combine (9) with our previous expression for E, the expected mean

[5] See, e.g., T. M. Whitin, *The Theory of Inventory Management*, Princeton, 1953, pp. 31–33.

sales per commodity to obtain an expression for minimum retailing costs. Using (7) rather than (6), for illustrative purposes, total sales are $\pi K/VC_dA(N)^2$ so that average sales will be this expression divided by N.

Substituting this for E in (9) and writing $b = \sqrt{2\pi KrT/VC_d}$ yields as the expression for minimum retailing cost

$$Q + b\sqrt{N}A(N) + NRT + a\sqrt{N}. \tag{10}$$

2. *The retailer's profits*

We may now obtain an expression for the retailer's total profits by subtracting his total costs (10) from his total revenues which can be obtained by multiplying the sales volume (7) by an appropriate price index. Let $s = (\pi K/VC_d)p^*$ where p^* is the price index; then

$$\text{Profits} = sA(N)^2 - Q - \sqrt{N}[bA(N) + a] - NRT.$$

As we have seen, we may expect $A(N)$ to have two positive roots at which points nothing will be sold and so profit will be negative. In between we may expect for reasonable values of the coefficients that the expression for profits will somewhere rise to a maximum which we can find by setting the first derivative equal to zero. This will indicate the optimum variety in his merchandise from the point of view of the retailer.

In particular, it is easy to see that the higher the handling and inventory costs, i.e., the higher a, r and T (and hence, the higher the value of b) the lower will be the optimal value of N, i.e. the smaller the variety it will pay to stock.

As in many operations research analyses the results have been formulated in terms for which there is no simply obtained quantitative empirical counterpart. In applying results like these, improvisation and ingenuity will no doubt be required to obtain even approximations to the true parameters. Moreover, this very preliminary model will no doubt have to be modified and tailored case by case to fit the facts of the situation, and even then computed results will have to be interpreted and employed only with extreme caution.

VI-25

AN AXIOMATIZATION OF UTILITY BASED ON THE NOTION OF UTILITY DIFFERENCES[1]

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1. Introduction

In the literature of economics (e.g., [1], [9], [14]) the notion of utility differences has been much discussed in connection with the theory of measurement of utility.[2] However, to the best of our knowledge, no adequate axiomatization for this difference notion has yet been given at a level of generality and precision comparable to the von Neumann and Morgenstern construction of a probabilistic scheme for measuring utility. (The early study of Wiener ([21]) is not axiomatically oriented.) The purpose of this paper is to present an axiomatization of this notion and to establish the expected representation theorem guaranteeing measurement unique up to a linear transformation.

Recent experimental work by economists and psychologists (see the bibliography in [8]) suggests there are cogent reasons for reviving the notion of utility differences in order clearly to separate utility and subjective probability. The interaction between probability and utility makes it difficult to make unequivocal measurements of either one or the other. The recent Mosteller and Nogee experiments ([15]) may be interpreted as measuring utility if objective probabilities are assumed or as measuring subjective probabilities if utility is assumed linear in money.

In [6] and [7] a detailed description is given of how utility may be experimentally measured by use of utility differences and a *single* chance event with subjective probability ½.

The scheme may be briefly described as follows.[3] Let E^* be a chance event with subjective probability ½, and suppose that the individual we are testing prefers outcome x to y, and outcome z to w. We present him with two alternative gambles, one of which he must choose. Gamble 1 is that if E^* occurs he gets x, and if E^* does not occur he gets w; Gamble 2 is that if E^* occurs he gets z, and if E^* does not occur he gets y. It seems intuitively reasonable to say that the individual should prefer Gamble 2 if and only if the utility difference between x and y is less than that between z and w. Once utility is measured by a procedure of this kind, we may measure subjective probabilities. (To some extent, this approach was anticipated in [17].)

Since the chance event E^* is fixed throughout the discussion, it does not play

[1] This work was supported in part by the Office of Ordnance Research, U. S. Army, and in part by the Stanford Value Theory Project.

[2] The formally similar notion of sensation differences is important in the literature of psychology. (e.g., [3], [12], [13], [19], [20].)

[3] The intuitive idea of this approach was primarily due to Professor Donald Davidson. It was suggested in [5] and has been the basis for the experiments reported in [6].

any formal role in our axiomatization and enters only via one particular empirical interpretation of the notion of utility differences. Consequently, interpretations of our primitive notions, completely divorced from any probability questions, are available for analyzing other approaches to utility theory. A justification for considering alternative schemes is the limited applicability of the probabilistic approach just described. It can and has been used in some laboratory experiments at Stanford (see [6]), but it is far from clear that it can be seriously applied to market behavior. An interpretation of utility differences in terms of amounts of money is an obvious alternative. We present such a scheme in the form of a *reduction* sentence (the general character of reduction sentences is discussed in [2]). For simplicity we consider a fixed individual, say, Jones, and we assume that a prior satisfactory analysis of preference (as opposed to preference differences) has already been given.

(1) IF: (i) *Jones prefers commodity x to commodity y, and commodity u to commodity v,* (ii) *Jones has in his possession commodities y and v, and* (iii) *Jones is presented with the opportunity of paying money to replace y by x and v by u,* THEN: *the utility difference between x and y is at least as great as that between u and v if and only if Jones will pay at least as much money to replace y by x as to replace v by u.*

An obvious objection to (1) is that it has the effect, so often argued against, of *measuring* utility in terms of money. However, the only assumption needed for (1) is that the relation between amounts of money and utility differences is monotonic increasing. A linear relation is *not* required. In our opinion such a monotonicity assumption is very reasonable for a wide variety of persons and situations.

An alternative reduction may easily be stated in terms of work. It should be clear that the choice of money or work is not meant to entail any special status for these two commodities. What is needed as a basis for constructing other reductions is simply the existence of a commodity flexible enough to serve in different situations and such that its marginal utility is either always positive or always negative in the situations under consideration.

In view of the many complex issues involved in assessing the workability, even in principle, of such reductions, it may be more useful to describe a particular experimental set-up which could be used to measure utility differences. For reasons which will become obvious, this scheme would not be directly applicable to market behavior, but on the other hand it does not presuppose any fixed relations between money and other commodities.

For definiteness, we consider six household appliances of approximately the same monetary value, for instance, a mixer, a deluxe toaster, an electric broiler, a blender, a waffle iron and a waxer. A housewife who does not own any of the six is chosen as subject. Two of the appliances are selected at random and presented to the housewife, say, the toaster and the waxer. She is then confronted with the choice of trading the toaster for the waffle iron, or the waxer for the blender. Presumably she will exchange the toaster for the waffle iron if and only if the utility difference between the waffle iron and the toaster is at least as great as the difference between the blender and the waxer (due account being taken of the algebraic sign of the difference). A sequence of such exchanges (repetitions

permitted) can easily be devised such that every utility difference is compared to every other. Our axioms specify for the set of choices sufficient ideal properties to guarantee the existence of a cardinal utility function.[4]

From another conceptual standpoint (as pointed out to us by our colleague, Professor Davidson), we may think of the housewife as expressing a simple preference between *pairs* of appliances. Thus if she trades the toaster for the waffle iron she has decided that she would rather have the pair (waffle iron, waxer) than the pair (toaster, blender). Put in these terms we are asking for a utility function φ of the Frisch and Fisher type ([10], [11]) such that one pair (x, y) is preferred to another (u, v) if and only if

$$\varphi(x) + \varphi(y) > \varphi(u) + \varphi(v).$$

The existence of such a function is taken to mean that "utilities are independent," that is, the commodities involved are neither complementary nor competitive with respect to each other. Viewed in this light, our axioms analyze the special conditions required for the existence of a cardinal utility function on a set of *independent* commodities. Whatever one's *a priori* feelings about the plausibility of the independence hypothesis there can be little doubt that the experiment just described would provide a means of empirically testing the hypothesis,[5] and thus would satisfy Samuelson's methodological demand ([18], p. 183):

> It may be argued that regarded purely as a working hypothesis the facts do not sharply contradict the independence assumption. A little investigation reveals that such a hypothesis has not been tested from this point of view. On the contrary, it is implicitly assumed from the beginning in the manipulation of the statistical data. Hence, one would have to go back to examine the original empirical data.

It is interesting to note that the problem of complementarity occupies a position in this interpretation analagous to the position occupied by the problem of a specific utility of gambling in a probabilistic interpretation.

It is also our opinion that many areas of economic and modern statistical theory do not warrant a behavioristic analysis of utility. In these domains, there seems little reason to be ashamed of direct appeals to introspection. For example, in welfare economics there are sound arguments for adopting a subjective view which would justify the determination of utility differences by introspective methods. Some psychological experiments on utility differences which essentially use introspective methods are reported in [4].

It is to be emphasized that the formal results presented in the remainder of this paper do not depend on any of the particular interpretations here proposed.

2. Primitive and Defined Notions

Our axiomatization is based on three primitive notions. The primitive K is a non-empty set, to be interpreted as a set of alternatives (objects, experiences,

[4] By considering just six items, we cannot get a realization of the axioms given in Section 3. However, by increasing the number of items, we would presumably be able to get a successively closer approximation.

[5] Some experiments are planned in collaboration with Professor Davidson.

events, or decisions) available to a given individual at a given time. The primitive Q is a binary relation whose field is K; the interpretation of Q is that $x\ Q\ y$ if and only if the individual does not prefer y to x. The third primitive is a quaternary relation R whose field is also K. In the intended interpretation $x, y\ R\ z, w$ if and only if the difference in preference between x and y is not greater than the difference in preference between z and w.

Our axiomatization assumes a rather complicated form if it is given only in terms of our three primitives. It is intuitively desirable to use some defined notions whose interpretation follows directly from that of the primitives.

Definition D1. *$x\ I\ y$ if and only if $x\ Q\ y$ and $y\ Q\ x$.* Obviously, I is the relation of *indifference.*

Definition D2. *$x\ P\ y$ if and only if not $y\ Q\ x$.* The relation P is the relation of *strict preference.*

Definition D3. *$x, y\ E\ z, w$ if and only if $x, y\ R\ z, w$ and $z, w\ R\ x, y$.* The interpretation of the quaternary relation E is that if x, y, z and w are alternatives, then $x, y\ E\ z, w$ if and only if the difference in preference between x and y is *equivalent* to the difference in preference between z and w.

Definition D4. *$x, y\ S\ z, w$ if and only if not $z, w\ R\ x, y$.* Clearly, $x, y\ S\ z, w$ if and only if the difference in preference between x and y is *strictly less* than the difference between z and w.

Definition D5. *$B(y, x, z)$ if and only if either $x\ P\ y$ and $y\ P\ z$, or $z\ P\ y$ and $y\ P\ x$.* The intuitive idea of *betweenness* is expressed by the relation B.

The above notions suffice for the statement of all but the last axiom, the Archimedean axiom. For the latter, one further quaternary relation is needed.

Definition D6. *$x, y\ M\ z, w$ if and only if $y\ I\ z$ and $B(y, x, w)$ and $x, y\ E\ z, w$.* The quaternary relation M appears to be a trivial specialization of the relation E. To clarify this situation, we introduce the notion of powers of M. The second power of M, for example, is the relation M^2 such that $x, y\ M^2\ z, w$ if and only if there exist elements u and v such that $x, y\ M\ u, v$ and $u, v\ M\ z, w$. The n^{th} power of M is defined recursively:

$$x, y\ M^1\ z, w \text{ if and only if, } x, y\ M\ z, w;$$

$$x, y\ M^n\ z, w \text{ if and only if there exist elements } u \text{ and } v \text{ such that}$$

$$x, y\ M^{n-1}\ u, v \text{ and } u, v\ M\ z, w.$$

The difference between powers of E and of M may be brought out by interpreting x, y, z, and w as points on a line. The interpretation of $x, y\ M^3\ z, w$, for instance, is that the intervals (x, y) and (z, w) are of the same length, and there are two intervals of this length between y and z. Of special significance is the fact that the interval (x, w) is four times the length of (x, y). On the other hand, in the case of the relation E^3 no specific length relation may be inferred for intervals (x, w) and (x, y).

As we shall see in Section 5, the proof of our representation theorem essentially depends on exploiting the properties of the powers of M.

3. Axioms

Using our primitive and defined notions, we now state our axioms for difference structures.

A system $\mathcal{K} = \langle K, Q, R \rangle$ *will be said to be a* DIFFERENCE STRUCTURE *if the following eleven axioms are satisfied for every* x, y, z, w, u, *and* v, in K:

Axiom A1. $x\ Q\ y$ *or* $y\ Q\ x$;

Axiom A2. *If* $x\ Q\ y$ *and* $y\ Q\ z$ *then* $x\ Q\ z$;

Axiom A3. $x, y\ R\ z, w$ *or* $z, w\ R\ x, y$;

Axiom A4. *If* $x, y\ R\ z, w$ *and* $z, w\ R\ u, v$ *then* $x, y\ R\ u, v$;

Axiom A5. $x, y\ R\ y, x$;

Axiom A6. *There is a* t *in* K *such that* $x, t\ E\ t, y$;

Axiom A7. *If* $x\ I\ y$ *and* $x, z\ R\ u, v$, *then* $y, z\ R\ u, v$;

Axiom A8. *If* $B(y, x, z)$ *then* $x, y\ S\ x, z$;

Axiom A9. *If* $B(y, x, z)$ *and* $B(w, u, v)$ *and* $x, y\ R\ u, w$ *and* $y, z\ R\ w, v$, *then* $x, z\ R\ u, v$;

Axiom A10. *If* $x, y\ S\ u, v$ *then there is a* t *in* K *such that* $B(t, u, v)$ *and* $x, y\ R\ u, t$;

Axiom A11. *If* $x, y\ R\ u, v$ *and not* $x\ I\ y$, *then there are elements* s *and* t *in* K *and a positive integer* n *such that* $u, s\ M^n\ t, v$ *and* $u, s\ R\ x, y$.

The interpretation of Axioms A1–A4 is obvious. Axiom A5 expresses a commutativity property of R and means essentially that for pairs of elements to stand in the relation R only their differences matter and not their relative order.

Axiom A6 means intuitively that between any two elements of K, there is a midpoint. This axiom represents a more reasonable assumption than, for instance, a formulation requiring that between any two elements there exist an element some arbitrary part, say $1/17^{\text{th}}$, of the distance between them. Indeed, the axiom as here stated, receives empirical corroboration in the field of psychology from the practice of "fractionation" and "bisection" experiments requiring the subject to select the tones in just the way described, and from the existence of laboratory equipment designed for such experimental use. (See, e.g., [19] and [20].) Also, the probabilistic experiments ([6]) described in the first section have demonstrated the practicality of finding such midpoints.

Axiom A10 means that if the difference between x and y is less than that between u and v, then there is an element t of K between u and v and the difference between x and y is not greater than the difference between u and t.

Axiom A11, the Archimedean axiom, means that if the difference between x and y is not greater than that between u and v, and if x is not indifferent to y, then there are n elements of K equally spaced in utility between u and v such that the difference between any consecutive two of these elements is not greater than the difference between x and y.

4. Elementary Theorems

A rather large number of elementary theorems is required for the complete proof of our representation theorem for difference structures. In the present paper, however, we are concerned merely to sketch the main outlines of such a

proof; and, for this purpose, it will be sufficient in this section to present definitions of certain relations, not needed for stating the axioms, but used in a key way to develop the required proof; and to state without proof several elementary theorems which describe typical properties of the relations defined, or which figure centrally in the sketched proof of the representation theorem. In particular, we omit completely a large group of theorems which develops the expected properties of Q and R and of the other simple "qualitative" relations (I, P, E, S, B) described in Section 2.

We first introduce the notion of the quaternary relation $N(a)$.

Definition D7. *$N(a)$ is the quaternary relation defined as follows*

i) *if $a = 1$, then $x, y\ N(a)\ u, v$ if and only if $x\ I\ u$ and $y\ I\ v$*

ii) *if $a \neq 1$, then $x, y\ N(a)\ u, v$ if and only if $x\ I\ u$ and there exists a z such that $x, y\ M^{a-1}\ z, v$.*

The interpretation of $N(1)$, of course, is obvious. To say for $a \neq 1$, that x, y $N(a)$ u, v means that x and u coincide, and that there are $a - 1$ equally spaced elements of K between u and v such that the difference between any two of them equals the difference between x and y. If x, y, u and v are interpreted as points on a line, this notion obviously corresponds to the intuitive notion of "laying off" an interval on another interval; that is, we interpret $x, y\ N(a)\ u, v$ intuitively as meaning that if we start from u, and "lay off" an interval of the length (x, y) a times in the appropriate direction, we obtain the interval (u, v). By means of the $N(a)$ relation, therefore, we are able to express the quantitative fact that the length of an interval (u, v) is a times the length of a subinterval (x, y).

The sort of "multiplication" of intervals characterized by the $N(a)$ relation possesses the expected properties; for example, we have the following theorem concerning ratios of intervals.

Theorem 1. *If $x, z\ N(a)\ x, y$ and $x, z\ N(ab)\ x, w$ then $x, y\ N(b)\ x, w$.*

Another theorem involving the $N(a)$ relation generalizes $A6$ and may be justified along similar lines. Characteristic of our system, it asserts that appropriate elements exist for dividing any interval into powers of 2.

Theorem 2. *If not $x\ I\ y$ then there is a z such that $x, z\ N(2^m)\ x, y$.*

Further $N(a)$-theorems state properties of "N-multiplication" for powers of 2. We have, for example, the usual law for addition of exponents:

Theorem 3. *If $x, w\ N(2^m)\ x, z$ and $x, z\ N(2^n)\ x, y$ then $x, w\ N(2^{m+n})\ x, y$.*

A crucial, but less obvious property is stated in the following theorem.

Theorem 4. *If $B(y, x, z)$ and $x, t\ N(2^m)\ x, y$ and $y, s\ N(2^m)\ y, z$ and $x, r\ N(2^m)\ x, z$ then $t, r\ E\ y, s$.*

We now define a relation in terms of which most of the proof of the representation theorem is carried through.

Definition D9. *$H(m, a; n, b)$ is the quaternary relation such that $x, y\ H(m, a; n, b)\ u, v$ if and only if there are elements z_1, z_2, w_1 and w_2 such that $x, z_1\ N(2^m)\ x, y$ and $u, w_1\ N(2^n)\ u, v$ and $x, z_1\ N(a)\ x, z_2$ and $u, w_1\ N(b)\ u, w_2$ and $x, z_2\ R\ u, w_2$.*

To say that $x, y\ H(m, a; n, b)\ u, v$ means intuitively that an $(a/2^m)^{\text{th}}$ part of the interval (x, y) is not greater than a $(b/2^n)^{\text{th}}$ part of the interval (u, v).

We may view our first theorem on this notion as enabling us to specify a partial

bound for the values of arguments satisfying the H-relation between two intervals.

Theorem 5. *If not* $x\ I\ y$ *and* $x, y\ H(m, a; n, b)\ u, v$, *then not* $u, v\ H(n, b; m + 1, a)\ x, y$.

Since the H-relation can be thought of intuitively as a special sort of inequality, we would expect to be able to prove many of the laws governing inequalities. Thus Theorem 6 expresses a kind of transitivity property and Theorem 7 an intuitively simple conservation property. Theorems 8, 9, 10 and 11 assert cancellation and multiplication laws.

Theorem 6. *If* $x, y\ H(m, a; n, b)\ u, v$ *and* $u, v\ H(n, b; p, c)\ r, s$ *then* $x, y\ H(m, a; p, c)\ r, s$.

Theorem 7. *If not* $x, y\ H(m, a; n, b)\ u, v$ *and* $w, z\ H(p, c; n, b)\ u, v$ *and* $a \leqq 2^m$ *and not* $x\ I\ y$, *then not* $x, y\ H(m, a; p, c)\ w, z$.

Theorem 8. *If* $x, y\ H(m, a; n, b)\ u, v$ *and* $ac \leqq 2^m$ *and* $bc \leqq 2^n$, *then* $x, y\ H(m, ac; n, bc)\ u, v$.

Theorem 9. *If* $x, y\ H(m, ac; n, bc)\ u, v$, *then* $x, y\ H(m, a; n, b)\ u, v$.

Theorem 10. *If* $x, y\ H(m, a; n, b)\ u, v$ *and either* $m \neq 0$ *or not* $x\ I\ y$, *then* $x, y\ H(m + c, a; n + c, b)\ u, v$.

Theorem 11. *If* $x, y\ H(m + c, a; n + c, b)\ u, v$ *and* $a \leqq 2^m$ *and* $b \leqq 2^n$ *then* $x, y\ H(m, a; n, b)\ u, v$.

Theorem 12 states an addition property for the arguments of the H-relation in the case of adjacent intervals.

Theorem 12. *If* $B(y, x, z)$ *and* $a + b \leqq 2^n$ *and* $x, y\ H(m, 1; n, a)\ u, v$ *and* $y, z\ H(m, 1; n, b)\ u, v$ *then* $x, z\ H(m, 1; n, a + b)\ u, v$.[6]

Finally, we state two existence theorems for arguments of the H-relation. These theorems are the form in which we make use of our purely qualitative continuity axiom (A10) and our Archimedean axiom (A11) respectively.

Theorem 13. *If* $x, y\ S\ u, v$, *then there are integers* b *and* n *such that* $b < 2^n$ *and* $x, y\ H(0, 1; n, b)\ u, v$.

Theorem 14. *If not* $u\ I\ v$, *then there is an integer* m *such that* $x, y\ H(m, 1; 0, 1)\ u, v$.

5. Representation Theorem

Our desired representation theorem is an immediate consequence of the following lemma. (As a matter of fact, it is rather customary in the theory of measurement to label a lemma of this sort the "theorem of adequacy" and not to state explicitly a representation theorem. Cf., e.g., [16], pp. 24–29.)

Fundamental Lemma. Let $\mathfrak{K} = \langle K, Q, R\rangle$ *be a difference structure. Then:*

A) *There exists a real-valued function* ϕ *defined on* K *such that for every* x, y, z, w *in* K,

i) $x\ Q\ y$ *if and only if* $\phi(x) \leqq \phi(y)$, *and*

ii) $x, y\ R\ z, w$ *if and only if* $|\phi(x) - \phi(y)| \leqq |\phi(z) - \phi(w)|$.

B) *if* ϕ_1 *and* ϕ_2 *are any two functions satisfying* (A), *then there exist real numbers* α *and* β *with* $\alpha > 0$ *such that for every* x *in* K, $\phi_1(x) = \alpha\phi_2(x) + \beta$.

[6] We are indebted to Professor Herman Rubin for the proof of this theorem.

Proof: *Part* A. We begin by choosing two elements u and v in K such that $u\ P\ v$ (if no such two elements exist, the proof is trivial). We next define for x and y in K the set of numbers $\mathcal{S}(x, y; u, v)$. A rational number r is in $\mathcal{S}(x, y; u, v)$ if and only if there are non-negative integers m and n and a positive integer b such that $b \leqq 2^n$ and $r = (b2^m)/2^n$ and $x, y\ H(m, 1; n, b)\ u, v$.

Let r and r' be positive rational numbers. Using Theorems 8, 10, 6, 9 and 11, in that order, we may easily prove that

(1) If $r \in \mathcal{S}(x, y; u, v)$ and $r < r'$ then $r' \in \mathcal{S}(x, y; u, v)$.

Using now principally Theorem 14 and Theorem 5 we may show that if not $x\ I\ y$ then the set $\mathcal{S}(x, y; u, v)$ has a positive number as a lower bound. Since by Theorem 14 $\mathcal{S}(x, y; u, v)$ is not empty, we conclude that it has a greatest lower bound. We use this fact to define the function $f_{(u,v)}$:

$$f_{(u,v)}(x, y) \text{ is the greatest lower bound of } \mathcal{S}(x, y; u, v).$$

Obvious arguments prove that

$$f_{(u,v)}(x, y) = 0 \text{ if and only if } x\ I\ y$$

and

$$f_{(u,v)}(u, v) = 1,$$

the choice of (u, v) thus corresponding to choice of a unit of length.

We obtain by an indirect argument from (1) that for any rational number r

(2) If $f_{(u,v)}(x, y) < r$ then $r \in \mathcal{S}(x, y; u, v)$,

and we are in a position to establish:

(3) If $x, y\ R\ z, w$ then $f_{(u,v)}(x, y) \leqq f_{(u,v)}(z, w)$.

(The proof is trivial in case $x\ I\ y$; hence we assume: not $x\ I\ y$.) Suppose, if possible, that $f_{(u,v)}(z, w) < f_{(u,v)}(x, y)$. Then there are integers m, n, b such that

$$f_{(u,v)}(z, w) < \frac{b2^m}{2^n} < f_{(u,v)}(x, y).$$

From (2) we then obtain:

$$z, w\ H(m, 1; n, b)\ u, v \quad \text{and not} \quad x, y\ H(m, 1; n, b)\ u, v.$$

Hence by Theorem 7, not $x, y\ H(m, 1; m, 1)\ z, w$, and thus by Theorem 10 and D9, not $x, y\ R\ z, w$, which contradicts the hypothesis of (3).

We next prove:

(4) If $f_{(u,v)}(x, y) \leqq f_{(u,v)}(z, w)$ then $x, y\ R\ z, w$.

Let

$$r = \frac{b_1 2^{m_1}}{2^{n_1}} \quad \text{be in} \quad \mathcal{S}(x, y; u, v)$$

and let

$$q = \frac{b_2 2^{m_2}}{2^{n_2}} \quad \text{be in} \quad \mathcal{S}(z, w; x, z).$$

Then we have: $b_1 \leqq 2^{n_1}$, $b_2 \leqq 2^{n_2}$, $x, y\ H(m_1, 1; n_1, b_1)\ u, v$, and $z, w\ H(m_2, 1; n_2, b_2)\ x, y$. Hence by Theorem 10, Theorem 8, and Theorem 6, $z, w\ H(m_1 + m_2, 1; n_1 + n_2, b_1 b_2)\ u, v$. We conclude that

(5) $$rq \quad \text{is in} \quad \mathcal{S}(z, w; u, v).$$

Now for the moment let

$$\alpha = f_{(u,v)}(x, y)$$
$$\beta = f_{(x,y)}(z, w)$$
$$\gamma = f_{(u,v)}(z, w).$$

Suppose, if possible, that $\alpha\beta < \gamma$. Then there is a positive ε such that $(\alpha + \varepsilon) \cdot (\beta + \varepsilon) = \gamma$. Clearly we may choose a number r in the open interval $(\alpha, \alpha + \varepsilon)$ and a number q in the open interval $(\beta, \beta + \varepsilon)$ such that r is in $\mathcal{S}(x, y; u, v)$ and q is in $\mathcal{S}(z, w; x, y)$. Since $rq < \gamma$, rq is not in $\mathcal{S}(z, w; u, v)$, but this contradicts (5), and we conclude that

(6) $$f_{(u,v)}(x, y) \cdot f_{(x,y)}(z, w) \geqq f_{(u,v)}(z, w).$$

Suppose now that not $x, y\ R\ z, w$. By Theorem 13 it follows that there is an n and a b with $b/2^n < 1$ such that $z, w\ H(0, 1; n, b)\ x, y$, and we conclude that $f_{(x,y)}(z, w) < 1$. Combined with (6), this result gives us: $f_{(u,v)}(x, y) > f_{(u,v)}(z, w)$, which contradicts our hypothesis, completing the proof of (4).

We now define the function $\phi_{(u,v)}$ as follows. For every x in K,

$$\phi_{(u,v)}(x) = \begin{cases} f_{(u,v)}(u, x), \text{ if } u\ Q\ x. \\ -f_{(u,v)}(u, x), \text{ if } x\ Q\ u. \end{cases}$$

We see at once that $\phi_{(u,v)}(u) = 0$, and thus our choice of u corresponds to the choice of an origin. (3) and (4) provide the basis for an obvious proof that

(7) $$x\ Q\ y \text{ if and only if } \phi_{(u,v)}(x) \leqq \phi_{(u,v)}(y).$$

To complete the proof of Part A we need to show that

(8) $$x, y\ R\ z, w \text{ if and only if } |\phi_{(u,v)}(x) - \phi_{(u,v)}(y)| \leqq |\phi_{(u,v)}(z) - \phi_{(u,v)}(w)|.$$

From (3) and (4) we see at once that it will be sufficient to prove

(9) $$f_{(u,v)}(x, y) = |\phi_{(u,v)}(x) - \phi_{(u,v)}(y)|.$$

Of the five possible cases that need to be considered for (9) we consider only the typical one where $x\ P\ y$ and $y\ P\ u$. For this case we must prove:

(10) $$f_{(u,v)}(x, y) + f_{(u,v)}(u, y) = f_{(u,v)}(u, x).$$

Suppose, if possible, that

$$f_{(u,v)}(x, y) + f_{(u,v)}(u, y) < f_{(u,v)}(u, x).$$

Then clearly there are integers m, n, b, b_1, b_2 such that

$$f_{(u,v)}(x, y) + f_{(u,v)}(u, y) < \frac{b2^m}{2^n} < f_{(u,v)}(u, x),$$

(11) $$f_{(u,v)}(x, y) < \frac{b_1 2^m}{2^n}$$

$$f_{(u,v)}(u, y) < \frac{b_2 2^m}{2^n}, \qquad \text{and}$$

$$b = b_1 + b_2 \leqq 2^n.$$

By (2) we have: $x, y\ H(m, 1; n, b_1)\ u, v$ and $y, u\ H(m, 1; n, b_2)\ u, v$. Hence by Theorem 12, $x, u\ H(m, 1; n, b_1 + b_2)\ u, v$, but from (11), we infer: not $x, u\ H(m, 1; n, b_1 + b_2)\ u, v$. On the basis of this contradiction, we conclude that

(12) $$f_{(u,v)}(x, y) + f_{(u,v)}(u, y) \geqq f_{(u,v)}(u, x),$$

and by an argument similar to the above we may show that equality holds in (12), thus establishing (9) for a typical case, and completing the proof of Part A.

Part B. Using elements u and v in K as in the proof of (A), we define functions h_1 and h_2 for every x in K by the equations:

$$h_1(x) = \frac{\phi_1(x) - \phi_1(u)}{\phi_1(v) - \phi_1(u)}$$

$$h_2(x) = \frac{\phi_2(x) - \phi_2(u)}{\phi_2(v) - \phi_2(u)},$$

where ϕ_1 and ϕ_2 are functions satisfying (A). Since $u\ P\ v$, we see at once that

$$h_1(u) = h_2(u) = 0$$

$$h_1(v) = h_2(v) = 1,$$

and that h_1 and h_2 satisfy (A). Thus in order to establish (B) it will be sufficient to prove that

(1) $$h_1 = h_2 .$$

We give the proof for the case where $u\ P\ x$ and $x\ P\ v$. Suppose, if possible, that $h_1(x) \neq h_2(x)$. For definiteness, let $h_1(x) < h_2(x)$. Then there is a positive ε such that

(2) $$h_2(x) = h_1(x) + \varepsilon.$$

We now consider the smallest integer, say, n^*, such that $\frac{1}{2}^{n^*} < \varepsilon$. (Since $h_1(x)$ and $h_2(x)$ are both between 0 and 1, $n^* \neq 0$.) By Theorem 2 there exists an

element, say, z^*, such that $u, z^* N(2^{n^*})\ u, v$. A simple argument shows that we must have: $z^* P x$.

Suppose now that there is an integer a such that $u, z^*\ N(a)\ u, x$. It is easy to prove by induction that we must then be able to infer:

(3) $$h_1(x) = h_2(x) = a/2^{n^*},$$

which contradicts (2).

Since on the supposition of (2) there is no such integer a, there must be an integer b and elements z_1 and z_2 such that

(4) $$\begin{cases} u, z^*\ N(b)\ u, z_1 \\ u, z^*\ N(b+1)\ u, z_2 \\ z_1\ P\ x \\ x\ P\ z_2\ . \end{cases}$$

Using the induction which yielded (3), we have from (4),

$$h_2(z_2) - h_1(z_1) = \frac{b+1}{2^{n^*}} - \frac{b}{2^{n^*}} < \varepsilon,$$

and we also obtain from (4):

$$h_1(z_1) < h_1(x) < h_1(z_2)$$
$$h_2(z_1) < h_2(x) < h_2(z_2).$$

Combining inequalities we conclude:

$$h_2(x) - h_1(x) < h_2(z_2) - h_1(z_1) < \varepsilon,$$

which contradicts (2).

The proof of (1) is completed by a consideration of the four other possible cases for the position of x with respect to u and v. (Two of the cases are trivial: $u\ I\ x$ and $v\ I\ x$.) Since (1) establishes (B), the proof of our lemma is finished.

We would not expect to have a strict isomorphism between an arbitrary difference structure $\mathcal{K} = \langle K, Q, R \rangle$ and some numerical structure, since distinct elements which stand in the relation I are assigned the same number. However by considering the coset algebra $\mathcal{K}/I = \langle K/I, Q/I, R/I \rangle$ of $\mathcal{K}$ under I, we may easily establish such an isomorphism. (Since I is obviously a congruence relation on K with respect to Q and R, it should be clear that K/I is the set of all I-equivalence classes and that Q/I and R/I are the relations between equivalence classes corresponding to Q and R.)

We define the quaternary relation T for real numbers as follows:

> if α, β, γ, and δ are real numbers, then $\alpha, \beta\ T\ \gamma, \delta$
>
> if and only if $|\alpha - \beta| \leqq |\gamma - \delta|$.

Let N be a set of real numbers. Then we call an ordered triple $\langle N, \leqq, T \rangle$ a *numerical difference structure* if N is closed under the formation of mid-points,

i.e., if α, β, are in N, then $(\alpha + \beta/2)$ is in N. We then obtain the following representation theorem as an immediate consequence of our lemma.

Representation Theorem. If $\mathcal{K} = \langle K, Q, R \rangle$ *is a difference structure, then* $\mathcal{K}/I = \langle K/I, Q/I, R/I \rangle$ *is isomorphic to a numerical difference structure. Moreover any two numerical difference structures isomorphic to* $\mathcal{K}/I$ *are related by a linear transformation.*

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VI–26

AGGREGATION OF UTILITY FUNCTIONS*[1]

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We are primarily concerned here with the question of integrability of the total demand in a model in which each consumer acts according to a cardinal utility function and has a fixed monetary income. It is well known that concavity of the various utilities is not sufficient to guarantee integrability, nor even to ensure rationality of the revealed preference. We show that if each personal utility function is homogeneous, in addition to satisfying the usual regularity conditions, then an aggregate utility function can be defined explicitly in terms of the given utilities. Furthermore, under the same assumptions we give a new characterization of equilibrium and show that equilibrium satisfactions are unique.

1. Introduction

The economic model to be considered is the following: In a market there are m consumers, each competing for one or more of a set of n commodities. Every consumer has a fixed positive monetary income[2] while production is outside the model, i.e., we do not consider negative quantities of the goods in question. A consumer's behavior is completely described by a personal utility function; when prices for the various goods are given, he will act so as to maximize his utility subject to his budget restriction. One of the first questions that arise is

i) For each set of prices we have the total demand. Can this demand be thought of as expressing the behavior of a single (fictitious) consumer acting according to a well-defined (aggregate) utility function?

The existence of an aggregate utility function would tell us, among other things, that the community revealed preference is rational, i.e., if a bundle x is preferred to a bundle y[3] and simultaneously y is preferred to x, then y is demanded whenever x is demanded and y can be purchased. In section IV we demonstrate a model for which the revealed preference is not rational, so that an aggregate utility function need not always exist.

Under certain circumstances the answer to i) is, as evidenced by relation (20)

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[1] This paper is based partly on results obtained while the author was a consultant with The RAND Corporation.

[2] In the classical exchange model each consumer holds a fixed commodity bundle for trading purposes. It can be shown that a fixed monetary income model can be thought of as a special case of the exchange model. The specialization is: If a consumer holds two commodities in quantities ξ and η, respectively, while another consumer holds ξ' and η' units of the same commodities then $\xi\eta' = \xi'\eta$.

[3] If a demand is given as a function of prices then the bundle x is (weakly) preferred to the bundle y providing there exists bundles $x_1, \cdots, x_k$ $(k \geqq 2)$ and price vectors $p_1, \cdots, p_{k-1}$ such that x_i is demanded at prices p_i, x_{i+1} can be purchased at prices $p_i (i = 1, \cdots, k - 1)$ and $x_1 = x$, $x_k = y$.

below, closely related to the existence of equlibrium prices[4] and it is relevant to ask:

ii) Does equilibrium exist for every supply?

It is a well known fact that any equilibrium distribution of the available supply is optimal (or "efficient") in the sense that by redistributing the available supply one cannot increase the utility of one consumer without decreasing the utility of some other consumer. Equilibrium prices need not, however, be unique and a given consumer's satisfaction (the value of his utility) may vary from one set of equilibrium prices to another. Consumers would then find it advantageous to enter into coalitions that would improve their position at the expense of other consumers. This difficulty in analyzing the market's stability can be settled by the following question:

iii) Are consumers' satisfactions unique at equilibrium?

For practical reasons we ask:

iv) How can equilibrium quantities be computed?

We propose to examine the preceding questions when all personal utilities are concave, homogeneous (of order 1), non-negative, continuous and non-constant. Theorems 1, 2 and 3 provide affirmative answers to iii), i) and ii), respectively,[5] although for ii) we require an essential hypothesis that the supply be positive. At the same time, iv) is answered (again for positive supply) by characterizing equilibrium quantities as solutions to a convex-homogeneous programming problem (Theorem 4). We have not, as would seem desirable, given necessary and sufficient conditions for the existence of an aggregate utility (nor for the existence of equilibrium). It can be said, however, that with very slight changes of the assumptions given above, models in which the revealed preference is not rational can be constructed and thus an aggregate utility does not exist (see Example 1 in Section IV).

Before proceeding further, we wish to remark about the assumptions imposed on each personal utility. Certainly all but concavity and homogeneity are self-explanatory. The assumption of homogeneity is probably the less realistic of the two; however, we may think of it as a second approximation. As a first approximation, it is frequently assumed that the utility functions are linear, simply because the theory of linear inequalities is so well developed and because an arbitrary concave function may be approximated by linear ones. Homogeneity, then, is a far more flexible assumption and no doubt is satisfied exactly in many situations. As for concavity, it is equivalent to super-additivity (once homogeneity is assumed). This is just the so-called "Wholesale Principle," where by grouping two or more orders together we obtain a utility which is no

[4] For a given supply a set of prices are called equilibrium providing the supply can meet the demand and all surplus commodities have zero prices (i.e., are free).

[5] The existence of equilibrium is demonstrated under less restrictive assumptions in [1] and [6].

less than the sum total of the utilities of the orders separately. Thus, to a man on a deserted road, for example, the value of a functioning automobile and 10 gallons of gasoline will usually be higher than the utility of an automobile without gasoline plus the utility of the 10 gallons of gasoline alone.

It is noteworthy that, unlike some treatments of Paretoan theory, we do not require personal utilities to be differentiable, although concavity implies differentiability almost everywhere (see [5]). By asking that utility functions be everywhere differentiable, such "reasonable" functions as the minimum of linear (homogeneous) functions would be excluded; the latter, however, meet all requirements imposed in this paper.

2. Definitions

Our model will be described within the framework of the theory of convex inequalities in a finite dimensional Euclidean space. R^n will denote the set of all real n-tuples, and R_+^n will denote the set of those n-tuples having each coordinate nonnegative. We shall use the customary matrix notation; i.e., if A and B are $m \times k$ and $k \times n$ matrices, respectively, then AB is the usual matrix product. Vector inequalities will mean that the inequality in question holds for each component. If f is a function from R_+^n to R (denoted $f:R_+^n \to R$), then f is *concave*, providing $f[\lambda x + (1 - \lambda)y] \geqq \lambda f(x) + (1 - \lambda)f(y)$ for all $x, y \in R_+^n$ ($x \in R_+^n$ is to be read "x is a member of R_+^n") and $\lambda \in [0, 1]$. f is (positively) *homogeneous*, providing $f(\lambda x) = \lambda f(x)$ whenever $x \in R_+^n$ and $\lambda \in R_+$. Continuity of f is with respect to the usual metric, i.e., if for $x \in R^n$ we denote the real number $\sqrt{xx}$ by $\| x \|$, then f is *continuous* providing that for each sequence

$$x_k \in R_+^n \ (k = 1, 2, \cdots)$$

such that $\| x_k - x \|$ converges to zero for some $x \in R_+^n$, it is also true that the corresponding $f(x_k)$ converge to $f(x)$. The statement "X is the set of all x's such that p holds" is abbreviated, as usual, to "$X = \{x \mid p\}$."

In the following discussion, we shall repeatedly apply the following duality theorem of homogeneous programming (see [3]).

LEMMA 1. Let A be an $n \times k$ matrix, $\phi:R_+^n \to R$, $\psi:R_+^k \to R$, where ϕ and ψ are homogeneous and continuous, ϕ is concave, and ψ is convex. Let

$$X = \{x \mid x \in R_+^n \text{ and } xAy \leqq \psi(y) \text{ whenever } y \in R_+^k\}$$

$$Y = \{y \mid y \in R_+^k \text{ and } xAy \geqq \phi(x) \text{ whenever } x \in R_+^n\}$$

and consider the statements

(a_1) if $y \in R_+^k$, $\quad Ay \geqq 0, \quad \psi(y) \leqq 0, \quad$ then $y = 0$;

(b_1) if $x \in R_+^n$, $\quad xA \leqq 0, \quad \phi(x) \geqq 0, \quad$ then $x = 0$;

(a_2) there is an $x_0 \in R_+^n$ such that $x_0Ay < \psi(y)$ whenever $y \in R_+^k$ and $y \neq 0$;

(b_2) there is a $y_0 \in R_+^k$ such that $xAy_0 > \phi(x)$ whenever $x \in R_+^n$ and a $\neq 0$.

The conclusions of the theorem are

(i) (a_1) and (a_2) are equivalent;
(b_1) and (b_2) are equivalent;

(ii) if $x_0 \in X$ maximizes ϕ on X and (a_1) holds, then there is a $y_0 \in Y$ which minimizes ψ on Y and $\psi(y_0) = \phi(x_0) = x_0Ay_0$;

(iii) if $y_0 \in Y$ minimizes ψ on Y and (b_1) holds, then there is an $x_0 \in X$ which maximizes ϕ on X and $\phi(x_0) = \psi_0(y_0) = x_0Ay_0$.

We are now in a position to give a precise description of the model with which we shall be concerned. There are m consumers denoted $B_1, \cdots, B_m$ competing for n *goods* denoted $G_1, \cdots, G_n$. The consumer B_i has a positive income of β_i units of money, and we assume that the monetary unit has been chosen so that $\sum_{i=1}^{m} \beta_i = 1$. The behavior of B_i is characterized by a *personal utility function* u_i, where $u_i: R_+^n \to R_+$ and each u_i is assumed to be concave, homogeneous, continuous, and nonconstant. Thus, if faced with a situation where the cost of one unit of G_j is π_j, B_i will demand any order $x = (\xi_1, \cdots, \xi_n) \in R_+^n$ which maximizes his utility subject to his budget constraint $xp = \sum_{j=1}^{n} \xi_j\pi_j \leqq \beta_i$. More formally, the *demand set* of B_i at *prices* $p = (\pi_1, \cdots, \pi_n) \in R_+^n$ is defined by

$$D_i(p) = \{x \mid x \text{ maximizes } u_i \text{ subject to } x \in R_+^n \text{ and } xp \leqq \beta_i\}. \tag{1}$$

Note that $D_i(p)$ may consist of a single element, or an infinite number of elements (although we see readily that it is convex then), or it may even be empty (as for instance when $p = 0$). We do know that if $p > 0$ then, u_i being continuous, $D_i(p)$ is nonempty.

As an immediate application of Lemma 1, we can prove that $x_i \in D_i(p)$ if and only if there exists a positive real number α_i such that

$$\begin{aligned} &\text{(i) } x_i \in R_+^n \\ &\text{(ii) } \alpha_i x_i p = \alpha_i\beta_i = u_i(x_i) \\ &\text{(iii) } \alpha_i xp \geqq u_i(x) \text{ whenever } x \in R_+^n. \end{aligned} \tag{2}$$

First, if (2) holds then $x_i \in R_+^n$, $x_ip = \beta_i$, and for any $x \in R_+^n$ such that $xp \leqq \beta_i$, we have $u_i(x) \leqq \alpha_i xp \leqq \alpha_i\beta_i = u_i(x_i)$. Thus, $u_i(x) \leqq u_i(x_i)$ and $x_i \in D_i(p)$. On the other hand, if $x_i \in D_i(p)$ then certainly $x_i \in R_+^n$; furthermore, if in Lemma 1 we let $k = 1$, $A = p$, $\phi(x) = u_i(x)$, and $\psi(y) = \beta_i y$, then (a_1) is obviously satisfied so that (ii) of that lemma holds with $x_i = x_0$. Letting $y_0 = \alpha_i$, we obtain (2).

In terms of the $D_i(p)$ we now define for every $p \in R_+^n$ the *community* (or *total*) *demand* $D(p)$:

$$D(p) = \left\{ x \mid \text{there exist } x_1, \cdots, x_m \text{ with } x_i \in D_i(p), i = 1, \cdots, m, \text{ and } x = \sum_{i=1}^{m} x_i \right\}. \tag{3}$$

If $x_i \in R_+^n$ for $i = 1, 2, \cdots, m$, then $x_1, \cdots, x_m$ is called a *distribution*. For a given supply $s = (\sigma_1, \cdots, \sigma_n)$, where σ_j denotes the supply of G_j, the distribution $x_1, \cdots, x_m$ is said to be a *feasible distribution*, providing $\sum_{i=1}^m x_i \leqq s$. The feasible distribution $x_1, \cdots, x_m$ is said to be an *equilibrium distribution* (for s), providing there exists a price vector $p \in R_+^n$ such that $x_i \in D_i(p)$ for every i and $sp \leqq 1$ (the last inequality expresses the fact that at equilibrium only free goods can be in oversupply); p is then called an *equilibrium price vector* (for s). As a consequence of Lemma 6, it follows that $x \in D(p)$ if and only if p is an equilibrium price vector for x. Thus the problem of finding equilibrium prices is simply that of "inverting" the set-valued function D_i; unfortunately the "simplicity" is theoretical rather than practical.

The feasible distribution $x_1^0, \cdots, x_m^0$ is said to be a *maximal distribution* (for s), providing it maximizes the *social-welfare function* ψ on the set of all feasible distributions, where ψ is defined for every distribution, $x_1, \cdots, x_m$ by

$$\psi(x_1, \cdots, x_m) = \prod_{i=1}^m u_i(x_i)^{\beta_i}. \tag{4}$$

If $u: R_+^n \to R$ and for $p \in R_+^n$ and we define

$$D_u(p) = \{x \mid x \text{ maximizes } u \text{ subject to } x \in R_+^n \text{ and } xp \leqq 1\}, \tag{5}$$

then u is called an *aggregate* (or *social*) *utility function* providing $D_u(p) = D(p)$ for every $p \in R_+^n$.

3. Statement and Proof of Principal Results

We shall prove the following:

THEOREM 1. For a given supply s, equilibrium satisfactions are unique; i.e., if $x_1, \cdots, x_m$ and $x'_1, \cdots, x'_m$ are two equilibrium distributions, then for each i we have $u_i(x_i) = u_i(x'_i)$.

THEOREM 2. There exists an aggregate utility function which is concave, homogeneous, and continuous.

THEOREM 3. *Maximality Principle.*

(i) Every equilibrium distribution is a maximal distribution.

(ii) If the supply is positive ($s > 0$), then every maximal distribution is an equilibrium distribution.

THEOREM 4. For a given supply s, consider the following problem:

Minimize qs subject to

$$q \in R_+^n \text{ and } q\sum_{i=1}^m y_i \geqq \prod_{i=1}^m u_i(y_i)^{\beta_i} \quad \text{for all distributions } y_1, \cdots, y_m. \tag{6}$$

We conclude that

(i) if q solves (6) and $s > 0$, then $qs > 0$ and $p = (1/qs)q$ is an equilibrium price vector;

(ii) if p is an equilibrium price vector with associated equilibrium distribution $x_1, \cdots, x_m$ and $\lambda = \prod_{i=1}^m u_i(x_i)^{\beta_i}$, then $\lambda > 0$ and $q = \lambda p$ solves (6).

Remark. In terms of Lemma 1, Theorem 4 can be interpreted by using the "dual" of (6), which is

$$\text{(7)}\qquad \begin{gathered} \text{maximize } \prod_{i=1}^{m} u_i(x_i)^{\beta_i} \text{ subject to} \\ x_i \in R_+^{\,n}, \quad \text{for all } i, \text{ and } \sum_{i=1}^{m} x_i \leqq s. \end{gathered}$$

This is precisely the condition for a maximal distribution; hence Theorem 4 is in a sense the dual of Theorem 3.

We first prove Theorem 1. Let $x_1, \cdots, x_m$ and $x'_1, \cdots, x'_m$ be two equilibrium distributions, and let p, p' be their associated equilibrium price-vectors. If $\alpha_1, \cdots, \alpha_m$ and $\alpha'_1, \cdots, \alpha'_m$ are the positive numbers given by (2), then

$$\text{(8)}\qquad \begin{aligned} \alpha_i x'_i p &\geqq u_i(x'_i) = \alpha'_i \beta_i \\ \alpha'_i x_i p' &\geqq u_i(x_i) = \alpha_i \beta_i . \end{aligned} \qquad i = 1, \cdots, m$$

Also, $\sum_{i=1}^{m} x_i \leqq s$ and $\sum_{i=1}^{m} x'i \leqq s$; therefore $\sum_{i=1}^{m} x'_i p \leqq sp \leqq 1$ and $\sum_{i=1}^{m} x_i p' \leqq sp \leqq 1$ so that

$$\text{(9)}\qquad \begin{aligned} 1 &\geqq \sum_{i=1}^{m} x'_i\, p \geqq \sum_{i=1}^{m} \frac{\alpha'_i}{\alpha_i}\, \beta_i \qquad \text{and} \\ 1 &\geqq \sum_{i=1}^{m} x_i\, p' \geqq \sum_{i=1}^{m} \frac{\alpha_i}{\alpha'_i}\, \beta_i . \end{aligned}$$

Now if α is a positive real number, then $\alpha + (1/\alpha) - 2 = (1/\alpha)(\alpha - 1)^2 \geqq 0$, and equality holds if and only if $\alpha = 1$. Adding the inequalities in (9) we have

$$\text{(10)}\qquad 2 \geqq \sum_{i=1}^{m} \left(\frac{\alpha'_i}{\alpha_i} + \frac{\alpha_i}{\alpha'_i} \right) \beta_i \geqq 2 \sum_{i=1}^{m} \beta_i = 2.$$

Hence $\alpha_i = \alpha'_i$ (because each $\beta_i > 0$), and from (8) we then have $u_i(x_i) = u_i(x'_i)$, as desired.

It should be noted that in fact p and p' are interchangeable; i.e., p corresponds to $x'_1, \cdots, x'_m$ and p' corresponds to $x_1, \cdots, x_m$.

Before proceeding with proofs of Theorems 2, 3, and 4, we require several lemmas.

LEMMA 2. If α, β, γ are real numbers with the property that $\alpha(1 - \lambda^{\beta}) \geqq \gamma(1 - \lambda)$ for all λ in some open neighborhood of $\lambda = 1$, then $\alpha\beta = \gamma$.

Proof. For $\lambda < 1$ we have $\alpha(1 - \lambda^{\beta})/(1 - \lambda) \geqq \gamma$, while for $\lambda > 1$ we have $\alpha(1 - \lambda^{\beta})/(1 - \lambda) \leqq \gamma$. However, as is readily seen,

$$\lim_{\lambda \to 1} (1 - \lambda^{\beta})/(1 - \lambda) = \beta;$$

thus $\gamma \leqq \alpha\beta \leqq \gamma$ and the conclusion follows:

LEMMA 3. Let $\beta_1, \cdots, \beta_m$ be positive real numbers such that $\sum_{i=1}^{m} \beta_i = 1$. Consider the function $f\colon R_+^{\,m} \to R_+$ defined by $f(\zeta_1, \cdots, \zeta_m) = \prod_{i=1}^{m} \zeta_i^{\beta_i}$. Then f is homogeneous, continuous, and concave.

Proof. Continuity and homogeneity are obvious. To show f concave, it suffices to prove that it is concave on the interior of R_+^m (because f is continuous). In the interior of R_+^m, f is differentiable and we may apply the well-known theorem (see [5], p. 88) that f is concave if the quadratic form Q of its second partial derivatives is negative-semidefinite. Now if $z = (\zeta_1, \cdots, \zeta_m)$ is in the interior of R_+^m (i.e., $z > 0$), then

$$Q_z(\eta_1, \cdots, \eta_m) = f(z)\left[\sum_{i,k=1}^{m} \frac{\eta_i \eta_k \beta_i \beta_k}{\zeta_i \zeta_k} - \sum_{i=1}^{m} \frac{\beta_i \eta_i^2}{\zeta_i^2}\right].$$

Since $f(z) > 0$, we need only show that for every $(\eta_1, \cdots, \eta_m)$ the expression in brackets is nonpositive. To this end we avail ourselves of the Cauchy-Schwartz inequality, which states that if $u, v \in R^m$ then $(uv)^2 \leqq (uu)(vv)$. Let u, v be the vectors whose i^{th} components are $\sqrt{\beta_i}$ and $\eta_i\sqrt{\beta_i}/\zeta_i$, respectively. Then

$$uu = \sum_{i=1}^{m} \beta_i = 1,$$

$$vv = \sum_{i=1}^{m} \frac{\beta_i \eta_i^2}{\zeta_i^2},$$

and

$$uv = \sum_{i=1}^{m} \frac{\eta_i \beta_i}{\zeta_i}.$$

Thus,

$$Q_z(\eta_1, \cdots, \eta_m) = f(z)[(uv)^2 - (uu)(vv)] \leqq 0. \qquad \text{(Q.E.D.)}$$

LEMMA 4. Let $\beta_1, \cdots, \beta_m$ be positive real numbers with $\sum_{i=1}^m \beta_i = 1$. For any $z = (\zeta_1, \cdots, \zeta_m) \in R_+^m$ such that $\sum_{i=1}^m \zeta_i \leqq 1$, we have

$$\prod_{i=1}^{m} \zeta_i^{\beta_i} \leqq \prod_{i=1}^{m} \beta_i^{\beta_i}, \tag{11}$$

and equality holds in (11) if and only if $\zeta_i = \beta_i$ for all i.

Proof. From Lemma 3 we know that f, as defined there, is continuous; thus there is a $z_0 = (\bar{\zeta}_1, \cdots, \bar{\zeta}_m)$ which maximizes f on the compact set of all $(\zeta_1, \cdots, \zeta_m) \in R_+^m$ such that $\sum_{i=1}^m \zeta_i \leqq 1$. A direct check reveals that (ii) of Lemma 1 may be applied, and we know then that there exists an $\eta \in R_+$ such that

$$\eta = \eta \sum_{i=1}^{m} \bar{\zeta}_i = \prod_{i=1}^{m} \bar{\zeta}_i^{\beta_i}$$

(12)

$$\eta \sum_{i=1}^{m} \zeta_i \geqq \prod_{i=1}^{m} \zeta_i^{\beta_i}, \quad \text{for all} \quad (\zeta_1, \cdots, \zeta_m) \in R_+^m.$$

Fixing a $k \in \{1, 2, \cdots, m\}$ and letting $\zeta_i = \bar{\zeta}_i$ if $i \neq k$ and $\zeta_k = \lambda\bar{\zeta}_k$ (where $\lambda \in R_+$), we obtain from (12)

$$\eta \sum_{i=1}^{m} \zeta_i = \eta(1 - \bar{\zeta}_k + \lambda\bar{\zeta}_k) \geqq \prod_{i=1}^{m} \zeta_i^{\beta_i} = \lambda^{\beta_k} \prod_{i=1}^{m} \bar{\zeta}_i^{\beta_i} = \eta\lambda^{\beta_k}. \tag{13}$$

Note that $\eta > 0$; otherwise, from (12), $f(z) \leqq 0$ for all $z \in R_+^m$, which is clearly impossible. Thus, from (13), $1 - \lambda^{\beta_k} \geqq \bar{\zeta}_k(1 - \lambda)$ for all $\lambda \in R_+$, and, by Lemma 2, $\bar{\zeta}_k = \beta_k$; this completes the proof.

LEMMA 5. The social welfare function ψ is concave, homogeneous, and continuous.

Proof. We have $\psi(x_1, \cdots, x_m) = \prod_{i=1}^m u_i(x_i)^{\beta_i}$ for each distribution $x_1, \cdots, x_m$. Since each u_i is homogeneous and continuous and $\sum_{i=1}^m \beta_i = 1$, ψ is homogeneous and continuous. Concavity of ψ follows from the concavity of each u_i and from the fact that the function f of Lemma 3 is concave and *non-decreasing* (i.e., if $0 \leqq z \leqq z'$, then $f(z) \leqq f(z')$, which is a consequence of the fact that each $\beta_i > 0$).

LEMMA 6. Let $g: R_+^n \to R$ be concave and bounded below; then g is nondecreasing.

Proof. Suppose $x, y \in R_+^n$, $x \geqq y$, and $g(x) < g(y)$. For every positive integer k, let $z_k = kx + (1 - k)y = k(x - y) + y$. Thus, $z_k \in R_+^n$ and $x = (1/k)z_k + (1 - (1/k))y$ for $k = 1, 2, \cdots$. From the concavity of g we then have $g(x) \geqq (1/k)g(z_k) + (1 - (1/k))g(y)$, or

$$g(z_k) \leqq k[g(x) - g(y)] + g(y),$$

which contradicts the boundedness of g. We can now prove Theorems 2 through 4.

Proof of Theorem 3. If $x_1, \cdots, x_m$ is an equilibrium distribution with associated equilibrium price vector p and $y_1, \cdots, y_m$ is a feasible distribution, then $\sum_{i=1}^m y_i p \leqq sp \leqq 1$, and from (2) and Lemma 4 we have

$$\prod_{i=1}^m u_i(y_i)^{\beta_i} \leqq \prod_{i=1}^m (\alpha_i y_i p)^{\beta_i} \leqq \prod_{i=1}^m (\alpha_i \beta_i)^{\beta_i} = \prod_{i=1}^m u_i(x_i)^{\beta_i}. \tag{14}$$

Thus $x_1, \cdots, x_m$ is a maximal distribution. On the other hand, if $x_1, \cdots, x_m$ is a maximal distribution and $s > 0$, then (in view of Lemma 5) we may apply Lemma 1 (ii), which tells us that there exists a $q \in R_+^n$ such that

(15)
$$q \sum_{i=1}^m x_i = \prod_{i=1}^m u_i(x_i)^{\beta_i} = sq \quad \text{and}$$
$$q \sum_{i=1}^m y_i \geqq \prod_{i=1}^m u_i(y_i)^{\beta_i} \quad \text{for every distributions} \quad y_1, \cdots, y_m.$$

If $sq = 0$, then (since $s > 0$) $q = 0$, which would mean that $\prod_{i=1}^m u_i(y_i)^{\beta_i} \leqq 0$ for all distributions $y_1, \cdots, y_m$, contradicting our assumption that each u_i is nonnegative and nonconstant. Let $p = (1/sq)q$; the expressions in (15) then become

(16)
$$p \sum_{i=1}^m x_i = sp = 1$$
$$p \sum_{i=1}^m y_i \geqq \prod_{i=1}^m \left[\frac{u_i(y)_i}{u_i(x)_i}\right]^{\beta_i} \quad \text{for every distribution} \quad y_1, \cdots, y_m.$$

We now fix an integer k, where $1 \leqq k \leqq m$, and in (16) we let $y_i = x_i$ when $i \neq k$, $y_k = x$. Thus for every $x \in R_+{}^n$, we have

$$(17)\qquad p\sum_{i=1}^m y_i = p\sum_{i=1}^m x_i - px_k + px = 1 + p(x - x_k) \geqq \left[\frac{u_k(x)}{u_k(x_k)}\right]^{\beta_k}.$$

Letting $x = \lambda x_k$, where $\lambda \in R_+$, we obtain from (17)

$$(18)\qquad px_k(\lambda - 1) \geqq \lambda^{\beta_k} - 1$$

so that, by Lemma 2, $px_k = \beta_k$. Now if $x \in R_+{}^n$ and $xp \leqq \beta_k$, then from (17) we have $[u_k(x)/u_k(x_k)]^{\beta_k} \leqq 1 + px - px_k \leqq 1 + \beta_k - \beta_k = 1$, i.e., $u_k(x) \leqq u_k(x_k)$ and $x_k \in D_k(p)$; the last in conjunction with $ps = 1$ and $\sum_{i=1}^m x_i \leqq s$ (this is so because $x_1, \cdots, x_m$ is maximal and hence feasible) shows that $x_1, \cdots, x_m$ is an equilibrium distribution. This completes the proof of Theorem 3.

Proof of Theorem 4. If q solves (6) and $s > 0$, then it can be readily verified that the assumptions of Lemma 1 (iii) are satisfied. Thus there exists a maximal distribution $x_1, \cdots, x_m$ such that $qs = q\sum_{i=1}^m x_i = \prod_{i=1}^m u_i(x_i)^{\beta_i} > 0$. Letting $p = (1/qs)q$ we obtain (16) which, as was shown in the proof of Theorem 3, tells us that $x_1, \cdots, x_m$ is an equilibrium distribution with associated equilibrium price vector p.

Let p, q, $x_1, \cdots, x_m$, and λ be as in (ii) of Theorem 4; $\lambda > 0$ then follows from (2). To prove that q solves (6), it suffices to show that q satisfies the constraint inequalities of (6) because then, since $qs = \lambda ps \leqq \prod_{i=1}^m u_i(x_i)^{\beta_i}$, for any q' satisfying the constraint inequalities of (6) we have

$$q's \geqq q'\sum_{i=1}^m x_i \geqq \prod_{i=1}^m u_i(x_i)^{\beta_i} \geqq qs$$

and thus qs is minimal. Let us assume then that there exists a distribution $y_1, \cdots, y_m$ such that $q\sum_{i=1}^m y_i < \prod_{i=1}^m u_i(y_i)^{\beta_i}$. Thus, by definition of p

$$p\sum_{i=1}^m y_i < \prod_{i=1}^m \left[\frac{u_i(y_i)}{u_i(x_i)}\right]^{\beta_i} \leqq \prod_{i=1}^m \left[\frac{\alpha_i y_i p}{\alpha_i \beta_i}\right]^{\beta_i} = \prod_{i=1}^m \left(\frac{y_i p}{\beta_i}\right)^{\beta_i},$$

i.e.,

$$(19)\qquad \left(\sum_{i=1}^m y_i p\right)\prod_{i=1}^m \beta_i^{\beta_i} < \prod_{i=1}^m (y_i p)^{\beta_i},$$

which can readily be seen to contradict Lemma 4. This completes the proof of Theorem 4.

We prove Theorem 2 by first giving an explicit definition of a social utility function and then proving that it has the required properties. For each $s \in R_+{}^n$, we let

$$(20)\qquad u(s) = \max\left\{\prod_{i=1}^m u_i(x_i)^{\beta_i} \;\middle|\; x_1, \cdots, x_m \text{ is a feasible distribution for } s\right\}.$$

In particular, $u(s)$ is precisely the minimal value of qs in (6) or, dually, the value of the social-welfare function at any maximal distribution.

(i) *u is homogeneous*

Note that for each i, $u_i(0) = 0$ because u_i is homogeneous and continuous; hence $u(0) = 0$. Thus if $x \in R_+^n$ and $\lambda = 0$, then $u(\lambda x) = 0 = \lambda u(x)$. Assume $\lambda > 0$. We know that there exist maximal (and hence feasible) distributions $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ for x and λx, respectively. It follows then that

$$\sum_{i=1}^{m} \lambda x_i \leqq \lambda x \quad \text{and} \quad \sum_{i=1}^{m} \frac{y_i}{\lambda} \leqq x; \tag{21}$$

therefore by the maximality of the two distributions

$$\begin{aligned} u(\lambda x) \geqq \prod_{i=1}^{m} u_i(\lambda x_i)^{\beta_i} = \lambda \prod_{i=1}^{m} u_i(x_i)^{\beta_i} = \lambda u(x) \geqq \lambda \prod_{i=1}^{m} u_i\left(\frac{y_i}{\lambda}\right)^{\beta_i} \\ = \prod_{i=1}^{m} u_i(y_i)^{\beta_i} = u(\lambda x), \end{aligned} \tag{22}$$

i.e., $u(\lambda x) = \lambda u(x)$.

(ii) *u is concave*

Let $x, y \in R_+^n$ and let $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ be maximal distributions for x and y, respectively. Thus $x + y \geqq \sum_{i=1}^m (x_i + y_i)$, because $x \geqq \sum_{i=1}^m x_i$ and $y \geqq \sum_{i=1}^m y_i$, and

$$\begin{aligned} u(x + y) \geqq \textstyle\prod_{i=1}^m u_i(x_i + y_i)^{\beta_i} \geqq \prod_{i=1}^m u_i(x_i)^{\beta_i} + \prod_{i=1}^m u_i(y_i)^{\beta_i} \\ = u(x) + u(y). \end{aligned}$$

(The last inequality is a consequence of Lemma 5.) We have shown then that $u(x + y) \geqq u(x) + u(y)$ which, together with homogeneity of u, implies that u is concave.

(iii) *u is continuous*

Let x^k, $k = 1, 2, \cdots$, be a sequence of points in R_+^n converging to some $x^0 \in R_+^n$. We wish to show that the corresponding functional values $u(x^k)$ also cnverge to $u(x^0)$. For $k = 0, 1, 2, \cdots$, let $x_1^k, \cdots, x_m^k$ be a maximal distribution for x^k. Thus $u(x^k) = \prod_{i=1}^m u_i(x_i^k)^{\beta_i}$ and $\sum_{i=1}^m x_i^k \leqq x^k$. The last inequality implies that the $x_1^k, \cdots, x_m^k$ are bounded; let $x_1, \cdots, x_m$ be one of their limit points. Then, since $\lim_{k\to\infty} x^k = x^0$, we must have $\sum_{i=1}^m x_i \leqq x^0$ and thus $u(x^0) \geqq \prod_{i=1}^m u_i(x_i)^{\beta_i}$.

We have just demonstrated that

$$u(x^0) \geqq \overline{\lim_{k\to\infty}}\; u(x^k). \tag{23}$$

Next, we assert that there is a sequence of nonnegative real numbers λ_k which converge to 1 such that $\lambda_k x^0 \leqq x^k$ for $k = 1, 2, \cdots$. If $x^0 = 0$, we let $\lambda_k = 1$ for all k's; otherwise we define λ_k by: $\lambda_k = \min_{\xi_j^0 > 0} (\xi_j^k/\xi_j^0)$, where we set $x^k = (\xi_1^k, \cdots, \xi_m^k)$. That the λ_k converge to 1 follows then from the fact that the x^k converge to x^0. Now $\sum_{i=1}^m \lambda_k x_i^0 \leqq \lambda_k x \leqq x^k$, and therefore

$$u(x^k) \geqq \textstyle\prod_{i=1}^m u_i(\lambda_k x_i^0)^{\beta_i} = \lambda_k u(x^0),$$

which shows that

$$u(x^0) \leqq \varliminf_{k\to\infty} u(x^k). \tag{24}$$

By combining (23) and (24), we obtain the desired result.

(iv) *for every* $p \in R_+^n$, $D(p)$ *is a subset of* $D_u(p)$

Let $x \in D(p)$ and let $x_1, \cdots, x_m$ be an associated distribution, i.e., $x_i \in D_i(p)$ for all i's and $\sum_{i=1}^m x_i = x$. We wish to show that if $y \in R_+^n$ and $yp \leqq 1$, then $u(y) \leqq u(x)$. Let $y_1, \cdots, y_m$ be maximal distribution for y; then, of course, $\sum_{i=1}^m y_i p \leqq yp \leqq 1$, and by Lemma 4

$$\prod_{i=1}^m (y_i p)^{\beta_i} \leqq \prod \beta_i^{\beta_i}. \tag{25}$$

Applying (2), we have:

$$\begin{aligned} u(x) &\geqq \prod_{i=1}^m u_i(x_i)^{\beta_i} & \\ &= \prod_{i=1}^m (a_i\beta_i)^{\beta_i} & \text{from (2)} \\ &\geqq \prod_{i=1}^m (a_i y_i p)^{\beta_i} & \text{from (25)} \\ &\geqq \prod_{i=1}^m u_i(y_i)^{\beta_i} & \text{from (2)} \\ &= u(y). & \text{(Q.E.D.)} \end{aligned}$$

(v) *for every* $p \in R_+^n$, $D_u(p)$ *is a subset of* $D(p)$

Suppose $x \in D_u(p)$, i.e., $u(x)$ is maximal subject to $x \in R_+^n$, $xp \leqq 1$. We apply Lemma 1 with $k = 1$, $\phi = u$, $A = p$, and $\psi(y) = y$; a direct check reveals that (a_2) is satisfied by letting $x_0 = 0$. From (ii) of the same lemma, we then know that there exists a nonnegative real number η such that

$$\begin{aligned} &\eta p x = \eta = u(x) && \text{and} \\ &\eta p y \geqq u(y), \quad \text{whenever} \quad y \in R_+^n. \end{aligned} \tag{26}$$

Note that η is actually positive; otherwise u would always be nonpositive, which is certainly not the case.

If $x_1, \cdots, x_m$ is a maximal distribution for x, then

$$\eta = u(x) = \prod_{i=1}^m u_i(x_i)^{\beta_i} > 0,$$

and for any distribution $y_r, \cdots, y_m$, we have from (26)

$$p\sum_{i=1}^m y_i \geqq \frac{1}{\eta} u\left(\sum_{i=1}^m y_i\right) \geqq \frac{1}{u(x)} \prod_{i=1}^m u_i(y_i)^{\beta_i} = \prod_{i=1}^m \left[\frac{u_i(y_i)}{u_i(x_i)}\right]^{\beta_i}. \tag{27}$$

This is precisely the second inequalty in (16) so that, as shown in the proof of Theorem 3, $x_i \in D_i(p)$ for $i = 1, \cdots, m$. Also, $1 = \sum_{i=1}^m x_i p \leqq xp \leqq 1$, thus if in $\sum_{i=1}^m x_i \leqq x$ we have strict inequality for some component, then the corresponding component of p is zero. In view of Lemma 6, this means that $x \in D(p)$; otherwise we could increase the component in question of any of the x_i's without altering the fact that $x_i \in D_i(p)$.

4. Examples

Example 1. We show here that if all u_i are not homogeneous, then an aggregate utility need not exist. There are two consumers and two goods with

$$u_1(\xi, \eta) = 52 \min\left(24\xi + 8\eta, \xi + 3\eta + \frac{246}{52}\right)$$

$$u_2(\xi, \eta) = 52 \min\left(\xi + 3\eta, 3\xi + \eta + \frac{4}{52}\right):$$

The incomes are given by $\beta_1 = 33/52$ and $\beta_2 = 19/52$. Clearly all conditions of Section II are satisfied except that the u_i are not homogeneous. It can be readily verified that for $p = (1, 3)$ and $q = (3, 1)$ the individual demands contain

$$x_1 = \frac{1}{52}(9, 8) \qquad \text{demanded by } B_1 \text{ at } p$$

$$x_2 = \frac{1}{52}(4, 5) \qquad \text{demanded by } B_2 \text{ at } p$$

$$y_1 = \frac{1}{52}(10, 3) \qquad \text{demanded by } B_1 \text{ at } q$$

$$y_2 = \frac{1}{52}(4, 7) \qquad \text{demanded by } B_2 \text{ at } q.$$

However, $(x_1 + x_2)q = 1$ while $(y_1 + y_2)p = 44/52 < 1$, and both u_i are nondecreasing, hence no "savings" are possible and $y_1 + y_2$ is definitely not demanded at prices p. The revealed preference is thus intransitive, and an aggregate utility function does not exist.

Example 2. We illustrate here Theorems 1 through 4 by means of a model consisting of two consumers and two commodities, with all utilities actually linear.

In general, if $u_i(\xi, \eta) = \alpha_{i1}\xi + \alpha_{i2}\eta$ (where α_{i1} and α_{i2} are positive real numbers) and B_i has income β_i, then for $p = (\pi_1, \pi_2) > 0$

$$D_i(p) = \begin{cases} \left(\dfrac{\beta_i}{\pi_1},\ 0\right) & \text{if } \pi_2\,\alpha_{i1} > \pi_1\,\alpha_{i2} \\[2ex] \left(0,\ \dfrac{\beta_i}{\pi_2}\right) & \text{if } \pi_2\,\alpha_{i1} < \pi_i\,\alpha_{i2} \\[2ex] \left\{\left(\dfrac{\lambda\beta_i}{\pi_1},\ \dfrac{(1-\lambda)\beta_i}{\pi_2}\right) \,\middle|\, \lambda \epsilon [0, 1]\right\} & \text{if } \pi_2\,\alpha_{i1} = \pi_1\,\alpha_{i2}. \end{cases}$$

Let us take a concrete example with incomes $\beta_1, \beta_2 > 0$, where $\beta_1 + \beta_2 = 1$ and

$$u_1(\xi, \eta) = 2\xi + \eta$$

$$u_2(\xi, \eta) = \xi + 3\eta.$$

It then follows that for $p = (\pi_1, \pi_2) > 0$,

$$D(p) = \begin{cases} \left(\dfrac{1}{\pi_1}, \; 0\right) & \text{if } 3\pi_1 < \pi_2 \\ \left\{\left(\dfrac{1-\lambda\beta_2}{\pi_1}, \; \dfrac{\lambda\beta_2}{\pi_2}\right) \middle| \lambda \in [0,1]\right\} & \text{if } 3\pi_1 = \pi_2 \\ \left(\dfrac{\beta_1}{\pi_1}, \; \dfrac{\beta_2}{\pi_2}\right) & \text{if } \pi_2 < 3\pi_1 < 6\pi_2 \\ \left\{\left(\dfrac{\lambda\beta_1}{\pi_1}, \; \dfrac{1-\lambda\beta_2}{\pi_2}\right) \middle| \lambda \in [0,1]\right\} & \text{if } \pi_1 = 2\pi_2 \\ \left(0, \; \dfrac{1}{\pi_2}\right) & \text{if } 2\pi_2 < \pi_1 \end{cases}$$

If the supply $s = (\sigma_1, \sigma_2) \in R_+^2$, then equilibrium exists if and only if $s \neq 0$. If $\sigma_1 = 0$ and $\sigma_2 > 0$, then $p = (\pi, 1/\sigma_2)$, where $\pi > 2/\sigma_2$, is an equilibrium price vector. Similarly, if $\sigma_1 > 0$, $\sigma_2 = 0$ then $p = (1/\sigma_1, \pi)$, where $\pi > 3/\sigma_1$, is an equilibrium price vector. However, if $s > 0$ then equilibrium prices are unique and are given as follows:

Case 1. $\dfrac{\beta_2}{\sigma_2} < \dfrac{3\beta_1}{\sigma_1} < \dfrac{6\beta_2}{\sigma_2}$; $\quad p = \left(\dfrac{\beta_1}{\sigma_1}, \; \dfrac{\beta_2}{\sigma_2}\right)$

Case 2. $\dfrac{\beta_2}{\sigma_2} \geqq \dfrac{3\beta_1}{\sigma_1}$; $\quad p = \dfrac{1}{\sigma_1 + 3\sigma_2}(1,3)$

Case 3. $\dfrac{\beta_1}{\sigma_1} \geqq \dfrac{2\beta_2}{\sigma_2}$; $\quad p = \dfrac{1}{2\sigma_1 + \sigma_2}(2,1)$.

Furthermore, equilibrium distributions and "pay-offs" are given by

Case 1. $x_1 = (\sigma_1, 0)$, $\quad x_2 = (0, \sigma_2)$, $\quad u_1(x_1) = 2\sigma_1$, $\quad u_2(x_2) = 3\sigma_2$,
$u(s) = (2\sigma_1)^{\beta_1}(3\sigma_2)^{\beta_2}$

Case 2. $x_1 = (\beta_1\sigma_1 + 3\beta_1\sigma_2, 0)$, $\quad x_2 = (\beta_2\sigma_1 - 3\beta_1\sigma_2, \sigma_2)$,
$u_1(x_1) = 2\beta_1(\sigma_1 + 3\sigma_2)$, $\quad u_2(x_2) = \beta_2(\sigma_1 + 3\sigma_2)$,
$u(s) = u_1(x_1)^{\beta_1}u_2(x_2)^{\beta_2} = (\sigma_1 + 3\sigma_2)(2\beta_1)^{\beta_1}\beta_2^{\beta_2}$

Case 3. $x_1 = (\sigma_1, \beta_1\sigma_2 - 2\beta_2\sigma_1)$, $\quad x_2 = (0, 2\sigma_1\beta_2 + \sigma_2\beta_2)$,
$u_1(x_1) = \beta_1(2\sigma_1 + \sigma_2)$, $\quad u_2(x_2) = 3\beta_2(2\sigma_1 + \sigma_2)$,
$u(s) = u_1(x_i)^{\beta_1}u_2(x_2)^{\beta_2} = (2\sigma_1 + \sigma_2)\beta_1^{\beta_1}(3\beta_2)^{\beta_2}$

Thus for $s = (\sigma_1, \sigma_2) \in R_+^2$,

$$u(s) = \begin{cases} (2\sigma_1 + \sigma_2)\beta^{\beta_1}(3\beta_2)^{\beta_2}, & \text{if } 2\beta_2\sigma_1 \leqq \beta_1\sigma_2 \\ (2\sigma_1)^{\beta_1}(3\sigma_2)^{\beta_2}, & \text{if } \beta_1\sigma_2 \leqq 2\beta_2\sigma_1 \leqq 6\beta_1\sigma_2 \\ (\sigma_1 + 3\sigma_2)(2\beta_1)^{\beta_1}\beta_2, & \text{if } 3\beta_1\sigma_2 \leqq \beta_2\sigma_1. \end{cases}$$

Hence the social utility function is proportional to u_1 when σ_1 is small compared with σ_2; it is proportional to σ_2 when σ_1 is large compared with u_2.

In the intermediate cone, $u(s)$ is "hyperbolic" (in the sense that the level curves of u in this region, if extended analytically to all of R_+^2, would be assymptotic to the co-ordinate axes).

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PART TWO
STOCHASTIC DECISION MODELS

VII

COMMENTARY ON PART TWO: STOCHASTIC DECISION MODELS

In most practical decision problems the decision maker is faced with uncertainties about the future—e.g., about future prices, demands, technological advances, and so on. The papers in this part present several models that have been proposed to cope with uncertainties. The papers are divided into two groups. The first group deals with general methods, the second group with applications to specific problems in inventory control.

Programming Under Uncertainty

Stochastic Programming

In 1955 Beale [4], Dantzig (Chapter 28), and Radner [14] independently pointed out that certain "stochastic" linear programming problems could be reduced to ordinary linear programming problems. The method of doing this is described below.

We begin by formulating the stochastic linear programming problem. Let A, b, c be respectively $m \times n$, $m \times 1$, $1 \times n$ matrices of real-valued random variables. Let x be a vector-valued function whose domain is the set of all possible values of the triple (A, b, c) and whose range is the set of all n-vectors of real numbers. The stochastic linear programming problem is that of choosing a function x in the above class that

(1) minimizes[1] $$E(cx)$$

subject to

(2) $$Ax = b$$

(3) $$x \geq 0.$$

An important special case is where the triple (A, b, c) has only finitely many values $(A_1, b_1, c_1), \cdots, (A_r, b_r, c_r)$ that occur with probabilities $p_1, \cdots, p_r$ ($\sum_{i=1}^{r} p_i = 1$) respectively. In this event, denote by x_i the value of the function x corresponding to (A_i, b_i, c_i). The problem (1), (2), (3) is now completely equivalent to the ordinary linear programming problem of finding n-vectors $x_1, \cdots, x_r$ of real numbers that

(1)′ minimize $$\sum_{i=1}^{r} p_i c_i x_i$$

[1] E stands for expected value.

subject to

$$(2)' \qquad A_i x_i = b_i \qquad (i = 1, \cdots, r)$$

$$(3)' \qquad x_i \geq 0 \qquad (i = 1, \cdots, r).$$

In the problem formulated above, x is allowed to depend on all elements of (A, b, c). In effect, the decision maker can observe the value of each element of (A, b, c) before choosing the values of his decision variables. In the most interesting applications, however, the decision maker is required to select the value of each coordinate of x after observing only a specified subset of the values of the elements of (A, b, c). For example, suppose decisions are made at several different points in time. Then decisions at one point will typically be made after observation of the values of fewer random variables then would be observed for decisions at a later point in time. This feature is characteristic of most of the papers in this part. A second example occurs in large organizations where complete circulation of information to all decision makers is not economical. In such cases decisions in different parts of the organization—for example, the sales and production departments of a manufacturer—would be based on different (possibly overlapping) subsets of the values of the elements of (A, b, c). Radner discusses an interesting example of this type in Chapter 27.

In the above examples we are asking that the class of admitted functions x be limited by restrictions of the following type: the jth coordinate of x, say x^j, is allowed to depend upon only a specified subset of the elements of (A, b, c). In the special case discussed above in which (A, b, c) takes on only finitely many values, the new restrictions still permit reduction of the stochastic linear programming problem to an ordinary linear programming problem. To see this reduction, let $T_j(A, b, c)$ be the triple formed by replacing each element of (A, b, c) upon which x^j is not allowed to depend by zero. Then we may ensure that x^j does not depend upon the collection of random variables indicated above by imposing the constraints

$$(4) \qquad x_i^j = x_k^j$$

for all i, j, and k such that $T_j(A_i, b_i, c_i) = T_j(A_k, b_k, c_k)$ where x_t^j is, of course, the jth coordinate of x_t.[2] The new problem of finding $x_1, \cdots, x_r$ to satisfy (1)′, (2)′, (3)′, (4) is still one in linear programming.

The approach outlined above for stochastic linear programming problems can be extended easily to stochastic non-linear programming problems. Specifically, suppose we replace cx and $Ax = b$ in (1) and (2) by $c(x)$ and $A(x) \leq b$ respectively, where $c(\cdot)$ is a real valued random function and $A(\cdot)$ is a vector-valued (m coordinate) random function. Then if the triple (A, b, c) has finitely many values, an ordinary non-linear programming problem analogous to (1)′,

[2] In actual computations, (4) would be used to eliminate variables from the problem (1)′–(3)′.

(2)′, (3)′, (4) can be formed. In the event that $c(\cdot)$ and $A(\cdot)$ are convex, known techniques of non-linear programming are available for solving the problem. If either $c(\cdot)$ or $A(\cdot)$ is not convex, the computational problem is always formidable.

There is one special but very interesting case in which an extreme simplification in the computations results. Specifically, suppose $Q(x, c)$ is a convex quadratic function of (x, c), with c being a vector of random variables. The problem is to find an x that

(1)″ minimizes $$EQ(x, c)$$

subject to (2). Observe that the non-negativity restriction on x is *not* retained and that A is not random. It turns out that one can solve (1)″, (2) by replacing c and b by their expected values and solving the resulting problem with the classical calculus. The resulting solution $\bar{x}$, say, is the expected value of the function x that is optimal for the original problem. This result is useful because it enables determination of the values of those coordinates of x that are not permitted to depend upon b or c. Values of other coordinates may be determined after the appropriate coordinates of b and c are observed. The result was first established by Simon [17] and is the principal result upon which the book by Holt et al. [12] is based.

Dynamic Programming Under Uncertainty

There is an alternative way to formulate stochastic decision problems in which decisions are made sequentially in time. Specifically, we suppose that a process can be in one of N "states" labeled $1, 2, \cdots, N$. Initially the process is in state 1, and it is desired to guide the process through a sequence of intermediate states to state N as profitably as possible. However, the decision maker can only partially control the evolution of the process from state to state. Instead, when the process is in state i, there are K decisions available, labeled $1, \cdots, K$. If decision k is chosen, the known probability that the process then goes to state j is p_{ij}^k. We suppose that the states are so numbered that the process cannot enter state j from state i if $j \leq i$. Thus,

$$\sum_{j=i+1}^{N} p_{ij}^k = 1 \qquad (i = 1, \cdots, N-1; k = 1, \cdots, K).$$

There is a cost c_{ik} incurred when the process is in state i and decision k is made. The problem is to choose decisions to be made in each state so as to minimize the total expected cost.

Let x_{ik} be the (unknown) *joint* probability that the process enters state i sometime during its evolution from state 1 to state N and that decision k is made at that time. The problem is then to choose x_{ik} that

(5) minimize $$\sum_{i,k} c_{ik} x_{ik}$$

subject to

$$
\begin{aligned}
&\sum_k x_{ik} &&= 1 \\
&\sum_k x_{jk} - \sum_{i<j} \sum_k p_{ij}^k x_{ik} &&= 0 \qquad (j = 2, \cdots, N-1)
\end{aligned} \tag{6}
$$

$$
x_{ik} \geq 0 \qquad (i = 1, \cdots, N-1; k = 1, \cdots, K). \tag{7}
$$

It is easy to see that one set of optimal dual variables $f_1, \cdots, f_{N-1} (f_N \equiv 0)$ for this linear programming problem satisfies

$$
f_i = \min_k \left\{ c_{ik} + \sum_{j=i+1}^{N} p_{ij}^k f_j \right\} \qquad (i = 1, \cdots, N-1). \tag{8}
$$

This formula is familiar in dynamic programming and permits $f_{N-1}, f_{N-2}, \cdots, f_1$ to be calculated recursively in the order given. Evidently, one can interpret f_i as the total expected cost incurred when the process starts in state i and optimal decisions are made thereafter. The optimal decision for state i is determined, as usual, as the minimizing value of k in (8).

The formulation given above generalizes the discrete dynamic programming model developed ((11)–(15)) in Part One. The formulation also extends to situations in which there are infinitely many states and decisions.

The first linear programming formulation of a stochastic decision problem along the above lines was given by Manne in the article reprinted as Chapter 29. He considered an infinite period problem in which revisitation of states is allowed and in which the objective is to minimize the long run average cost per unit time. D'Epenoux [7] then showed that a similar formulation was possible for the infinite stage model in which future costs are discounted and the objective is to minimize the total expected discounted cost. In fact the model (5)–(8) given above is really a special case of his model.

The first complete account of the models described above—although not from a linear programming viewpoint—is given in the excellent book of Howard [13].

There has been no systematic comparison of the relative advantages of the stochastic programming and dynamic programming approaches to stochastic decision problems. However, it is known that each approach is computationally more efficient than the other for certain classes of problems. We remark also that both approaches lead to very large programming problems in interesting applications. This fact probably accounts for the relatively small number of practical applications of these models. The development of the decomposition principle [6] is a significant step in attempting to exploit the structure appearing in these (and other) large scale programming problems to simplify computations.

Stochastic Constraints

In formulating stochastic decision problems one sometimes wishes to require that a certain inequality hold but must omit the restriction because it would

occasionally lead to infeasibility. In such cases an alternative is to require that the inequality hold with a given probability. This kind of restriction is called a stochastic constraint.

Stochastic constraints can be accommodated in either the stochastic programming or dynamic programming formulations of decision problems. In the stochastic programming formulation we replace (2), (3) with

$$P(a_i x \leq \beta_i) \geq \alpha_i \qquad (i = 1, \cdots, m) \tag{9}$$

where a_i is the ith row of A, β_i is the ith coordinate of b, and the α_i are scalars with $0 \leq \alpha_i \leq 1$.

Unfortunately it seems in general to be difficult to solve problems of this sort. Charnes, Cooper, and Symonds in Chapter 33 and Charnes and Cooper in Chapter 30 have suggested approaches that are suitable in special cases. More recent work along the same lines is given in Charnes and Cooper [5] and Thompson et al. [10c]. We describe here a method that is given in Charnes and Cooper [5] and generalizes an idea that first appeared in the paper reprinted as Chapter 33.

We impose the following assumptions:

(i) the a_i and c are given fixed vectors;
(ii) $b = (\beta_i)$ is a vector of jointly normally distributed random variables; and
(iii) x is restricted to functions of the special form

$$x = Db + y \tag{10}$$

where D is a given $n \times m$ matrix and y is an n-vector that is not permitted to depend upon b but is allowed to vary.

Now substituting (10) into (9) we get

$$P(a_i y \leq \beta_i - a_i Db) \geq \alpha_i \qquad (i = 1, \cdots, m). \tag{11}$$

Clearly $\beta_i - a_i Db$ is normally distributed with mean $\mu_i \equiv E(\beta_i) - a_i DE(b)$ and variance σ_i^2.[3] Thus letting $\Phi(t)$ denote the standard normal cumulative distribution, (11) becomes

$$1 - \Phi\left(\frac{a_i y - \mu_i}{\sigma_i}\right) \geq \alpha_i \qquad (i = 1, \cdots, m). \tag{12}$$

Let K_i be such that $\Phi(K_i) = 1 - \alpha_i$. Then since Φ is monotone increasing, (12) is equivalent to

$$a_i y \leq \mu_i - K_i \sigma_i \qquad (i = 1, \cdots, m). \tag{13}$$

In a similar manner, substituting (10) into (1) gives the equivalent objective function

$$\text{minimizes} \qquad cy. \tag{14}$$

Now the problem of finding an n-vector of real numbers y satisfying (13), (14)

[3] Since $\beta_i - a_i Db$ is linear in the β_j, the variance is easily calculated.

has been reduced from a problem with stochastic constraints to an ordinary linear programming problem.

Stochastic constraints can also be accommodated in the dynamic programming formulation of decision problems. In this case a typical stochastic constraint takes the form of requiring that a subset S of the states $2, \cdots, N-1$ be entered with probability not exceeding $\alpha (0 \leq \alpha \leq 1)$. This constraint requires, for example, that we add to the system (5)–(7) the inequality

$$\sum_{i \in S} \sum_{k} x_{ik} \leq \alpha. \tag{15}$$

In this event the dual variables do not ordinarily satisfy (8), so that a simple recursive solution of the dual problem is not possible. However, the simplex method or any other algorithm that is available for solving linear programming problems could be used to solve (5), (6), (7), (15).

Inventory Control with Stochastic Demands

During the last fifteen years, there has been a steady stream of research papers on inventory problems with stochastic demands. Several books [2, 11, 12, 16] that give accounts of this work are now available. The stimulus for this work was the pioneering paper of Arrow, Harris, and Marschak [3]. They formulated a dynamic single product model (among others) with the following characteristics: The demands occurring in each of an unbounded sequence of periods are independent and identically distributed random variables. An (s, S) ordering policy is followed. This ordering policy requires that if the amount x of inventory on hand before ordering in a period is less than s, an order for $S - x$ units is placed; otherwise no order is placed. There is no lag in delivery of orders; and unsatisfied demand in a period is lost. A cost structure (ordering, holding, and penalty costs) is imposed and a method of computing the long run (equivalent) average cost per period is developed using renewal theory. The resulting average cost may then be minimized to find an (s, S) policy that is optimal. Additional investigations along these lines are given by Beckman in Chapter 34.

The basic paper of Arrow, Harris, and Marschak inspired Dvoretzky, Kiefer, and Wolfowitz [8, 9] to apply ideas used in decision theory [19] and sequential analysis [18] to formulate and solve (at least formally) very general forms of the basic inventory model outlined above. In particular they showed (in principle) how to determine ordering rules that are optimal among *all* rules—not just those of the (s, S) type. In so doing, they introduced the functional equation approach of dynamic programming to this area of application.

Subsequent investigations of dynamic stochastic inventory models have been directed toward establishing simple sufficient conditions on the demand distributions and cost structure that ensure the optimality of simple ordering policies (e.g., the (s, S) policy). These studies have also led to simple and efficient computational procedures for determining optimal policies. Scarf [16] gives an excellent recent survey of these results.

Investigations of this type were initiated by Bellman, Glicksberg, and Gross

TABLE II
Characteristics of Inventory Models

Chapter	31	32	33	34	35	36
Number of periods						
Single						x
Finite	x	x	x		x	
Unbounded	x	x		x	x	
Demand process						
Stationary	x			x	x	
Non-stationary		x	x			
Known distribution	x	x	x	x	x	x
Unknown distribution		x				
Special distributions		x	x	x		x
Review						
Periodic	x	x	x	x	x	
Continuous				x		
Delivery lead time						
Zero	x		x			
Non-negative		x		x	x	
Unsatisfied demand						
Backlogged		x	x	x	x	
Lost	x	x				
Costs						
Stationary	x	x		x	x	
Non-stationary			x			
Ordering						
Linear	x	x			x	
Fixed + linear				x	x	x
Convex	x	x	x			
Storage and penalty						
Linear	x		x	x		x
Fixed + linear	x					
Convex		x			x	
System Organization						
Single facility	x	x	x	x		
Parallel facilities						x
Series structure					x	

in the article reprinted as Chapter 31. They showed that the (s, S) (with $s = S$) policy is optimal for the Arrow-Harris-Marschak model under fairly weak assumptions, provided that the ordering cost is linear. For the unbounded period model, an optimal value of $S(=s)$ is found as the root of a single transcendental equation. The analysis exploits the special structure of the appropriate functional equation. This work is generalized by Karlin [2a] and Karlin and Scarf [2b] in several ways—e.g., to allow backlogging of unsatisfied demands and a delivery lag. Further generalizations to the case where the demand distribu-

tions vary over time and where the demand distributions are initially known only up to an unknown parameter are given by Karlin in Chapter 32. Scarf [15] established the brilliant result that an (s, S) policy is optimal among *all* policies, provided that the ordering cost is the sum of a cost proportional to the amount ordered and a fixed cost for placing an order, that the holding and penalty cost function is convex, and that unsatisfied demand is backlogged.

A different approach to finding optimal inventory policies for finite period models is proposed by Charnes, Cooper, and Symonds in Chapter 33. They use stochastic constraints and seek an optimal linear decision rule of the type discussed earlier. This approach permits reduction of the problem to a deterministic inventory problem of the type described in the Commentary on Part One. Thus the special algorithms (e.g., Chapter 12) for solving deterministic inventory problems can be used to solve the stochastic problem.

All of the models discussed above have been concerned with stocking decisions made at a single point. In Chapter 35 Clark and Scarf consider a generalization in which there are several echelons in which inventory is stored. And in Chapter 36 and in Allen [1], Allen analyzes a one period model in which the problem is to determine how stocks of a single product at several locations can be redistributed to minimize combined shortage and redistribution costs.

The main features of the models discussed above are given in Table II.

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VIII–27

THE APPLICATION OF LINEAR PROGRAMMING TO TEAM DECISION PROBLEMS*[1]

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In a team decision problem there are two or more *decision variables*, and these different decisions can be made to depend upon *different* aspects of the environment, or *information variables*, the resulting payoff being a random variable. The choice of optimal rules for selecting information variables and for making decisions is the central problem of the economic theory of teams. This paper shows, by means of an example, how linear programming can be applied to obtain optimal team decision functions in the case in which the payoff to the team is a convex polyhedral function of the decision variables.

1. Introduction[2]

In a *team decision problem* there are two or more *decision variables*, and these different decisions can be made to depend upon different aspects of the environment, i.e., upon different *information variables*. The choice of optimal rules for selecting information variables and for making decisions is the central problem of the economic theory of teams. In a previous paper [1], Marschak has given an introduction to the main concepts of this theory. In the present paper I shall show how the technique of linear programming can be used to solve a typical class of team decision problems.

The "character" of a decision problem is determined by the form of the function that is to be maximized, which I shall call the *payoff function*. Much of the available data about business leads naturally to the formulation of decision problems in terms of what might be called *convex polyhedral* payoff functions; i.e., problems for which the space of decision variables can be divided into regions, whose boundaries are linear, such that within each region the payoff is a linear function of the decision variables. As is well known, such a problem is amenable to linear programming, and as I have shown in another paper [2], the introduction of probabilistic uncertainty, and of the further complications of a team situation, does not destroy the linear character of a programming problem, although it may result in a substantial increase in the "size" of the problem.

In this paper I will illustrate these ideas by means of an example; a general

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formulation has been given in the paper just referred to, but the reader will probably have no trouble in providing such a generalization himself. The example to be used is about as simple as it can be without sacrificing any of the three features that I want to illustrate, namely, (1) uncertainty, (2) the fact that different decision variables can be made to depend upon different information variables, and (3) a nondegenerate convex polyhedral payoff function. (Therefore, the reader should not expect too much in the way of realism!)[3]

Within the framework of the example, I shall (1) show how to apply linear programming to compute optimal decision rules for any particular structure of information and communication; (2) compare different structures of information and communication; (3) discuss an effect of joint constraints on the decision variables, in a partly "decentralized" team; and (4) point out the relationship between team decision problems and sequential decision problems.

2. An Example

Consider a "firm" with two activities, which I will label "production" and "promotion," and suppose that the levels of expenditure on these two activities must be chosen for one period to come. Let a denote the amount of money alloted to production, and let xa be the resulting quantity produced (there is only one commodity concerned), where x can be interpreted as the "productivity" of the production activity. Similarly, let b denote the amount of money alloted to promotion, and let yb be the resulting demand generated. The quantity actually sold will therefore be the smaller of the two quantities, xa and yb; if both the product and the demand generated are "perishable," and if the units are chosen so that the price of the commodity is 1, then the profit resulting from the pair of expenditures (a, b) is

$$\min(xa, yb) - (a + b).$$

If the business were at all profitable, then the firm would of course expand its scale of operation indefinitely, were it not for the fact that its immediate supply of capital is limited. This limit is not absolute, but there is a substantial cost attached to obtaining more capital than is immediately available. Letting k denote the capital limit, and $(1 + f)$ denote the cost per dollar of additional capital, the firm's profit, as a function of the decision variables a and b, is given by the payoff function

$$u(a, b; x, y) = \min(xa, yb) - (a + b) - f\max(0, a + b - k). \tag{1}$$

If $0 < xy/x + y - 1 < f$, then the function u just defined is indeed convex and polyhedral, and its contours are shown in figure 1.

It is easy to see that, for given x and y, the function u attains its maximum when

$$ax = by, \qquad a + b = k, \tag{2}$$

[3] A. Manne has aptly described this type of example as "allegorical" rather than "realistic".

that is to say, when

$$a = \frac{yk}{x + y}, \qquad b = \frac{xk}{x + y}, \tag{3}$$

and the maximum value of u is

$$\max_{a,b} u = \left(\frac{xy}{x + y} - 1\right) k.$$

Suppose now that the firm is uncertain about the actual values of the "productivity" parameters, x and y, that will prevail during the period in question, and that these values can be predicted accurately only at some substantial cost. Two extreme alternatives suggest themselves. The firm could pay the cost and obtain the relevant information, and then make the appropriate decisions, as given by equation (3). This alternative will be called the case of *full information.* On the other hand, the firm could rely only upon its knowledge of the probability distribution of x and y, which is assumed to be known, and choose that pair of

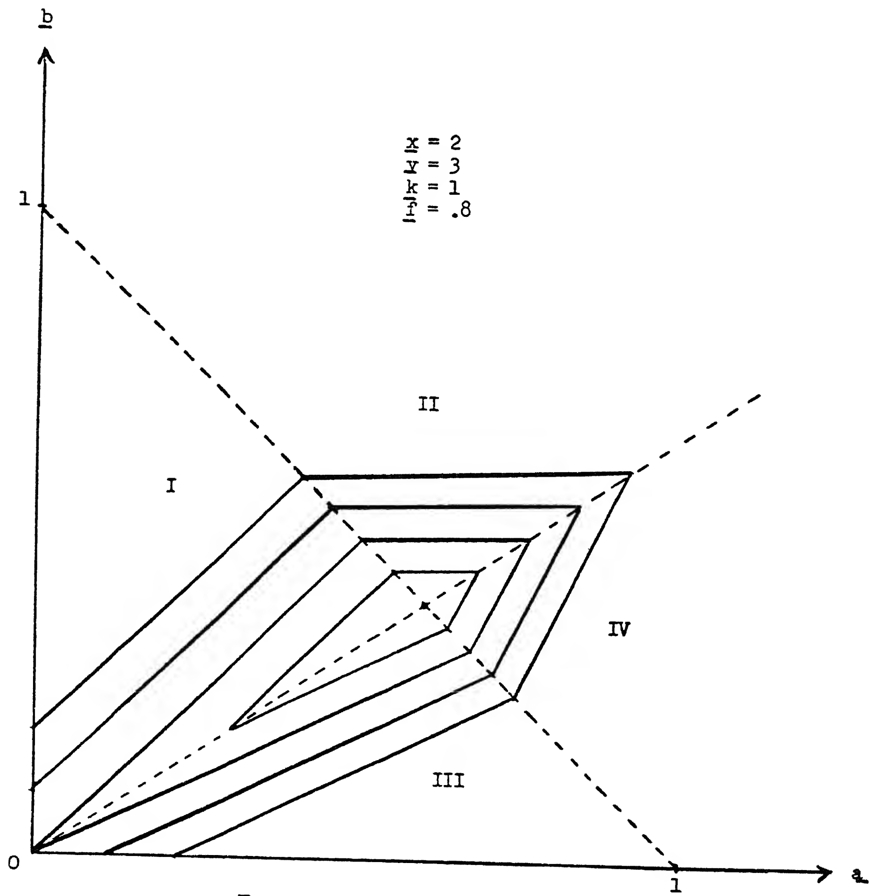

Fig. 1. Iso-profit curves for $u(a, b; x, y)$

expenditures that maximizes the expected, or average, profit. This will be called the case of *routine operation*. Each of these two alternatives involves a different structure of information. Which alternative is the better depends upon which one results in the higher expected profit, net of the cost of information.

A third, intermediate, alternative is suggested under a circumstance that has been described by Marschak as "cospecialization of action and information." In this case, it costs less for the person in charge of production to get the needed information about the parameter x than it does for the person in charge of promotion to do so, and the reverse holds for the parameter y. If, in addition, communication between these two persons is costly, it may be desirable to have the decision about the variable a made by the production manager only in the light of knowledge about x, and the decision about the variable b made by the promotion manager only in the light of knowledge about y, all however according to a decision rule agreed upon in advance. This last alternative will be called the case of *decentralization*.

The possibility of costly communication may seem far-fetched in the context of this simple example; however, if instead of a and b one thinks of two fairly complicated sequences of decision, with each person (or department) getting new information all the time, then it might indeed be costly to achieve a complete exchange of information between the two.

As a primary step toward solving the over-all problem of choosing both a best information structure and best decision rules, one must, at least in principle, solve the various "sub-optimizing" problems of choosing the best decision rules for given information structures, and this is the type of problem that will be considered in detail in the rest of this paper. Before doing so (in the next section), it may be helpful to look at the results for some given numerical values of the parameters.

Suppose that x and y are statistically independent, and can each take on one of two values, with equal probability, the values being given in table 1. Suppose, furthermore, that the amount of free capital (k) equals 1,000 dollars, and that the cost of additional capital ($1 + f$) is 2.7 dollars per dollar. (It is clear from equations (2) and (3) that the value of k merely determines the scale of operation, and does not influence the relative expenditures.) The maximum possible expected profit for each of the three alternative information structures described above is given in Table 2.

In the "routine" case, the decision rules are, in a sense, degenerate; a best

TABLE 1

Joint Probability Distribution of x and y

x	y = 2.8	y = 3.6
3.0	1/4	1/4
3.4	1/4	1/4

single value of a and a best single value of b are to be chosen. In the "decentralized case," however, a *pair* of values (a_1, a_2) and a pair of values (b_1, b_2) are to be chosen, where a_1 denotes the expenditure that will be made by the production manager if he learns that x will have the value 3.0, a_2 is the expenditure corresponding to $x = 3.4$, etc. In the "full information" case, there are four values a_{ij} and four values b_{ij} to be chosen, where a_{ij} denotes the expenditure that will be made on production corresponding to the pair of parameter values (x_i, y_j), etc. Table 3 shows the best decision rules for each of the three information structures, and Table 4 shows the resulting allocations of resources.

An interesting feature of the "decentralized" case (for this numerical example) is that, under the best decision rules, the capital limit is actually exceeded by a small amount whenever a and b both take on their largest values, an event that occurs with probability $\frac{1}{4}$. On the other hand, in the "routine" and "full information" cases one could as well, from the beginning, have imposed the constraint

$$a + b = k \tag{4}$$

and taken the payoff function to be

$$u(a, b; x, y) = \min (xa, yb) - (a + b).$$

However, imposing the constraint (4) on the decision functions in the "decentralized" case would be too stringent. In the present numerical example such a constraint would reduce the expected profit from 541 to 534, or by 7.5% of the difference in expected profit between the routine and full information cases. In general, if different decisions are based upon different information variables, as in the "decentralized case", a certain degree of lack of complete coordination will typically be introduced, and to require that a given constraint never be violated may turn out to be uneconomical when the true cost of a violation is actually weighted against the possible advantages.

Even in this simple example there are many other conceivable information structures besides the three already mentioned. Some of these will be mentioned

TABLE 2

Maximum Expected Profit

	Routine	Decentralized	Full Information
Maximum expected profit................	503	541	592

TABLE 3

Best Decision Rules

Routine	Decentralized	Full Information	
$a = 486$	$a_1 = 512$	$a_{11} = 483$	$b_{11} = 517$
$b = 514$	$a_2 = 452$	$a_{12} = 452$	$b_{12} = 548$
	$b_1 = 548$	$a_{21} = 545$	$b_{21} = 455$
	$b_2 = 426$	$a_{22} = 514$	$b_{22} = 486$

TABLE 4

Allocations of Resources Under Best Decision Rules

Parameter Values	Best Decision Rules		
	Routine	Decentralized	Full information
$x = 3.0$ $y = 2.8$	$a = 486$ $b = 514$	512* 548*	483 517
3.0 3.6	486 514	512 426	452 548
3.4 2.8	486 514	452 548	545 455
3.4 3.6	486 514	452 426	514 486

* Total expenditure exceeds immediate supply of capital.

in Section 4. In the next section I will take up the problem of computing best decision rules for any given information structure.

3. Computing the Optimal Decision Rules

The procedure for computing the optimal decision rules for a given information structure involves converting the team decision problem into an equivalent linear programming problem. The following discussion is in terms of two decision variables and two random parameters, in order to make more transparent its relation to the example of the previous section, but the generalization to any number of decision variables and random parameters is obvious.

Suppose that

$$u(a, b; x, y) = \min_n f_n(a, b; x, y), \qquad (n = 1, \cdots, N) \tag{5}$$

where, for every n, x, and y, f_n is linear in a and b. Suppose also that x and y take on only a finite number of values, with probabilities $p(x, y)$. Furthermore, let

$r = R(x, y)$ be the information on which action a is based,
$s = S(x, y)$ be the information on which action b is based,
A denote any function of r (a decision rule for a),
B denote any function of s (a decision rule for b),
Z denote any function of x and y.

Then the following two maximization problems are equivalent, in the sense that the maximum values are the same, and (A, B) is a solution of Problem I if and only if there is a Z such that (Z, A, B) is a solution of Problem II.

Problem I. Choose A and B so as to maximize

$$Eu(A[R(x, y)], B[S(x, y)]; x, y), \tag{6}$$

subject to $A(r)$, $B(s)$ nonnegative.

Problem II. Choose Z, A and B so as to maximize

$$EZ(x, y), \tag{7}$$

subject to $Z(x, y)$, $A(r)$, $B(s)$ nonnegative, and to the further constraints that

$$Z(x, y) \leqq f_n(A[R(x, y)], B[S(x, y)]; x, y) \tag{8}$$

for every n, x and y. (Note: the symbol E denotes mathematical expectation with respect to the random parameters x and y.)

Since $EZ(x, y) = \sum_{x,y} p(x, y)Z(x, y)$ is a linear function of the "variables" $Z(x, y)$, and since the constraints (8) are linear in $Z(x, y)$, $A(r)$ and $B(s)$, Problem II is a standard linear programming problem.

Returning to the example of the previous section, let the function u be given by equation (1); then it is easy to see that u can be expressed in the form (5) by taking

$$\begin{aligned} f_1 &= (x - 1)a - b, \\ f_2 &= (x - 1 - f)a - (1 + f)b + fk, \\ f_3 &= -a + (y - 1)b, \\ f_4 &= -(1 + f)a + (y - 1 - f)b + fk. \end{aligned} \tag{9}$$

(These 4 functions correspond to the regions I–IV, respectively, in Fig. 1.)

Consider the decentralization example; there one has the information structure

$$R(x, y) = x, \qquad S(x, y) = y. \tag{10}$$

Suppose, furthermore, that x and y can each take on one of two values, as in the numerical example; then A will take on one of two values, say a_1 and a_2, according as x equals x_1 or x_2; and likewise for B. $Z(x, y)$, however, will take on one of four values, say z_{ij}, corresponding to the four pairs (x_i, y_j). In this case Problem II takes the form:

Choose z_{ij}, a_i, b_j, so as to maximize $\sum_{ij} p_{ij}z_{ij}$, subject to z_{ij}, a_i, b_j non-

TABLE 5

Constraint Matrix

z_{11}	z_{12}	z_{21}	z_{22}	a_1	a_2	b_1	b_2	$\leqq$
E	0	0	0	$-G_1$	0	$-H_1$	0	F
0	E	0	0	$-G_1$	0	0	$-H_2$	F
0	0	E	0	0	$-G_2$	$-H_1$	0	F
0	0	0	E	0	$-G_2$	0	$-H_2$	F

where

$$E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad G_i = \begin{bmatrix} x_i - 1 \\ x_i - 1 - f \\ -1 \\ -1 - f \end{bmatrix}, \quad H_j = \begin{bmatrix} -1 \\ -1 - f \\ y_j - 1 \\ y_j - 1 - f \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ fk \\ 0 \\ fk \end{bmatrix} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}$$

Corresponds to region (I–IV) on Fig. 1

negative, and the set of linear constraints presented in matrix ("detached coefficient") form in Table 5.

The fortunate pattern of 1's and 0's in the left half of the constraint matrix of Table 5 is characteristic of a linear problem derived from one with a polyhedral profit function; from a computational point of view, the addition of the variables z does not represent a significant increase in the number of variables. The scattering of 0's throughout the right half of the constraint matrix is typical of a team decision problem.

More generally, for the decentralized case in this example, if x can take on I values, and y can take on J values, then for Problem II there will be $IJ + I + J$ variables, and $4IJ$ constraints. Because of the special structure of this problem, the dual will always be considerably easier to solver than the primal form. In order to solve the dual, it should not take substantially more computing effort than a linear programming model with $I + J$ constraint equations.

4. Interpretation of Sequential Decision Problems as Team Problems

Thus far in this paper the different decision variables in a team decision problem have been interpreted as the decisions of different *persons*. Another class of problems with the same formal structure arises from sequential decision problems for even a single "person" (e.g., inventory and production scheduling problems). In this case the different decision variables correspond to decisions taken at different points of time. Thus, if there is a decision to be made in each of two successive time periods, and information about the parameter values also tends to become known sequentially, then, using the notation of the lemma of Section 3, either of the following information structures is likely to be relevant:

$$(11)\qquad \begin{cases} R(x, y) = \text{constant} \\ S(x, y) = x \end{cases}$$

or

$$(12)\qquad \begin{cases} R(x, y) = x \\ S(x, y) = (x, y) \end{cases}$$

The technique of Section 3 applies just as well, of course, to these information structures as it did to the ones considered there. However, the special "triangular" character of the information structures that arise in single-person sequential problems often leads to computational simplifications that do not apply to team problems in general. On the other hand, it is clear that sequential or "dynamic" elements can be incorporated into a team decision problem, without altering the basic mathematical framework.

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VIII–28

LINEAR PROGRAMMING UNDER UNCERTAINTY

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Summary

The essential character of the general models under consideration is that activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that are required to be determined; those in the second (or later) stages can not be determined in advance since they depend on the earlier stages and the random or uncertain demands which occur on or before the latter stage. It is important to note that the set of activities are assumed to be *complete* in the sense that, whatever be the choice of activities in the earlier stages (consistent with the restrictions applicable to their stage), there is a possible choice of activities in the latter stages. In other words *it is not possible to get in a position where the programming problem admits of no solution.*

The initial work on this paper was stimulated by discussions with A. Ferguson who proposed that linear programming methods be extended to include the case of uncertain demands for the problem of optimal allocation of a carrier fleet to airline routes to meet an anticipated demand distribution. The application of the theory found in this paper to his problem (discussed later under Example 4) will be the subject of a separate joint paper. The case of certain demands was discussed earlier [4].

A complete computation procedure is given for a special class of two-stage linear programming models in which allocations in the first stage are made to meet an uncertain but known distribution of demands occurring in the second stage. This case, applicable to many practical problems constitutes the principal part of the paper. Next, a class of models is considered where the activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that can be determined in advance because those in the second and later stages depend on the outcome of random events. Theorems on convexity of the objective (cost) functions are established for the general m-stage case.

Example 1: *Minimum Expected Cost Diet.* A nutrition expert wishes to advise his followers on a minimum cost diet without prior knowledge of the prices [6]. Since prices of food (except for general inflationary trends) are likely to show variability due to weather conditions, supply, etc., he wishes to assume a distribution of possible prices rather than a fixed price for each food, and determine a diet that meets specified nutritional requirements and minimizes expected costs.

Let x_j be the quantity of j^{th} food purchased in pounds, p_j its price, and a_{ij} be the quantity of the i^{th} nutrient (e.g., vitamin A) contained in a unit quantity of the j^{th} food, and b_i the minimum quantity required by an individual for good health. Then the x_j must be chosen so that

$$\sum_{j=1}^{n} a_{ij}x_j \geqq b_i \qquad x_j \geqq 0 (i = 1, 2, \cdots, m) \tag{1}$$

and the cost of the diet will be

$$C = \sum_{j=1}^{n} p_j x_j . \tag{2}$$

The x_j are chosen before the prices are known so that the expected costs of such a diet are clearly

$$\text{Exp } C = \sum_j \bar{p}_j x_j \tag{3}$$

where $\bar{p}_j$ is its expected price. Since the $\bar{p}_j$ are known in advance, the best choices of x_j are those which satisfy (1), minimize (3). Hence in this case expected prices may be used in place of the distribution of prices and the usual linear programming problem solved.[1]

Example 2: *Shipping to an Outlet to Meet an Uncertain Demand.*

Let us consider a simple two-stage case: A factory has 100 items on hand which may be shipped to an outlet at the cost of \$1 apiece to meet an uncertain demand d_2. In the event that the demand should exceed the supply, it is necessary to meet the unsatisfied demand by purchases on the local market at \$2 apiece. The equations that the system must satisfy are

$$\begin{aligned} 100 &= x_{11} + x_{12} \\ d_2 &= x_{11} \qquad + x_{21} - x_{22} \\ C &= x_{11} \qquad + 2x_{21} \end{aligned} \qquad (x_{ij} \geqq 0) \tag{4}$$

where x_{11} = number shipped from the factory, x_{12} = number stored at factory;
x_{21} = number purchased on open market, x_{22} = excess of supply over demand;
d_2 = unknown demand uniformly distributed between 70 and 80;
C = total costs.

It is clear that whatever be the amount shipped and whatever be the demand d_1, it is possible to choose x_{21} and x_{22} consistent with the second equation. The unused stocks $x_{12} + x_{22}$ are assumed to have no value or are written off at some reduced value (like last year's model automobiles when the new production comes

[1] In some applications, however, it may not be desirable to minimize the expected value of the costs if the decision has too great a variation in the actual total costs. H. Markowitz [5] in his analysis of investment portfolios develops a technique for computing for each possible expected value the minimum variance. This enables the investor to sacrifice some of his expectation to control his risks.

in). To illustrate some of the concepts of this paper, a solution will be presented later.

Example 3: *A Three-Stage Case.*

For this purpose it is easy to construct an extension of the previous example by allowing the surpluses x_{12} and x_{22} to be carried over to a third stage, i.e.,

(5)

1st stage	100	$=$	x_{11}	$+ x_{12}$						
2nd stage	d_2	$=$	x_{11}		x_{21}	$- x_{22}$				
	70	$=$		$- x_{12}$			$+ x_{23}$	$+ x_{24}$		—
3rd stage	d_3	$=$				$+ x_{22}$	$+ x_{23}$		$+ x_{31}$	$- x_{32}$
	C	$=$	x_{11}		$+ 2x_{21}$		$+ x_{23}$		$+ 2x_{31}$	

where x_{23} = number shipped from factory in 2nd stage, x_{24} = number stored at factory in 2nd stage;
70 = number produced 2nd stage;
d_3 = unknown demand in 3rd stage uniformly distributed between 70 or 80;
x_{31} = number purchased on the open market in 3rd stage, x_{32} = excess of supply over demand in 3rd stage.[2]

It will be noted that the distribution of d_3 is independent of d_2. However, the approach which we shall use will apply even if the distribution of d_3 depends on d_2. This is important in problems where there may be some postponement of the *timing of demand.* For example, it may be anticipated that the potential refrigerator buyers will buy in November or December. However, those buyers who failed to purchase in November, will affect the demand distribution for December.

Example 4: *A Class of Two-Stage Problems.*

In the Ferguson problem and in many supply problems the total costs may be divided into two parts: first the costs of assigning various resources to several destinations j and second the costs (or lost revenues) incurred because of the failure of the total amounts $u_1, u_2, \cdots, u_n$ assigned to meet demands at various destinations in unknown amounts $d_1, d_2, \cdots, d_n$ respectively.

The special class of two-stage programming problems we are considering has the following structure.[3] For the first stage:

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (x_{ij} \geqq 0) \tag{6}$$

$$\sum_{i=1}^{m} b_{ij} x_{ij} = u_j \tag{7}$$

[2] No solution for this example will be given in this paper. For this case perhaps the simplest approach is through the techniques of dynamic programming; see R. Bellman [1].

[3] The remarks of this section apply if (6) and (7) are replaced more generally by $AX = a$, $BX = U$ where X is the vector of activity levels in the first stage, A and B are given matrices, a a given initial status vector, and $U = (u_1, u_2, \cdots, u_n)$.

where x_{ij} represents the amount of i^{th} resource assigned to the j^{th} destination and b_{ij} represents the number of units of demand at destination j that can be satisfied by one unit of resource i. For the second stage

$$d_j = u_j + v_j - s_j \qquad (j = 1, 2, \cdots, n) \tag{8}$$

where v_j is the shortage[4] of supply and s_j is the excess of supply.

The total cost function is assumed to be of the form

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} \alpha_j v_j \tag{9}$$

i.e., depends linearly on the choice x_{ij} and on the shortages v_j (which depend on assignments u_j and the demands d_j).

Our objective will be to minimize total expected costs.[5] Let $\phi_j(u_j \mid d_j)$ be the minimum costs at a destination if the supply is u_j and the demand is d_j. It is clear that

$$\phi_j(u_j \mid d_j) = \begin{cases} \alpha_j(d_j - u_j) & \text{if} \quad d_j \geqq u_j \\ 0 & \text{if} \quad d_j < u_j \end{cases} \tag{10}$$

where α_j is the coefficient of proportionality. We shall now give a result due to H. Scarf.

Theorem: The expected value of $\phi_j(u_j \mid d_j)$, denoted by $\phi_j(u_j)$ is a convex function of u_j.

Proof: Let $p(d_j)$ be the probability density of d_j, then

$$\begin{aligned} \phi_j(u_j) &= \alpha_j \int_{x=u_j}^{+\infty} (x - u_j)p(x)\,dx \\ &= \alpha_j \int_{x=u_j}^{+\infty} xp(x)dx - \alpha_j u_j \int_{x=u_j}^{+\infty} p(x)\,dx \end{aligned} \tag{11}$$

whence differentiating $\phi(u)$

$$\phi_j'(u_j) = -\alpha_j \int_{x=u_j}^{+\infty} p(x)\,dx. \tag{12}$$

It is clear that $\phi_j'(u_j)$ is a non-decreasing function of u_j with $\phi_j''(u_j) \geqq 0$ and that $\phi_j(u_j)$ is convex. An alternative proof (due also to Scarf) is obtained by applying a lemma which we shall use later on.

Lemma: If $\phi(x_1, x_2, \cdots, x_n \mid \theta)$ is a convex function over a fixed region Ω for

[4] Equation (8) should be viewed more generally then simply as a statement about the shortage and excess of supply. In fact, given any u_j and d_j, there is an infinite range of possible values of v_j and s_j satisfying (8). For example, v_j might be interpreted as the amount obtained from some new source (perhaps at some premium price) and s_j the amount not used. When the cost form is as in (9), it becomes clear that in order for c to be a minimum the values of v_j and s_j will have the more restrictive meaning above.

[5] H. Markowitz in his analysis of portfolios considers the interrelation of the variance with the expected value. See [5].

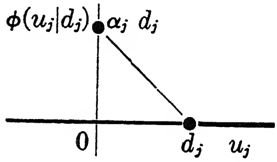
every value of θ, then any positive linear combination of such functions is also convex in Ω.

In particular if θ is a random variable with probability density $p(\theta)$, then expected value of ϕ

(13) $$\phi(x_1, x_2, \cdots, x_n) = \int_{-\infty}^{+\infty} \phi(x_1, x_2, \cdots, x_n|\theta)p(\theta)\, d\theta$$

is convex. For example from (10), $\phi(u_j \mid d_j)$, plotted below, is convex.

(14)



From the lemma the result readily follows that $\phi_j(u_j)$ is convex.

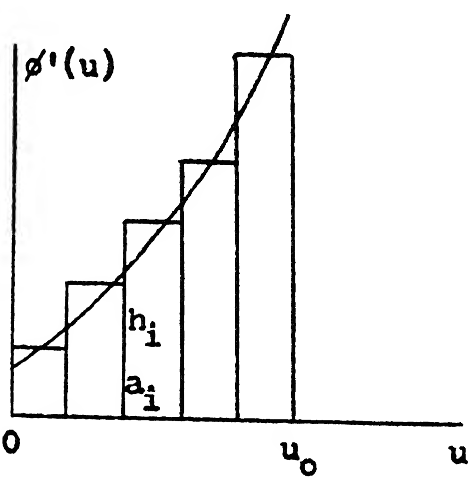
From the basic theorem the expected value of the objective function is

(15) $$\text{Exp } C = \sum c_{ij}x_{ij} + \sum_{j=1}^{n} \alpha_j\phi_j(u_j)$$

where $\phi_j(u_j)$ are *convex functions.* Thus the original problem has been reduced to minimizing (15) subject to (6), (7).

This permits application of a well-known device for *approximating* such a problem by a standard linear programming problem in the case the objective function can be represented by a sum of convex functions. See for example [3] or Charnes and Lemke, [2]. To do this one approximates the derivative of $\phi(u)$ in some sufficiently large range $0 \leqq u \leqq u_0$ by a step function

(16)



involving k steps where size of the i^{th} base is a_i and its height is h_i ; where $h_1 \leqq$

$h_2 \leqq \cdots \leqq h_k$ because ϕ is convex. An approximation for $\phi(u)$ is given by

(17) $$\phi(u) \doteq \phi(0) + \text{Min} \sum_1^k h_i \Delta_i$$

subject to

(18) $$u = \sum_{i=1}^k \Delta_i\,, \qquad 0 \leqq \Delta_i \leqq a_i\,.$$

Indeed, it is fairly obvious that the approximation achieves its minimum by choosing $\Delta_1 = a_1$, $\Delta_2 = a_2$, $\cdots$ until the cumulative sum of the Δ_i exceeds u for some $i = r$; Δ_r is then chosen as the value of the residual with all remaining $\Delta_{r+i} = 0$. In other words, we have approximated an integral by the sum of rectangular areas under the curve up to u, i.e.,

(19) $$\phi(u) = \phi(0) + \int_0^u \phi'(x)\,dx \doteq \sum_{i=1}^r h_i a_i + h_r \Delta_r\,.$$

The next step is to replace $\phi(u)$ by $\sum_1^k h_i \Delta_i$, u by $\sum_1^k \Delta_i$ in the programming problem and add the restrictions $0 \leqq \Delta_i \leqq a_i$. If the objective is minimization of total costs, it will, of necessity, for whatever value of $u = \sum_1^n \Delta_i$ and $0 \leqq \Delta_i \leqq a_i$, minimize $\sum_1^k h_i \Delta_i$. Thus, this class of two-stage linear programming problems involving uncertainty can be reduced to a standard linear programming type problem. In addition, simplifying computational methods exist when variables have upper bounds such as $\Delta_i \leqq a_i$; see [3].

Example 5: *The Two-Stage Problem with General Linear Structure.*

We shall prove a general theorem on convexity for the two-stage problem that forms the inductive step for the multi-stage problem. We shall say a few words about the significance of this convexity later on. The assumed structure of the general[6] two-stage model is

(20) $$\begin{aligned} b_1 &= A_{11}X_1 \\ b_2 &= A_{21}X_1 + A_{22}X_2 \\ C &= \phi(X_1, X_2 \mid E_2) \end{aligned}$$

where A_{ij} are known matrices, b_1 a known vector of initial inventories. For example

$a_i = \sum_{j=1}^m x_{ij}$	here $b_1 = (a_1, a_2, \cdots, a_m)$
	here $X_1 = (x_{11}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, \cdots, x_{mn})$
$d_j = \sum_{i=1}^m b_{ij}x_{ij} + v_j - s_j$	here $b_2 = (d_1, d_2, \cdots, d_n)$
$C = \sum\sum c_{ij}x_{ij} + \sum \alpha_j v_j$	here $X_2 = (v_1, v_2, \cdots, v_n, s_1, s_2, \cdots, s_n)$

b_2 an unknown vector whose components are determined by a chance mechanism.[7]

[6] A special case of the general model given in (20) is found in Example 4.

[7] The chance mechanism may be the "market," the "weather."

(Mathematically, E_2 is a sample point drawn from a multidimensional sample space with known probability distribution); X_1 is the vector of nonnegative activity levels to be determined in the first stage, while X_2 is the vector of nonnegative activity levels for the second stage. It is assumed that whatever be the choice of X_1 satisfying the first-stage equations and whatever be the particular values of b_2 determined by chance, there exists at least one vector X_2 satisfying the second-stage equations. The total costs C of the program are assumed to depend on the choice of X_1, X_2, and parametrically on E_2. The basic problem is to choose X_1 and later X_2 in the second stage such that the expected value of C is a minimum.

Theorem: If $\phi(X_1, X_2 \mid E_2)$ is a convex function in X_1, X_2 whatever be X_1 in Ω_1, i.e., satisfying the 1st stage restrictions and whatever be X_2 in $\Omega_2 = \Omega_2(X_1 \mid b_2)$, i.e., satisfying the 2nd stage restrictions given b_2 and X_1, then there exists a convex function $\phi_0(X_1)$ such that the optimal choice of X_1 subject to $b_1 = A_{11}X_1$ is found by minimizing $\phi_0(X_1)$ where

$$\begin{aligned} \phi_0(X_1) &= \underset{E_2}{\operatorname{Exp}} \, [\underset{X_2 \in \Omega_2}{\operatorname{Inf}} \, \phi(X_1, X_2 \mid E_2)], \\ \operatorname{Exp} C &= \underset{X_1 \in \Omega_1}{\operatorname{Inf}} \, \phi_0(X_1); \end{aligned} \tag{21}$$

the expectation (Exp) is taken with respect to the distribution of E_2 and the greatest lower bound (Inf)[8] is taken with respect to all $X_2 = \Omega_2(X_1 \mid E_2)$.

Proof:[9] In order to minimize the Exp $\phi_1(X_1, X_2 \mid E_2)$, it is clear that once X_1 has been selected, E_2 determined by chance, that X_2 must be selected so that $\phi(X_1, X_2 \mid E_2)$ is minimized for fixed X_1 and E_2. Thus, the costs for *given* X_1 and E_2 is given by

$$\phi_1(X_1 \mid E_2) = \underset{X_2 \in \Omega_2}{\operatorname{Inf}} \, \phi(X_1, X_2 \mid E_2). \tag{22}$$

The expected costs for a *given* X_1 is then simply the expected value of $\phi_1(X_1 \mid E_2)$ and this we denote by $\phi_0(X_1)$. The optimal choice of X_1 to minimize expected costs C is thus reduced to choosing X_1 so as to minimize $\phi_0(X_1)$. There remains only to establish the convexity property. We shall show first that $\phi_1(X_1 \mid E_2)$ for bounded ϕ_1 is convex for X_1 in Ω_1. If true, then applying the lemma, the result that $\phi_0(X_1)$ is convex readily follows. Let us suppose that $\phi_1(X_1 \mid E_2)$ is not convex, then there exist three points in $\Omega_1 : X_1', X_1'', X_1''' = \lambda X_1' + \mu X''$ $(\lambda + \mu = 1, 0 \leqq \lambda \leqq 1)$ that violate the condition for convexity, i.e.,

$$\lambda\phi_1(X_1' \mid E_2) + \mu\phi_1(X_1'' \mid E_2) < \phi_1(X_1''' \mid E_2) \tag{23}$$

[8] The greatest lower bound instead of minimum is used to avoid the possibility that the minimum value is not attained for any admissible point $X_2 \in \Omega_2$ or $X_1 \in \Omega_1$. In case where the latter occurs, it should be understood that while there exists no X_i where the minimum is attained, there exists X_i for which values as close to minimum as desired are attained.

[9] This proof is along lines suggested by I. Glicksberg.

or

$$\lambda\phi_1(X_1' \mid E_2) + \mu\phi_1(X_1'' \mid E_2) = \phi_1(X_1''' \mid E_2) - \epsilon_0 \qquad \epsilon_0 > 0. \tag{24}$$

For any $\epsilon_0 > 0$, however, there exists X_2' and X_2'' such that

$$\begin{aligned} \phi_1(X_1' \mid E_2) &= \phi(X_1', X_2' \mid E_2) - \epsilon_1 & 0 \leqq \epsilon_1 < \epsilon_0 \\ \phi_1(X_1'' \mid E_2) &= \phi(X_1'', X_2'' \mid E_2) - \epsilon_2 & 0 \leqq \epsilon_2 < \epsilon_0 . \end{aligned} \tag{25}$$

Setting $X_2''' = \lambda X_2' + \mu X_2''$ we note because of the assumed linearity of the model (20) that $(\lambda X_2' + \mu X_2'') \epsilon\, \Omega_2(\lambda X_1' + \mu X_1'' \mid E_2)$ and hence by convexity of ϕ

$$\lambda\phi(X_1', X_2' \mid E_2) + \mu\phi(X_1'', X_2'' \mid E_2) \geqq \phi(X_1''', X_2''' \mid E_2) \tag{26}$$

whence by (25)

$$\lambda\phi_1(X_1' \mid E_2) + \mu\phi_1(X_1'' \mid E_2) \geqq \phi(X_1''', X_2''' \mid E_2) - \lambda\epsilon_1 - \mu\epsilon_2 \tag{27}$$

and by (24)

$$\phi_1(X_1''' \mid E_2) \geqq \phi(X_1''', X_2''' \mid E_2) - \lambda\epsilon_1 - \mu\epsilon_2 + \epsilon_0 \;(0 \leqq \lambda\epsilon_1 + \mu\epsilon_2 < \epsilon_0) \tag{28}$$

which contradicts the assumption that $\phi_1(X_1''' \mid E_2) = \operatorname{Inf} \phi(X_1''', X_2 \mid E_2)$. The proof for unbounded ϕ is omitted.

Example 5: *The Multi-Stage Problem with General Linear Structure.*

The structure assumed is

$$\begin{aligned} b_1 &= A_{11}X_1 \\ b_2 &= A_{21}X_1 + A_{22}X_2 \\ b_3 &= A_{31}X_1 + A_{32}X_2 + A_{33}X_3 \\ b_4 &= A_{41}X_1 + A_{42}X_2 + A_{43}X_3 + A_{44}X_4 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m &= A_{m1}X_1 + A_{m2}X_2 + A_{m3}X_3 + \cdots \cdots A_{mm}X_m \\ C &= \phi(X_1, X_2, \cdots, X_m \mid E_2, E_3, \cdots, E_m) \end{aligned} \tag{29}$$

where b_1 is a known vector; b_i is a chance vector ($i = 2, \cdots, m$) whose components are functions of a point E_i drawn from a known multi-dimensional distribution; A_{ij} are known matrices. The sequence of decisions is as follows: X_1, the vector of nonnegative activity levels in the 1st stage, is chosen so as to satisfy the first stage restrictions $b_1 = A_{11}X_1$; the values of components of b_2 are chosen by chance by determining E_2; X_2 is chosen to satisfy the 2nd stage restrictions $b_2 = A_{21}X_1 + A_{22}X_2$, etc. iteratively for the third and higher stages. It is further assumed that:

(1) The components of X_j are nonnegative;

(2) There exists at least one X_j satisfying the j^{th} stage restraints, whatever be

the choice of $X_1, X_2, \cdots, X_{j-1}$ satisfying the earlier restraints or the outcomes $b_1, b_2, \cdots, b_m$.

(3) The total cost C is a convex function in $X_1, \cdots, X_m$ which depends on the values of the sample points $E_2, E_3, \cdots E_m$.

Theorem: An equivalent $(m - 1)$ *stage programming problem with a convex pay-off function can be obtained by dropping the* m^{th} *stage restrictions and replacing the convex cost function* ϕ *by*

$$\phi_{m-1}(X_1, X_2, \cdots, X_{m-1} \mid E_2, \cdots, E_{m-1}) \tag{30}$$
$$= \operatorname*{Exp}_{E_m} \ \operatorname*{Inf}_{X_m \in \Omega_m} \ \phi(X_1, X_2, \cdots, X_m \mid E_2, \cdots, E_m)$$

where Ω_m *is the set of possible* X_m *that satisfy the* m^{th} *stage restrictions.*

Since the proof of the above theorem is identical to the two-stage case no details will be given. The fact that a cost function for the $(m - 1)$ stage can be obtained from the m^{th} stage is simply a consequence that optimal behavior for the m^{th} stage is well defined, i.e., given any state, e.g., $(X_1, X_2, \cdots, X_{m-1})$, at the beginning of this stage, the best possible actions can be determined and the minimum expected cost evaluated. This is a standard technique in "dynamic programming." For the reader interested in methods built around this approach the reader is referred to R. Bellman's book on dynamic programming [1].

While the existence of convex functions has been demonstrated that permit reduction of an m-stage problem to equivalent m-1, m-2, $\cdots$, 1-stage problems, it appears hopeless that such functions can be computed except in very simple cases. The convexity theorem was demonstrated not as a solution to an m-stage problem but only in the hope that it will aid in the development of an efficient computational theory for such models. It should be remembered that any procedure that yields a local optimum will be a true optimum if the function is convex. This is important because multi-dimensional problems in which non-convex functions are defined over non-convex domains lead as a rule to local optimum and an almost hopeless task, computationally, of exploring other parts of the domain for the other extremes.

Solution for Example 2: *Shipping to an Outlet to Meet an Uncertain Demand.*

Let us consider the two-stage case given earlier (4). It is clear that, if supply exceeds demand $(x_{11} > d_2)$, that $x_{21} = 0$ gives minimum costs and, if $x_{11} \leqq d_2$, that $x_{21} = d_2 - x_{11}$ gives minimum costs. Hence

$$\operatorname*{Min}_{x_{21}} \phi = \begin{cases} x_{11} & \text{if} \quad x_{11} > d_2 \\ x_{11} + 2(d_2 - x_{11}) & \text{if} \quad x_{11} \leqq d_2 . \end{cases} \tag{31}$$

Since d_2 is assumed to be uniformly distributed between 70 and 80

$$\operatorname*{Exp}_{d_2} [\operatorname*{Min}_{x_{21}} \phi] = \begin{cases} -x_{11} + 150 & \text{if} \quad x_{11} \leqq 70 \\ 77.5 + \frac{1}{10}(75 - x_{11})^2 & \text{if} \quad 70 < x_{11} \leqq 80 \\ x_{11} & \text{if} \quad 80 \leqq x_{11} \end{cases} \tag{32}$$

This function is clearly convex and attains its minimum 77.5, which is the expected cost, at $x_{11} = 75$. Since $x_{11} = 75$ is in the range of possible values of x_{11} as determined by $100 = x_{11} + x_{12}$ this is clearly the optimal shipment. In this case it pays to ship $x_{11} = \bar{d}_2 = 75$, the expected demand.

It can be shown by simple examples that one cannot replace, in general, the chance vectors b_i by $\bar{b}_i$, the vector of expected values of the components of b_i. Nevertheless, this procedure, which is quite common, probably provides an excellent starting solution for any improvement technique that might be devised. For example, in the problem of Ferguson (application of Example 4), using as a start the solution based on expected values of demand, it was an easy matter to improve the solution to an optimal one whose expected costs were 15% less.

Solution for Example 5: *The General Two-Stage Case.*

When the number of possibilities for the chance vector b_2 is $b_2^{(1)}, b_2^{(2)}, \cdots, b_2^{(k)}$ with probabilities $p_1, p_2, \cdots, p_k$, $(\sum p_i = 1)$, it is not difficult to obtain a direct linear programming solution for small k, say $k = 3$. Since this type of structure is very special, it appears likely that techniques can be developed to handle large k. For $k = 3$, the problem is equivalent to determining vectors X_1 and vectors $X_2^{(1)}, X_2^{(2)}, X_2^{(3)}$ such that

$$
\begin{aligned}
b_1 &= A_{11}X_1 \\
b_2^{(1)} &= A_{21}X_1 + A_{22}X_2^{(1)} \\
b_2^{(2)} &= A_{21}X_1 \qquad\qquad + A_{22}X_2^{(2)} \\
b_2^{(3)} &= A_{21}X_1 \qquad\qquad\qquad\qquad + A_{22}X_2^{(3)}
\end{aligned}
\tag{33}
$$

$$\text{Exp } C = \gamma_1 X_1 + p_1\gamma_2 X_2^{(1)} + p_2\gamma_2 X_2^{(2)} + p_3\gamma_3 X_2^{(3)} = \text{Min}$$

where for simplicity we have assumed a linear objective function.

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VIII–29

LINEAR PROGRAMMING AND SEQUENTIAL DECISIONS*†

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Using an illustration drawn from the area of inventory control, this paper demonstrates how a typical sequential probabilistic model may be formulated in terms of (a) an initial decision rule and (b) a Markov process, and then optimized by means of linear programming. This linear programming technique may turn out to be an efficient alternative to the functional equation approach in the numerical analysis of such problems. Regardless of computational significance, however, it is of interest that there should be such a close relationship between the two traditionally distinct areas of dynamic programming and linear programming.

1. Summary

Using an illustration drawn from the area of inventory control, this paper demonstrates how a typical sequential probabilistic model may be formulated in terms of (a) a decision rule, specifying order quantities as a function of initial stock levels, and (b) a Markov process in which the transition probabilities depend both upon the decision rule and also upon the probability distribution of demands. Optimization of the decision rule is accomplished by means of linear programming.

In contrast with the linear programming studies of Dantzig [4] and Radner [10], the time horizon considered here is infinite rather than finite. For a study very closely related to this one, the reader is referred to a paper written by R. Howard [7].

The essential idea underlying this linear programming formulation is that the "state" variable i (initial stock level) and the "decision" variable j (order quantity) are introduced as subscripts to the unknowns x_{ij}. These unknowns x_{ij} represent the *joint* probabilities with which the state variable takes on the value of i and the decision variable the value of j. With an infinite time horizon, it is then possible to derive equilibrium distributions (steady state probabilities) of inventory levels, production quantities, and shortage levels. The requirements of statistical equilibrium furnish the linear restraints, and the objective function to be minimized consists of the expected cost level corresponding to the equilibrium probabilities.

Although the particular application described is a rather specialized one, there seem to be quite a number of dynamic programming problems in which this computational technique may prove to be an efficient alternative to the usual iterative method for solving functional equations. As yet, there is only

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a limited amount of evidence available for comparing the effectiveness of the two approaches from the viewpoint of numerical analysis. Regardless of computational significance, however, it is of interest that there should be such a close relationship between the two traditionally distinct areas of dynamic programming and linear programming.

2. Formulation of the Problem

This is a single-item inventory problem in which the initial stock on hand at the beginning of each "month" is, in Bellman's terminology, the "state variable." [2, p. 81] The size of initial inventory will be indicated by the subscript i. The quantity produced within the month is the "decision variable," and the amount produced will be indicated by the subscript j. Our problem is to obtain an optimal sequential decision rule—that is, to specify a value of j for each value taken on by i.

The sum of initial inventory *plus* the quantity produced will be known as the "available stock," and its size will be denoted by k.

The quantity demanded during the month is a serially independent random variable, n. The symbol p_n represents the probability with which n units will be demanded.

The size of month-end terminal inventories will be indicated by t. If backlogs of demand are to be ruled out, $t = \max(0, k - n)$.

Once that a decision rule and a demand probability distribution have been specified, the inventory process may be regarded as a Markov chain. From this chain may be calculated the equilibrium probability distribution of inventory levels, of production quantities, and of shortage levels. It will be assumed that the decision rule is to be specified in such a way as to minimize the expected monthly costs corresponding to these equilibrium probabilities. (Note that this objective is closely related to, but by no means identical with that of minimizing *discounted* expected costs.)

The relevant costs here consist of the sum of the expected value of three components: (1) those costs related to the initial inventory levels i, (2) those related to the production quantities j, and (3) those related to the shortage levels $(n - k)$. Symbolically, total costs are expressed as follows:

$$\mathcal{E}C_1(i) + \mathcal{E}C_2(j) + \mathcal{E}C_3(n - k) \tag{1}$$

No convexity restrictions are imposed upon any of the three functions $C_1(i)$, $C_2(j)$, and $C_3(n - k)$.[1] Convexity is, in effect, brought about by supposing that mixed strategies are available. In other words, the conditional probability of taking action j (given that the initial inventory is at level i) may lie anywhere in the closed interval between zero and unity.[2]

[1] It is a serious limitation of the Holt-Modigliani-Simon production smoothing model that all cost functions must be quadratic. [6] No such assumption is required in the case discussed here.

[2] In an accompanying note by Harvey Wagner [12], it is shown that even though probability mixtures are permissible, there will always be an optimal solution consisting solely of "pure" strategies.

Some fairly light restrictions are imposed upon the quantities i, j, k, n, and t. First, they must be non-negative integers. Second, there must exist a positive integer T, an upper limit upon inventory accumulation, such that:

$$t = \max(0, k - n) \leqq T$$

The linear programming problem described below will involve $T + 1$ equations. In order for the simplex computations to be carried out with present-day electronic machine programs, it would be necessary to choose units in such a way that the integer T does not exceed something of the order of 200.

3. Some Definitions

DF: y_i = probability that a month's initial stock equals $i \cdot (\sum_i y_i = 1.)$
DF: y'_t = probability that a month's terminal stock equals $t \cdot (\sum_t y'_t = 1.)$
Statistical equilibrium requires:

$$y_t = y'_t \qquad (t = 0, 1, \cdots, T) \tag{2}$$

DF: x_{ij} = joint probability with which the initial stock equals i and the production quantity equals j.

$$\therefore \sum_j x_{ij} = y_i \qquad (i = 0, 1, \cdots, T) \tag{3}$$

and

$$\sum_{i,j} x_{ij} = 1 \tag{4}$$

DF: z_k = probability that the available stock equals k

$$\therefore z_k = \sum_{\substack{i,j:\\ i+j=k}} x_{ij} \qquad (k = 0, 1, \cdots, T) \tag{5}$$

DF: p_n = probability that n units are demanded within the month.
N.B. The probabilities p_n are independent of any choices made by the decision-maker. The probabilities x_{ij}, y_i, y'_t, and z_k, however are directly under his control. (Note that once the joint probabilities x_{ij} have been specified, it is straightforward to reconstruct the decision rule—i.e., the conditional probability of taking action j, given the initial stock level i.)

4. Relationships between the Individual Probabilities

Since the random variable n is independent of the available stock k, and since $t = \max(0, k - n)$:

$$(6) \qquad \begin{aligned} y'_0 &= \sum_{\substack{k,n:\\ k-n \leqq 0}} p_n z_k \\ y'_t &= \sum_{\substack{k,n:\\ k-n=t}} p_n z_k \qquad (t = 1, 2, \cdots, T) \end{aligned}$$

By (5):

$$(7)\qquad \begin{aligned} y'_0 &= \sum_{\substack{i,j,n:\\ i+j-n\leqq 0}} p_n x_{ij} \\ y'_t &= \sum_{\substack{i,j,n:\\ i+j-n=t}} p_n x_{ij} \qquad (t = 1, 2, \cdots, T) \end{aligned}$$

By (2) and (3), we finally arrive at the interdependence relationships between the individual unknowns x_{ij}:

$$(8.0)\qquad \sum_j x_{0j} = \sum_{\substack{i,j,n:\\ i+j-n\leqq 0}} p_n x_{ij}$$

$$(8.t)\qquad \sum_j x_{tj} = \sum_{\substack{i,j,n:\\ i+j-n=t}} p_n x_{ij} \qquad (t = 1, 2, \cdots, T)$$

Equations (8.0) — (8.T) may each be interpreted as a requirement of statistical equilibrium. In each of these equations, the left-hand side measures the probability with which the *initial* monthly inventory level will be t, and the right-hand side the probability with which the terminal level will equal t. Statistical equilibrium implies that these two probabilities must coincide.

The unknowns in the linear programming model are the joint probabilities x_{ij}. The constraints consist of the usual non-negativity conditions upon the x_{ij}, together with equations (4) and (8.1)–(8.T). Equation (8.0) is redundant, and need not be included explicitly within the constraint set.

5. Expected Costs

The cost coefficient associated with each of the x_{ij} will be known as c_{ij}. The total cost expression to be minimized by means of the simplex procedure is as follows:

$$(9)\qquad \sum_{i,j} c_{ij} x_{ij}$$

How do we assign values to the coefficients c_{ij} so as to be consistent with the minimand given previously by expression (1)? Note that:

$$\begin{aligned} \mathcal{E}C_1(i) &= \sum_i y_i C_1(i) = \sum_{i,j} x_{ij} C_1(i) \\ \mathcal{E}C_2(j) &= \sum_{i,j} x_{ij} C_2(j) \\ \mathcal{E}C_3(n-k) &= \sum_{i,j} x_{ij} \sum_n p_n C_3(n-i-j) \end{aligned}$$

The cost coefficient c_{ij} associated with the unknown x_{ij} is therefore constructed as follows:

$$(10)\qquad c_{ij} = C_1(i) + C_2(j) + \sum_n p_n C_3(n-i-j)$$

6. A Numerical Example

In order to construct a numerical example, it is necessary to assign values to the demand probabilities, to the three cost functions, and to the upper limit

placed upon inventory accumulation. For illustrative purposes, we will work with the following:

$$p_0 = \tfrac{2}{3} \qquad C_1(i) = i \qquad T = 3$$
$$p_1 = 0 \qquad C_2(j) = 3j$$
$$p_2 = \tfrac{1}{3} \qquad C_3(n - i - j) = \max\,[0, 6(n - i - j)]$$

In addition, it will be assumed that the production capacity is at most one unit per month (i.e., j = either 0 or 1). Note that the mean demand level amounts to only $\frac{2}{3}$ of this capacity limit. There is, however, a $\frac{1}{3}$ probability that demand will actually amount to twice the production limit.

Table 1 contains a calculation of the cost coefficients for this problem, and Table 2 indicates the constraint matrix in detached coefficients form. In transcribing equations (8.1)–(8.3) into this matrix, the right-hand side shown earlier in the text has been subtracted from the left-hand side. Equation (8.1), for example, has been transformed as follows:

$$\sum x_{1j} - \sum_{\substack{i,j,n:\\ i+j-n=1}} p_n x_{ij} = 0$$

TABLE 1

Calculation of the cost coefficients c_{ij}

Identification subscripts (i, j)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)
Inventory costs $= C_1(i) = i$	0	0	1	1	2	2	3
Production costs $= C_2(j) = 3j$	0	3	0	3	0	3	0
Shortage costs $= \Sigma_n p_n C_3(n - i - j) = \Sigma_n p_n \max\,[0, 6(n - i - j)]$	4	2	2	0	0	0	0
Total cost coefficient $= c_{ij}$	4	5	3	4	2	5	3

TABLE 2

Detached coefficients matrix

Identification subscripts (i, j)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	Constant terms
Equation (4)	1	1	1	1	1	1	1	= 1
Equation (8.1)	0	$-\frac{2}{3}$	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	= 0
Equation (8.2)	0	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	1	0	= 0
Equation (8.3)	0	0	0	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	= 0
Optimal activity levels, x_{ij}	—	$\frac{1}{3}$	—	$\frac{2}{9}$	$\frac{4}{9}$	—	ε*	
Conditional probability of ordering quantity j—given an inventory level of i. $\dfrac{x_{ij}}{\sum_j x_{ij}}$	0	1	0	1	1	0	1*	

* To eliminate the question of degeneracy, it is convenient to regard the value of $x_{3,0}$ as ε, a "small" positive quantity.

Also shown in Table 2 is the optimal linear programming solution to the problem. According to this calculation, the initial inventory will be at a zero level during $\frac{1}{3}$ of the months, at a unit level $\frac{2}{9}$ of the time, and at a level of two during the remaining $\frac{4}{9}$.[3] The conditional probabilities derived from this solution indicate the following decision rule: Whenever the initial inventory has dropped to a level of either zero or unity, one unit of production is ordered. At higher initial levels, no production takes place at all. Note that no mixed strategies are indicated despite the fact that this option was built into the model.

7. Some Observations

(1) There are a number of paths by which one may prove that it will always be optimal to adopt pure strategies. One way is sketched out in the accompanying note by Harvey Wagner. Another—and perhaps a more intuitive way—is to follow the line of reasoning by which Dvoretzky, Kiefer, and Wolfowitz dismiss mixed strategies in a problem of this sort. This is a problem in which the demand probabilities p_n are known in advance to the decision-maker, and do not have to be estimated by him. [5, p. 191 n.] Hence the conclusion that in a two-person game in which the decision-maker has "found out" his opponent's strategy, it will never hurt him to restrict his own choice of strategies to pure ones.[4]

(2) The choice of an upper limit, T, upon inventory accumulation is admittedly an arbitrary one. If, after finding an optimal solution for a given value of T, and observing that $x_{T-j,j} = 0$ for all j, it is entirely possible that a further increase in the value of T will lower the minimand still further. It is a simple matter to construct pathological cost functions that will yield this result. Lest the reader become too concerned over this potential snare, it is worth pointing out that there are a number of applications in which there exist very real upper limits upon the accumulation of inventory, e.g., the reservoir capacity of a hydroelectric system.

(3) It is not altogether legitimate to have brushed aside the question of initial conditions for the Markov process. If the optimal matrix in the linear programming solution is a "decomposable" one, the initial conditions will clearly govern the ultimate statistical equilibrium. The most direct way to circumvent this difficulty would be to assume that the initial conditions lie within the control of the decision-maker—at least to the extent that he may choose them so as to start off within any one of the subsystems into which the larger system splits up.

(4) It is possible to attach an economic interpretation to the implicit prices (dual variables) associated with the linear programming solution. They represent

[3] The average monthly cost associated with this solution equals $(\frac{1}{3})$ (5) + $(\frac{2}{9})$ (4) + $(\frac{4}{9})$ (2) = 31/9. It is of some interest to compare this cost level with that of the do-nothing basic feasible solution—one in which the unknown x_{00} equals unity, all other unknowns are set at zero, and the resulting monthly costs amount to 4.

[4] I am indebted to J. Marschak for having pointed out the applicability of this line of reasoning to the problem at hand.

the amount by which total costs would be altered if the initial inventory were at the t th level rather than at zero.[5] Apparently, they are related to the solution of Bellman's functional equation for the inventory problem. [2, pp. 159–164] This being so, it should be a comparatively simple matter to use them in order to link together a non-stationary finite-horizon model with a stationary one having an infinite horizon.

8. Areas of Application

Among the applications that suggest themselves, the following stochastic models would seem to be of the most interest:

(1) *Changes in the rate of production.* A number of studies have been concerned with systems in which the costs depend not only upon the rate of production (as in the example above), but also upon the rate of change of that level. (E.g., [6].) This kind of problem could be attacked through the same methods outlined here by defining the "state variable" i as a *pair* of numbers: one representing the initial inventory level and the other the rate of production during the immediately preceding period. With this one change in interpretation, things would proceed in essentially the same way that has been suggested here. The only serious difficulty might arise from the computational costs involved in an increase in the number of equations within the linear programming model. Instead of just one equation for each of the $(T + 1)$ levels of inventory, there would now be r equations—one for each of the r discrete rates of production that were considered. Altogether, the programming matrix would contain $r \cdot (T + 1)$ rows.

(2) *Seasonal storage of inventories.* Several recent papers have been focussed upon the problem of optimization under conditions of seasonally fluctuating demands (e.g., the demand for heating oil [3]) or of supplies (e.g., the supply of water for hydroelectric installations [9]). In order for a linear programming model to reflect such seasonal fluctuations in the probability distribution of demands or of supplies, the state variable i would again have to represent a pair of numbers—the first indicating the season of the year and the second the inventory level at the beginning of the particular season. The conditions of statistical equilibrium would then imply equality between probabilities for the terminal inventories of one season and the initial inventories of the one following. With s seasons and $(T + 1)$ inventory levels in each, a total of $s(T + 1)$ equations would be involved. Even with time subdivided into 12 individual months and with 10 levels of inventory considered during each month, the computational requirements would still remain modest—a 120-equation system.

(3) *Multi-location inventory problems.* In the event that inventories are scattered among several geographical locations, it may no longer be appropriate to describe the system in terms of a single state variable—the aggregate quantity

[5] For the numerical solution shown in Table 2, the implicit prices associated with equations (8.1)–(8.3) are, respectively, $-7/3$, $-13/3$, and $-11/3$. These values serve to measure the comparative advantage of beginning the Markov process with an inventory level of 1, 2, or 3 units.

held in stock. Instead, a separate quantity must be specified for each location.[6] If, then, there are stocks held at l different locations, the state variable i will have to be regarded as an l-tuplet of numbers. With $(T + 1)$ alternative inventory levels at each individual location, the linear programming model would contain no less than $(T + 1)^l$ distinct equations. As far as any realistic problems are concerned, it must be conceded that this number of equations could become hopelessly large. Even with just four locations and five inventory levels at each, the system would contain 625 equations! The most obvious way to reduce the size of such problems would be to devise some judicious scheme for aggregation into a manageable number of geographical areas.

(4) *Delivery lags.* Each of the cases described thus far has been based upon the assumption that delivery lags are short—that any production ordered at the beginning of a period will be available to satisfy whatever demand takes place within the period. With long delivery lags, these models hardly seem to be appropriate.

A number of authors [1, 8] have shown, however, that there is a simple way to analyze a problem in which there are long but fixed delivery lags—that is, no randomness in the time required for delivery. (This formulation guarantees that all currently outstanding orders will have been received prior to the arrival of any order placed currently.) In addition to non-random delivery lags, these authors also assume that a shortage in supply is reflected in a temporary backlog rather than in a permanent loss of demand.

With these assumptions, the appropriate state variable required in order to describe the system is no longer the actual inventory on hand, but rather the sum of that inventory plus all outstanding orders. To adapt this suggestion to the linear programming model discussed here, all that needs to be done is to reinterpret the state variable i as "stock on hand plus orders outstanding." The probability p_n would be regarded as the probability that n units were demanded during whatever time interval is required for the delivery of an order. This variant upon the inventory model is equally well adapted to the case in which time is regarded as a discrete or as a continuous parameter.

9. An Unresolved Difficulty

The minimand employed here represents the average level of costs per unit of time, and completely ignores the dating of these costs. Time discounting is neglected—just as in many other treatments of the inventory problem. The only justification for this procedure must be that the mean interval between successive recurrences of any given inventory level—that this mean interval is short relative to the discount factor.

In cases involving equipment analysis, however, this simplification seems quite unpalatable. The interval between successive replacements of a piece of equipment is likely to be measured in years rather than months [11]. With

[6] Essentially the same problem arises if, instead of one commodity in several locations, we are concerned with planning for several different commodities at a single location.

such models, the "present worth" form of minimand appears essential. It will be of considerable interest to see whether the current linear programming formulation of Markov processes can be extended to the case of time discounting.[7]

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[7] Since these lines were written, F. d'Epenoux has succeeded in producing a linear programming formulation for the case of time discounting. His paper appears in *Revue Française de Recherche Opérationnelle*, 4, No. 14, 1960: "Sur un Problème de Production et de Stockage dans l'Aléatoire", p. 3.

VIII–30

CHANCE-CONSTRAINED PROGRAMMING*†[1]

A. CHARNES[2] AND W. W. COOPER[3]

A new conceptual and analytical vehicle for problems of temporal planning under uncertainty, involving determination of optimal (sequential) stochastic decision rules is defined and illustrated by means of a typical industrial example. The paper presents a method of attack which splits the problem into two non-linear (or linear) programming parts, (i) determining optimal probability distributions, (ii) approximating the optimal distributions as closely as possible by decision rules of prescribed form.

Introduction

The problem of stochastic (or better, chance-constrained) programming is here defined as follows: Select certain random variables as functions of random variables with known distributions in such a manner as (a) to maximize a functional of both classes of random variables subject to (b) constraints on these variables which must be maintained at prescribed levels of probability. More loosely, the problem is to determine optimal stochastic decision rules under these circumstances. An example is supplied in [2]. Temporal planning in which uncertainty elements are present, but in which management has access to "control variables" with which to influence outcomes, is a general way of characterizing these problems. Thus, queuing problems in which the availability of servers, customers, or both are partly controllable fall within this classification. It should be noted, that the constraints to be maintained at the specified levels of probability will typically be given in the form of inequalities.

The method of attack which will be outlined in this paper consists of splitting the problem into two parts: (i) determining distributions which maximize the functional, subject to the probability constraints; (ii) approximating the distributions so determined as closely as possible (in some sense) by functions of the known random variables of some prescribed or admissible class. The functions so determined can be regarded as approximations to the optimal stochastic decision rule from the admissible class of such rules.

Specifically, for discrete distributions, and piecewise linear functionals along

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[1] This paper provides the background for the invited lecture presented by the authors at the Fifth International Meeting, The Institute of Management Sciences, Philadelphia, October 17, 1958.

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with linear inequalities involving the random variables—to be maintained at prescribed levels of probability—we factor the original problem into two new problems: (1) a problem which determines the coefficients of the step functions comprising the optimal discrete distributions—or, alternately, the discrete probability frequencies—and (2) another problem which determines the parameters of the optimal decision rule in the sense of a "best approximation".

To fix the ideas, we present first a formulation of a partly controllable situation which has heretofore been formulated and treated (inadequately) by queuing models. We consider terminal tankage facilities supplied by a refinery and with pickup by tankers.

Let

R_j = amount of oil sent to the tankage facilities in the jth period
S_j^α = amount of oil which is picked up from the facilities in the jth period by the αth type of tanker.
I_j = inventory on hand at the beginning of the jth period.
T_A = tankage (i.e., storage facilities) available.

Thus,

$$I_j = I_0 + \sum_{l=1}^{j-1} \left[R_l - \left(\sum_\alpha S_l^\alpha\right) \right] \tag{1}$$

relates the inventory I_j to the initial inventory, I_0, and the rates of input and withdrawal.

The objective is to minimize the expected total cost of input and withdrawal including such features as the cost of changing input rates, demurrage, charter and dispatch. Formally this may be stated as

$$\text{minimize} \quad E[C(\mathbf{R}, \mathbf{S})]$$

where the vectors $\mathbf{R}$ and $\mathbf{S}$ indicate that cost is to be considered over all periods. The expectation sign, E, indicates that these vector variables are to be considered stochastically in assessing the expected value of the total costs, C.

This general formulation can be given a variety of specific forms and intepretations. For example,

$$C(\mathbf{R}, \mathbf{S}) = \sum_j C_j(R_j) + \sum_j w_j \,|R_j - R_{j-1}| + \sum_j d_j A_j^- \tag{2a}$$

where

$$I_j + R_j - \sum_\alpha S_j^\alpha = A_j = A_j^+ - A_j^-$$

$$A_j^+, A_j^- \geqq 0.$$

and

$$A_j^- = \frac{|A_j| - A_j}{2} = \begin{cases} 0 & \text{when } A_j \geqq 0 \\ -A_j & \text{when } A_j < 0 \end{cases}$$

$$A_j^+ = \frac{|A_j| + A_j}{2} = \begin{cases} 0 & \text{when } A_j \leqq 0 \\ A_j & \text{when } A_j > 0 \end{cases}$$

In this case

$$\text{(2b)}\quad \begin{aligned} &\textstyle\sum_j C_j(R_j) \text{ refers to costs of handling due to the amount } R_j\,,\\ &\textstyle\sum_j w_j \mid R_j - R_{j-1} \mid \text{ is the component of total costs due to changing the rate in period } j,\\ &\textstyle\sum_j d_j A_j^- \text{ are demurrage costs,} \end{aligned}$$

and other details may be added as necessary.

In this model the S_j are random variables. However the random components are stated in terms of deviations from scheduled amounts. In addition, the model is of a conditional stochastic variety so that past data, developing experience and forecasts of the future all enter into determining the optimum R_j.

The direct problem is stated in terms of meeting the objective specified in (2) subject to the following constraints.

$$\text{(3a)}\qquad \Pr\{I_j + R_j \geqq \sum_\alpha S_j^\alpha\} \geqq \beta_j\,, \qquad j = 1, 2, \cdots$$

$$\text{(3b)}\qquad \Pr\{I_j + R_j \leqq T_A\} \geqq \gamma_j\,, \qquad j = 1, 2, \cdots$$

where "Pr" means "probability" and β_j and γ_j are the prescribed confidence levels desired for *each* period $j = 1, 2, \cdots$.

The long range problem is concerned with evaluating T_A in terms of the effects on expected cost C. Notice, however, the two significant features associated with this evaluation in (3b): One, there is a valuation element associated with varying T_A while the γ_j are fixed. Two, there is a valuation element associated with cost effects on the risks of not meeting schedules or varying R as different levels of confidence are specified. The former is associated with cost reductions (or increases) arising from varying T_A (and hence R and S in response thereto) at given levels of risk. Hence both risk and service may be evaluated in various combinations when studying the alteration of tankage.

We next illustrate the factoring procedure by means of the following simpler example:

$$\max E \sum_{j=1}^{N} \left\{ P_j(S_j)S_j - (c_j + T_j)R_j - K_j\left(\frac{I_j + I_{j+1}}{2}\right)\right\}$$

subject to

$$\text{(1)}\quad \begin{array}{ll} \text{(i)} & \Pr\{I_j + R_j \geqq S_j + I_{\min}\} \geqq \alpha_j \\ \text{(ii)} & \Pr\{I_j + R_j - S_j \leqq I_{\max}\} \geqq \beta_j \\ \text{(iii)} & R_j \geqq 0 \end{array}$$

where, in period $j = 1, 2, \cdots, N$,

I_j = Inventory on hand at start
R_j = Production rate to be scheduled
S_j = Sales demand
$I_{\min}$ = Minimum inventory to be maintained
$I_{\max}$ = Storage capacity
K_j = Inventory carrying charge
c_j, T_j = Production and Transport Cost (per unit), respectively,
$P_j(S_j)$ = Unit sales price as function of sales demand

and E indicates expectation. I.e., the objective is to maximize the expected net return over an N period planning horizon subject to the probabilistic constraints which are to be honored in each of the $j = 1, 2, \cdots, N$ periods.

This problem—indeed, a more general one involving a convex functional—was treated in [2] by means of a restricted class of decision rules which made it possible to transform the problem into a deterministic one (involving certainty equivalents) which could be solved by a specially developed convex programming algorithm. The purpose of the present paper is, by contrast, to suggest a new analytic method which offers the possibility of handling a much wider class of decision rules.

For this example—as well as for the class of rules which will be considered—the observable stochastic variables are independent. Because the decision on R_j must be made before S_j is observed, the admissible class of decision rules for R_j (or $A_j = I_j + R_j$) can involve, as random variables, only $S_1, \cdots, S_{j-1}$. The A_j may thus be considered statistically independent of the respective S_j. This means that the distribution of $A_j - S_j$ is given by a convolution of the distributions of A_j and $-S_j$.

In considering the first of the two parts into which the problem is to be split we shall transform it into a mixed integer programming problem for determining the relative frequencies of distributions for the A_j. By definition of I_j and A_j,

$$(2)\qquad \begin{aligned} R_j &= A_j - A_{j-1} + S_{j-1} \qquad \text{and} \\ I_j &= A_{j-1} - S_{j-1} \end{aligned}$$

Thus, by well-known properties of the expectation operator, E, the maximand is reduced to a linear function of the (yet-to-be determined) relative frequencies of the A_j.

Similarly, the expression

$$(3)\qquad \Pr\{A_j - S_j \geqq x\} = \int_x^\infty \hat{f}_j(y) g_j(x - y)\, dy,$$

where $\hat{f}_j$ and g_j are the density functions respectively, for $-S_j$ and A_j, is a linear function of the λ_{jr}, the (unknown) relative frequency of the rth possible amount for A_j. Thus, (1·i) and (1·ii) go over into linear inequalities involving the λ_{jr}. To these we must append the conditions

$$(4)\qquad \begin{aligned} \sum_r \lambda_{jr} &= 1 \\ \lambda_{jr} &\geqq 0 \end{aligned}$$

so that the λ_{jr} may be interpreted as relative frequencies. Also, since (1·iii) may be rewritten

$$(5.1)\qquad A_j \geqq A_{j-1} - S_{j-1}$$

we may interpret it as

$$(5.2)\qquad \min A_j \geqq \max (A_{j-1} - S_{j-1})$$

in order to transform the non-negativity requirement to a condition on the frequency functions.

As is known, the density function for $A_{j-1} - S_{j-1}$ has its relative frequencies as linear functions of the $\lambda_{j-1,r}$'s; let these be denoted by $\lambda'_{j-1,k}$. The requirement (5.2) can then be expressed by

$$
\begin{aligned}
& h_{ji} \leqq \sum_{k=1}^{i} \lambda'_{j-1,k} && i = 1, 2, \cdots, n \\
& && j = 1, 2, \cdots, N \\
(6) \qquad & \sum_{r=1}^{s} \lambda_{jr} \leqq h_{js} && s = 1, 2, \cdots, n \\
& 0 \leqq h_{ji} \leqq 1
\end{aligned}
$$

and h_{ji} shall be an integer.

We have thus transformed the first part of the problem into the form: maximize a linear function of the λ_{jr} subject to the linear conditions given by (4) through (6), plus the requirements that the h_{ji}'s shall be integers. General methods for such mixed-integer problems have been provided by E. M. L. Beale [1] and R. Gomory [3].

The solution to the first part thus leaves us with a solution to a problem which is less restricted than the one originally stated. It should also be obvious that more general piecewise linear functionals can be comprehended via this mode of attack (with at worst mixed integer requirements) and that more complicated linear stochastic constraints may be handled where suitable variable transformations—e.g., analogous to those from the R_j, I_j to the A_j—permit a translation into convolutions (hence linear inequality conditions) for the unknown relative frequencies.

Knowing now the solution to the first part, we next seek to approximate as closely as possible the distributions for the A_j by means of functions

$$
(7) \qquad A_j = A_j(S_1, \cdots, S_{j-1})
$$

where the functions are from some specified admissible class; the class of possible stochastic decision rules any one of which will prescribe the value of A_j given $S_1, \cdots, S_{j-1}$.[4] For example, the functions

$$
(8) \qquad A_j = \alpha_{j0} + \sum_{r=1}^{j-1} \alpha_{jr} S_r
$$

comprise the class of linear decision rules.

For this class—i.e., (8)—the problem of approximation is probably best carried out in terms of characteristic functions. E.g., the characteristic function corresponding to the decision rule for A_j—i.e., $D(A_j)$—is given by [4]

$$
(9) \qquad \phi_{D(A_j)}(t) = \prod_{r=0}^{j-1} \phi_{S_r}(\alpha_{jr} t),
$$

[4] This class may possibly be extended to include still other variables with known distributions that might improve the fit.

a product of the known characteristic functions of the S_r involving the unknown α_{jr} in their argument. By choice of the α_{jr} we seek to approximate $\phi_{A_j}(t)$, the characteristic function for the distribution obtained as a solution to the first problem, in such a manner that the distribution (or density) function corresponding to $\phi_{D(A_j)}(t)$ has (as closely as possible) the desired characteristics of the distribution of A_j.

Evidently there are many ways of specifying the latter problem. For example, one may approximate some subset of the cumulants, or semi-invariants; this is equivalent to approximating the mean and (or) other selected moments (and associated characteristics) of the distribution of A_j as closely as possible. Another possibility is to minimize

$$\int_{-\infty}^{\infty} \left[\phi_{A_j}(t) - \prod_{r=0}^{j-1} \phi_{S_r}(\alpha_{jr}\, t)\right]^2 dt.$$

By Parseval's Theorem this integral is equal to

$$2\pi \int_{-\infty}^{\infty} [p_{A_j}(x) - p_{D(A_j)}(x)]^2\, dx$$

where $p_{A_j}(x)$ is the density function corresponding to $\phi_{A_j}(t)$ and $p_{D(A_j)}(x)$ is the density function corresponding to the decision rule. Still another possibility would be to weight the dispersion of $p_{D(A_j)}(x)$ from $p_{A_j}(x)$ by the relative frequencies of $p_{A_j}(x)$—e.g., to minimize

$$2\pi \int_{-\infty}^{\infty} p_{A_j}^2(x)[p_{A_j}(x) - p_{D(A_j)}(x)]^2\, dx.$$

By Parseval's Theorem, and the fact that the Fourier transform of the product of two functions is the convolution of their individual transforms, this is equivalent to

$$\text{minimizing} \quad \int_{-\infty}^{\infty} [\{\phi_{A_j}^{*}(\phi_{A_j} - \phi_{D(A_j)})\}^{(t)}]^2\, dt,$$

where the * denotes the convolution operation. All three of these possibilities are classical non-linear minimization problems since the α_{jr} are completely unrestricted.

As should now be clear, the problem of stochastic (chance-constrained) programming involves difficulties of an order incommensurate to that of "certainty" programming. These difficulties stem fundamentally from the probabilistic constraints, which experience (let alone theory) has made clear, are *not* adequately represented as some have done by applying the expectation operator to the stochastic form. It is hoped that the conceptual framework and approximation ideas above will stimulate additional research on models and methods of this character which are essential to insight into and progress on management problems of a temporal nature involving conditional decisions.

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IX-31

ON THE OPTIMAL INVENTORY EQUATION

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1. Summary

The purpose of this paper is to discuss a number of functional equations which arise in the "optimal inventory" problem. This is a particular case of the general problem of ordering in the face of an uncertain future demand.
Actually, an important aspect of the problem is that of determining a suitable criterion of cost, one which is both realistic and analytically malleable.

In the following sections we shall consider various sets of assumptions which yield various functional equations, all belonging to a common family.

2.1 Finite Total Time Period

The first process we shall consider is one involving the stocking of only one item, where we may order at each of a finite number of equally-spaced times and we must fulfill the demand at these same times or pay a penalty. We shall further assume that there is no delay in filling an order or a demand.

It is important to emphasize that we have made the, in many cases, unrealistic assumption that the distribution of demand is the same at each stage of the process. Fortunately, as is easily verified following the argumentation below, although this changes the parameters describing the optimal policy, it does not affect the basic structure of the optimal policy, namely constant stock level.

Let us suppose that we know completely the following functions. Again in realistic situations, the determination of these functions may constitute a major difficulty.

For the case where the cost functions are taken proportional to the quantity ordered we obtain complete solutions for the case of an arbitrary number of items and arbitrary distribution of demand.

2. Introduction

In this paper we wish to consider a number of interesting analytic problems arising in the study of inventory and stock control. The origin of this work is a paper by Arrow, Harris, and Marschak, [1], which contains the first mathematical formulation of problems of this genre. Following this are the detailed papers of Dvoretzky, Kiefer, and Wolfowitz, [4], [5], containing existence and uniqueness theorems for the class of functional equations which arise, and a discussion of statistical estimation problems connected with the determination of demand functions. Up to the present, relatively few complete solutions of general classes of these problems have been given and little seems to have been published on the more realistic problems involving stockpiles of many different items with correlated demand functions.

In the pages that follow we shall obtain complete solutions, in the sense that the structure of the optimal policy will be completely determined, for some significant cases where simple, but quite realistic and useful, assumptions are made.

The usefulness of these explicit solutions is great. Apart from the fact that they may be used to obtain approximate solutions to problems of more intricate type these exact solutions frequently lay bare the combinations of essential parameters which are most meaningful.

Quite often, the analytic representation of the solution possesses a quite elegant and simple economic interpretation which, when verbalized, permits one to obtain a good approximation to the optimal policy in more complicated processes, cf. (2), for further discussion of this point.

It is, furthermore, often true that the determination of an optimal policy may depend upon far less than would seem to be the case at first sight. Thus, in one class of problems in which we wish to minimize expected cost, it turns out that we need only know expected outcomes, cf. (2). Another example of this phenomenon occurs below where only the cumulative distribution of demand plays a role in determining optimal policy.

Apart from the results we obtain, the methods we employ possess an independent interest. They have already been employed in connection with other classes of functional equations in the theory of dynamic programming, cf. (2), and appear quite useful in applications in other fields such as the calculus of variations. What stands out quite vividly is that the method of successive approximations is not only useful in the production of existence and uniqueness theorems, to which dull task it is usually relegated, but is a powerful analytic tool for the discovery and proof of properties of the solution of a functional equation.

The paper is divided into three parts. The first part contains a discussion of the characteristic features of the problem we shall discuss and the assumptions we shall make, together with a derivation of the functional equations which arise from various combinations of features and assumptions.

In §3 we consider the problem of existence and uniqueness of solutions of these equations, and the convergence of successive approximations. In particular, we show that we can always obtain monotone convergence by choosing an initial approximation in "policy space", cf. (2).

Although existence and uniqueness have been treated by DKW, (4), we feel that it is worthwhile to present another proof here since the convergence of the successive approximations and the uniqueness of the solution play paramount roles in our further discussion. Our proof is distinct from theirs.

In §4 we present the simple observation which guides all our subsequent analysis.

Part II is devoted to a discussion of a number of models in which the optimal policy is characterized by the principle of constant stock level. In particular, this is the case, in the multi-dimensional as well as the one-dimensional case, if all ordering costs are directly proportional to the amounts ordered. We also consider a number of cases where the penalty cost for ordering to meet an ex-

cess of demand over supply contains a fixed administrative or "red tape" cost. This is the usual model which has been treated. Here the results are less complete, due to the fact that the optimal policies seem to have much more complicated structures.

If we introduce "red-tape" or "set-up" costs which are independent of the quantity ordered, the problems that arise become much more formidable and escape our methods. Furthermore, it can be shown, by means of examples, that the structure of the optimal policies changes radically.

Part III considers two processes with more complicated optimal policies. One arises from the consideration of a convex cost function for initial ordering, and the other from a time-lag between order and delivery. The solution to both problems is obtained by means of the use of successive approximations.

Part I—Mathematical Formulation

3. Formulation of the General Problem

The problem we shall discuss in various related forms is a particular case of the general problem of decision-making in the face of an uncertain future. The version we shall consider is concerned with the problem of stocking a supply of items to meet an uncertain demand.

The situation is as follows: at various specified times we have an opportunity to order supplies of a certain set of items, where the cost of ordering depends upon the number ordered of each item, and where there may or may not be some fixed administrative costs which are independent of the number ordered. At various other times, demands are made upon the stocks of these items. The interesting case is where these demands are not known in advance, but where we do know the joint distribution of demands. The incentive for ordering lies in a penalty which is assessed whenever the demand of an item exceeds the supply. Different penalties are levied in different fields of activity which means that a number of different models must be considered. An important case is where the penalty is directly proportional to the excess of demand over supply.

Speaking loosely, we wish to determine the ordering policy at each stage which will minimize some average function of the total cost of carrying on the activity.

(1)
(a) $\phi(s)\,ds$ = the probability that the demand will lie between s and $s + ds$.[1]
(b) $k(z)$ = the cost of ordering z items initially to increase the stock level.
(c) $p(z)$ = the cost of ordering z items to meet an excess, z, of demand over supply, the *penalty cost*.

Let x denote the stock level at the initiation of the process. Assuming that there are n stages we will order a quantity y_1 at the first, where y_1 depends upon

[1] We shall avoid Stieltjes integrals throughout to simplify the discussion. It will readily be seen that all our results carry over to the more general situation with suitable attention to possible non-uniqueness of roots of certain equations we shall derive.

x, y_2 at the second stage, where y_2 depends upon the new stock level, and so on. A set $(Y_1, Y_2, \cdots, Y_n)$ of functions, determining for each $k \leqq n$ the amount $Y_k = Y_k(x_k)$ to be ordered at the k^{th} stage as a function of the stock level x_k will be called a policy. Corresponding to each policy, there will be a certain expected total cost for this n-stage process, consisting of initial ordering and penalty costs.

The problem we set ourselves is that of determining the policy, or policies, which minimize the expected total cost. A policy which yields this minimum expected cost is called *optimal.*[2]

At any stage, the problem is characterized completely by two state variables, x, the supply of stock, and n, the number of remaining stages. Let us then define

(2) $f_n(x)$ = expected total cost for an n-stage process starting with an initial supply x and using an optimal ordering policy.

Let us now proceed to obtain a functional equation for $f_n(x)$. We have, for the one-stage process, a cost equal to

$$k(y - x) + \int_y^\infty p(s - y)\phi(s)\,ds, \tag{3}$$

if a quantity $y - x \geqq 0$ is ordered.

Since y is to be chosen to minimize the expected cost, we see that $f_1(x)$ is given by

$$f_1(x) = \operatorname*{Min}_{y \geqq x} \left[k(y - x) + \int_y^\infty p(s - y)\phi(s)\,ds \right]. \tag{4}$$

In general, for $n \geqq 2$ we nave

$$\begin{aligned} f_n(x) = \operatorname*{Min}_{y \geqq x} \Bigg[k(y - x) &+ \int_y^\infty p(s - y)\phi(s)\,ds + f_{n-1}(0) \int_y^\infty \phi(s)\,ds \\ &+ \int_0^y f_{n-1}(y - s)\phi(s)\,ds \Bigg], \end{aligned} \tag{5}$$

upon enumerating the possibilities, cf. AHM, [1].

2.2 Unbounded Time Period—Discounted Cost

If we wish to consider an unbounded period of time over which this process operates, we must introduce some device to prevent infinite costs from entering.

The most natural such device is that of discounting the future costs, using a fixed discount ratio, a, for each period. This possesses a certain amount of economic justification and a great deal of mathematical virtue, particularly in its invariant aspect.

If we set

(6) $f(x)$ = expected total discounted cost starting with an initial supply x and using an optimal policy,

[2] Another criterion, of probably greater importance, which we shall not discuss here, is that of minimizing the probability that the cost exceeds a fixed level.

we obtain, by the same enumeration of possibilities, in place of (5) the functional equation

$$f(x) = \operatorname*{Min}_{y \geqq x} \left[k(y - x) + a \int_y^\infty p(s - y)\phi(s)\, ds + af(0) \int_y^\infty \phi(s)\, ds + a \int_0^y f(y - s)\phi(s)\, ds \right]. \tag{7}$$

The advantage of (7) over (5) lies in the fact that it contains $f(x)$, one function of one variable, in place of a sequence of functions, $\{f_n(x)\}$.

2.3 Unbounded Time Period—Partially Expendable Items

If we assume that some of the items supplied upon demand may be partially recovered, so that a demand of s items results in a return of bs items, $0 \leqq b \leqq 1$, which may be used again, the analogue of (7) is

$$f(x) = \operatorname*{Min}_{y \geqq x} \left[k(y - x) + a \int_y^\infty p(s - y)\phi(s)\, ds + a \int_y^\infty f(bs)\phi(s)\, ds + a \int_0^y f(y - s + bs)\phi(s)\, ds \right]. \tag{8}$$

2.4 Unbounded Time Period—One Period Lag in Supply

Let us now assume that when we order a quantity z it does not become available until one period later. If the current supply is x and y was on order from the period before, $x + y$ will be available to meet the next demand. The functional equation corresponding to (7) is now of more complicated form

$$f(x) = \operatorname*{Min}_{z \geqq 0} \left[kz + a \int_x^\infty p(s - x)\phi(s)\, ds + af(z) \int_x^\infty \phi(s) ds + a \int_0^x f(x - s + z)\phi(s) ds \right]. \tag{9}$$

2.5 Unbounded Time Period—Two Period Lag

If we have a two period lag, we have two-stage variables which describe the state of the process,

(1) x = quantity of stock available to meet the next demand
y = quantity to be delivered one period after that.

Hence we define

(2) $f(x, y)$ = expected total cost with x and y as above using an optimal policy.

Then $f(x, y)$ satisfies the equation

$$f(x, y) = \operatorname*{Min}_{z \geqq 0} \left[kz + a \int_x^\infty p(s - z)\phi(s)\, ds + af(y, z) \int_x^\infty \phi(s)\, ds + a \int_0^x f(x - s + y, z)\phi(s)\, ds \right]. \tag{3}$$

We shall not consider this equation here, although it is amenable to the same techniques we apply to the others.

3. Existence and Uniqueness Theorems

There is a uniform technique, the method of successive approximations, first exploited by Picard, for obtaining results concerning the existence and uniqueness of solutions of functional equations. It is particularly important for our discussion since we shall consistently determine properties of the solution by demonstrating that all the approximations possess these properties.

Let us illustrate how one could obtain existence and uniqueness for the general class of equations of which the above are particular examples by considering equation (7) in section 2.2. A more extended treatment of this general class of equations may be found in (3).

To simplify the notation, let us set

$$T(y, x, f) = k(y - x) + a\int_y^\infty p(s - y)\phi(s)\,ds + af(0)\int_y^\infty \phi(s)\,ds + a\int_0^y f(y - s)\phi(s)\,ds. \tag{1}$$

Then equation (2.7) has the form

$$f(x) = \min_{y \geqq x} T(y, x, f). \tag{2}$$

Let us impose the following conditions

(3)

(a) $\phi(s) \geqq 0, \int_0^\infty \phi(s)\,ds = 1,$

(b) *$p(s)$ is continuous, monotone increasing, and* $\int_0^\infty p(s)\phi(s)\,ds < \infty,$

(c) *$k(y)$ is continuous for $y \geqq 0$ and* $k(0) = 0,$

(d) $0 < a < 1.$

Under these conditions, we have the result

Theorem 1. *There is a unique bounded solution to* (2). *This solution, $f(x)$, is continuous. Let $f_0(x)$ be any non-negative bounded continuous function defined over $0 \leqq x < \infty$, and define the sequence $\{f_n(x)\}$ as follows for $n = 0, 1, \cdots,$*

$$f_{n+1}(x) = \min_{y \geqq x} [T(y, x, f_n)]. \tag{4}$$

Then

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Proof: Let us begin by showing that the sequence $f_n(x)$ is uniformly bounded.

If $|f_0(x)| \leqq M$ for $x \geqq 0$, we have

$$
(5) \quad \begin{aligned}
|f_1(x)| \leqq T(x, x, f_0) &\leqq a \int_0^\infty p(s)\phi(s)\,ds + \\
&\qquad aM\left[\int_y^\infty \phi(s)\,ds + \int_0^y \phi(s)\,ds\right] \\
&\leqq a \int_0^\infty p(s)\phi(s)\,ds + aM.
\end{aligned}
$$

From this we see by means of an inductive argument that

$$
(6) \quad |f_n(x)| \leqq \left(a \int_0^\infty p(s)\phi(s)\,ds + M\right) \Big/ (1 - a).
$$

Now to establish convergence. For each n, we see, as a consequence of our assumptions that $f_n(x)$ and $T(y, x, f_n)$ are continuous functions of x and y for $y \geqq x$. Let $y_n = y_n(x)$ be a value of y for which $T(y, x, f_n)$ attains its minimum. This value is never infinite, since $T(\infty, x, f_n) = \infty$ for all $x \geqq 0$. For the sake of definiteness, let y_n be the smallest such value.

We have then

$$
(7) \quad \begin{aligned}
f_{n+1} &= T(y_n, x, f_n) \leqq T(y_{n-1}, x, f_n) \\
f_n &= T(y_{n-1}, x, f_{n-1}) \leqq T(y_n, x, f_{n-1})
\end{aligned}
$$

Combining these inequalities we have

$$
(8) \quad \begin{aligned}
f_{n+1} - f_n &\geqq T(y_n, x, f_n) - T(y_n, x, f_{n-1}) \\
&\leqq T(y_{n-1}, x, f_n) - T(y_{n-1}, x, f_{n-1}),
\end{aligned}
$$

whence

$$
(9) \quad \begin{aligned}
|f_{n+1} - f_n| \leqq \operatorname{Max}\Big[& a \int_0^{y_n} |f_n(y_n - s) - f_{n-1}(y_n - s)|\,\phi(s)\,ds \\
& + a\,|f_n(0) - f_{n-1}(0)| \int_{y_n}^\infty \phi(s)\,ds,\ a \int_0^{y_{n-1}} |f_n(y_{n-1} - s) \\
& - f_{n-1}(y_{n-1} - s)|\,\phi(s)\,ds + a\,|f_n(0) - f_{n-1}(0)| \int_{y_{n-1}}^\infty \phi(s)\,ds\Big]
\end{aligned}
$$

Hence

$$
(10) \quad \begin{aligned}
\operatorname*{Max}_{0 \leqq x} |f_{n+1} - f_n| &\leqq a \operatorname*{Max}_{0 \leqq x} |f_n - f_{n-1}| \int_0^\infty \phi(s)\,ds \\
&\leqq a \operatorname*{Max}_{0 \leqq x} |f_n - f_{n-1}|.
\end{aligned}
$$

Consequently, the series $\sum_{n=0}^\infty (f_{n+1}(x) - f_n(x))$ converges uniformly for all

$x \geqq 0$ and $f_n(x)$ converges to $f(x)$, a bounded solution of (2). Since each element of the sequence $\{f_n(x)\}$ is continuous, $f(x)$ is continuous.

To establish uniqueness, let $F(x)$ be another bounded solution of (2) and use the same technique as above (8) for the two equations

$$(11) \qquad \begin{aligned} F(x) &= \operatorname*{Min}_{y \geqq x} T(y, x, F) \\ f_{n+1}(x) &= \operatorname*{Min}_{y \geqq x} T(y, x, f_n). \end{aligned}$$

We readily see that $F(x) - f_n(x) \to 0$ as $n \to \infty$. Hence $F(x) \equiv f(x)$.

Observe that if we take

$$(12) \qquad \begin{aligned} f_1(x) &= \operatorname*{Min}_{y \geqq x} \left[k(y - x) + a \int_y^\infty p(s - y)\phi(s)\, ds \right], \\ f_2(x) &= \operatorname*{Min}_{y \geqq x} T(y, x, f_1), \end{aligned}$$

and so on, we readily obtain $f_2 \geqq f_1$, and thus monotone *increasing* convergence.

On the other hand, if we approximate in "policy space"; e.g., choose $y = x$ continually, we have for $f_1(x)$ the functional equation

$$(13) \qquad f_1(x) = T(x, x, f_1),$$

and then $f_2(x)$ defined by

$$f_2(x) = \operatorname*{Min}_{y \geqq x} T(y, x, f_1)$$

Clearly $f_2(x) \leqq f_1(x)$ for all $x \geqq 0$ and thus we obtain monotone *decreasing* convergence.

4. A Simple Formal Observation

In this section we wish to present the fundamental formal analytic property of functional equations of the form

$$(1) \qquad u(x) = \operatorname*{Min}_{y} v(x, y), \; y \in R(x),$$

upon which all our subsequent work depends.

In general, the variation will be over some region, $R(x)$, dependent upon x. Let us assume that the minimum is attained inside the region, and that v is differentiable. Then at the minimizing value of y we have

$$(2) \qquad 0 = v_y$$

This determines a function $y(x)$, which need not be single-valued; however, let us assume we may select one such value $y(x)$ for each x so that the resulting function is differentiable.

Then, for this function y we have

$$(3) \qquad u(x) = v[x, y(x)]$$

The crucial observation is now that

$$u'(x) = v_x + v_y\, dy/dx = v_x, \tag{4}$$

since $v_y = 0$ by (2).

Similarly, if

$$u(x_1, x_2) = [\operatorname*{Min}_{y_1, y_2} v(x_1, x_2, y_1, y_2)],\ (y_1, y_2) \in R(x_1, x_2), \tag{5}$$

and we assume that the minimum is attained on the inside, we have

$$\begin{aligned} u_{x_1} &= v_{x_1}, \\ u_{x_2} &= v_{x_2} \end{aligned} \tag{6}$$

at the minimizing points.

Let us now apply these remarks to the functional equation of (2.2.7), under the assumption that $k(z) = kz$, a linear function of z and that $p(z) = pz$. We have

$$\begin{aligned} f(x) = \operatorname*{Min}_{y \geqq x} \Bigg[ky - kx + a \int_y^\infty p(s - y)\phi(s)\, ds + af(0) \int_y^\infty \phi(s)\, ds \\ + a \int_0^y f(y - s)\phi(s)\, ds \Bigg]. \end{aligned} \tag{7}$$

If the minimum is attained at a point $y > x$, we have at this point

$$k - ap \int_y^\infty \phi(s)\, ds + a \int_0^y f'(y - s)\phi(s)\, ds = 0, \tag{8}$$

an equation independent of x!

Furthermore, for this value of $y = y(x)$, we have

$$f'(x) = -k. \tag{9}$$

These two results, correctly combined and interpreted, furnish the clues to the solution of all the problems we consider. We shall discuss them in more detail in §6, and in §7 we shall utilize the multi-dimensional analogues.

Part II—Constant Stock Level

5. Preliminaries

In this part of the paper we shall consider several processes characterized by the principle of constant stock level. The common feature of these models is the assumption that the cost of initial ordering is directly proportional to the amount ordered. As we shall point out in the following part, the addition of an administrative cost, in general, changes the nature of the optimal policy completely.

In §6, we obtain the complete solution, for an arbitrary distribution function $\phi(s)$, for the case where the penalty cost is also directly proportional to the number ordered. In §7 we extend this result to the multi-dimensional case, and show

that the solution for the case where there are many items subject to a joint distribution of demand possesses the very important property of sub-optimality.[3]

Turning from the consideration of these processes involving unbounded time intervals, we consider the finite process described by (2.1.5) and show that again the assumption of direct proportionality entails a principle of constant stock level at each stage.

We now enter territory which is much rougher when we consider the case where the penality cost includes a "red-tape" term which is independent of the amount ordered. The form of the solution now seems to depend upon the form of the demand function. Several important classes of distribution functions fall within the categories we can handle precisely.

Finally we indicate the form of the general solution without, however, being able to make any use of it.

6. Proportional Cost—One-Dimensional Case

In this section we present the solution of the case where both cost functions are directly proportional to the amounts ordered.

Theorem 2. *Consider the equation*

$$(1)\qquad \begin{aligned} f(x) = \operatorname*{Min}_{y \geqq x} \Big[k(y - x) &+ a \int_y^\infty p(s - y)\phi(s)\, ds + af(0) \int_y^\infty \phi(s)\, ds \\ &+ a \int_0^y f(y - s)\phi(s)\, ds \Big] \end{aligned}$$

where we impose the conditions

$$(2)\qquad \begin{aligned} &\text{(a) } k \text{ and } p \text{ are positive constants,} \\ &\text{(b) } \phi(s) > 0, \quad \int_0^\infty \phi(s)\, ds = 1, \quad \int_0^\infty s\phi(s)\, ds < \infty, \\ &\text{(c) } 0 < a < 1, \\ &\text{(d) } ap > k. \end{aligned}$$

Let $\bar{x}$ *be the unique root of*

$$(3)\qquad k = ap \int_y^\infty \phi(s)\, ds + ak \int_0^y \phi(s)\, ds.$$

Then the optimal policy has the form[4]

$$(4)\qquad \begin{aligned} &\text{(a) for} \quad 0 \leqq x \leqq \bar{x}, \quad y = \bar{x}, \\ &\text{(b) for} \quad x \geqq \bar{x}, \quad y = x. \end{aligned}$$

In other words, the optimal stock level is $\bar{x}$.

[3] By "sub-optimality" we mean here that the optimal stock level for any item can be assigned independently of the levels assigned to the other items.

[4] It was pointed out by the referee that equation (3) has the following simple interpretation. The run-out probability must be set at the level where the marginal cost for holding inventory will be balanced against the marginal penalty for run-out.

If $ap \leqq k$, the solution is given by $y = x$ for $x \geqq 0$; i.e., *never order*.

In order to understand the genesis of this solution, let us proceed heuristically. If we obtain a plausible solution and then verify directly that it satisfies the equation in (1) above, the uniqueness theorem tells us that it is *the* solution.

As pointed out in §4, if the minimum occurs at $y > x$, the minimizing values of y must be roots of the equation

$$k + a\left[-p\int_y^\infty \phi(s)\,ds + \int_0^y f'(y-s)\phi(s)\,ds\right] = 0, \tag{5}$$

where

$$f'(x) = -k. \tag{6}$$

Now let us pull ourselves up by our bootstraps! If the solution has the conjectured form, the complicated term, $\int_0^y f'(y-s)\phi(s)\,ds$, may be replaced by the simpler term $-k\int_0^y \phi(s)\,ds$ so that the equation in (5) may be replaced by the simpler equation

$$k - ap\int_y^\infty \phi(s)\,ds - ak\int_0^y \phi(s)\,ds = 0, \tag{7}$$

which determines y, without involving $f'(x)$, as yet unknown.

Since $\int_0^\infty \phi(s)\,ds = 1$, this equation reduces to

$$\int_0^y \phi(s)\,ds = (ap-k)/a(p-k), \tag{8}$$

an equation which possesses exactly one root under the assumption that $\phi(s) > 0$.

Having determined $\bar{x}$ as the root of (8), we proceed to determine $f(x)$ as follows.

For $0 \leqq x \leqq \bar{x}$ we have

$$\begin{aligned} f(x) = k(\bar{x}-x) + a\Bigg[\int_{\bar{x}}^\infty p(s-\bar{x})\phi(s)\,ds + f(0)\int_{\bar{x}}^\infty \phi(s)\,ds \\ + \int_0^{\bar{x}} f(\bar{x}-s)\phi(s)\,ds\Bigg], \end{aligned} \tag{9}$$

and $f'(x) = -k$, or,

$$f(x) = f(0) - kx. \tag{10}$$

Substituting (10) in (9), and setting $x = 0$, we obtain the following result for $f(0)$,[5]

$$f(0) = \left[k\bar{x} + pa\int_{\bar{x}}^\infty (s-\bar{x})\phi(s)\,ds - ak\int_0^{\bar{x}} (\bar{x}-s)\phi(s)\,ds\right] \Big/ (1-a) \tag{11}$$

To determine $f(x)$ for $x \geqq \bar{x}$ we employ the equation

[5] Note that the value of $\bar{x}$ given in (8) is the value of $\bar{x}$ which minimizes $f(0)$.

$$(12)\quad f(x) = a\left[\int_x^\infty p(s-x)\phi(s)\,ds + f(0)\int_x^\infty \phi(s)\,ds + \int_0^x f(x-s)\phi(s)\,ds\right],$$

which we write

$$(13)\qquad f(x) = u(x) + a\int_0^x f(x-s)\phi(s)\,ds,$$

where $u(x)$ is a known function of x. This, in turn, we write

$$(14)\quad f(x) = u(x) + a\int_0^{x-\bar{x}} f(x-s)\phi(s)\,ds + a\int_{x-\bar{x}}^x f(x-s)\phi(s)\,ds.$$

In the interval $[x - \bar{x}, x]$, $f(x-s)$ is known, hence we may write, combining the $u(x)$ term and the second integral

$$(15)\qquad f(x) = v(x) + a\int_0^{x-\bar{x}} f(x-s)\phi(s)\,ds, \quad x \geqq \bar{x}$$

If we now set $x - \bar{x} = z$ and $f(\bar{x}+z) = g(z)$, we see that $g(z)$ satisfies the equation

$$(16)\qquad g(z) = v(\bar{x}+z) + a\int_0^z g(z-s)\phi(s)\,ds, \quad z \geqq 0,$$

a simple renewal equation which can be solved by iteration if we wish.

Actually, it is much simpler to differentiate (12) first and then proceed as above. It seems to be a general characteristic of functional equations in the theory of dynamic programming that the derivatives satisfy simpler equations, and are the more basic quantities.

Let us turn now to a proof that the conjectured solution is actually a solution. Call the bounded function obtained above $F(x)$ and the constant in (11), C. Then $F(x)$ is completely determined by the following equations

$$\text{(a)}\quad F(x) = C - kx, \qquad 0 \leqq x \leqq \bar{x}$$

$$(17)\quad \text{(b)}\quad F(x) = a\left[\int_x^\infty p(s-x)\phi(s)\,ds + F(0)\int_x^\infty \phi(s)\,ds + \int_0^x F(x-s)\phi(s)\,ds\right], \quad x \geqq \bar{x},$$

Let us begin by showing that $F(y) + ky$ is non-decreasing for $y \geqq \bar{x}$. We have, using the expression for $F(x)$ given in (17b), for $x > \bar{x}$

$$\begin{aligned} F'(x) &= -ap\int_x^\infty \phi(s)\,ds + a\int_0^x F'(x-s)\phi(s)\,ds \\ (18)\qquad &= -ap\int_x^\infty \phi(s)\,ds + a\int_0^{x-\bar{x}} F'(x-s)\phi(s)\,ds \\ &\qquad + a\int_{x-\bar{x}}^x F'(x-s)\phi(s)\,ds. \end{aligned}$$

In the interval $[x - \bar{x}, x]$, we have $0 \leqq x - s \leqq \bar{x}$, and hence $F'(x - s) = -k$. Thus, for $x \geqq \bar{x}$,

$$F'(x) = -ap \int_x^\infty \phi(s)\, ds - ka \int_{x-\bar{x}}^x \phi(s)\, ds + a \int_0^{x-\bar{x}} F'(x - s)\phi(s)\, ds, \tag{19}$$

or

$$\begin{aligned} F'(x) + k = \left[k - ap \int_x^\infty \phi(s)\, ds - ak \int_0^x \phi(s)\, ds\right] \\ + a \int_0^{x-\bar{x}} [F'(x - s) + k]\, \phi(s)\, ds. \end{aligned} \tag{20}$$

The expression $u(x) = k - ap\int_x^\infty \phi(s)\, ds - ak\int_0^x \phi(s)\, ds$ is zero at $x = \bar{x}$ and increasing thereafter. Setting $x - \bar{x} = z$ and $F'(x + z) + k = g(z)$, we see that $g(z)$ satisfies the equation

$$g(z) = u(\bar{x} + z) + \int_0^z g(z - s)\phi(s)\, ds, \quad z \geqq 0. \tag{21}$$

Hence $g(z)$ is positive for $z > 0$, as we see from the Neumann solution.

Let us now show that $F(x)$ satisfies the equation

$$\begin{aligned} F(x) = \operatorname*{Min}_{y \geqq x} \Big[k(y - x) + a\Big[\int_y^\infty p(s - y)\phi(s)\, ds + F(0) \int_y^\infty \phi(s)\, ds \\ + \int_0^y F(y - s)\phi(s) ds\Big], \end{aligned} \tag{22}$$

or

$$\begin{aligned} F(x) + kx = \operatorname*{Min}_{y \geqq x} \Big\{ky + a\Big[\int_y^\infty p(s - y)\phi(s)\, ds \\ + F(0) \int_y^\infty \phi(s)\, ds + \int_0^y F(y - s)\phi(s)\, ds\Big]\Big\}. \end{aligned} \tag{23}$$

Now for $y \geqq \bar{x}$, $\{\cdots\} = ky + F(y)$ by (17(b)), and since this function is non-decreasing (23) clearly holds for $x \geqq \bar{x}$. On the other hand for $y \leqq \bar{x}$,

$$\begin{aligned} \{\cdots\} = ky + a\Big[\int_y^\infty p(s - y)\phi(s)\, ds + F(0) \int_y^\infty \phi(s)\, ds \\ + \int_0^y (F(0) - k(y - s))\phi(s)\, ds\Big] = ky + a\Big[\int_y^\infty p(s - y)\phi(s)\, ds \\ + F(0) - k \int_0^y (y - s)\phi(s)\, ds\Big] \end{aligned} \tag{24}$$

and thus has the derivative

$$\{\cdots\}' = k + a\left[-\int_y^\infty p\phi(s)\, ds - k \int_0^y \phi(s)\, ds\right]; \tag{25}$$

moreover, by (3), this derivative vanishes at $y = \bar{x}$. Since $\{\cdots\}'' = (ap - k)\phi(y)$ almost everywhere, and this quantity is non-negative, $\{\cdots\}' \leqq 0$ for $y \leqq \bar{x}$ or $\{\cdots\}$ is non-increasing on $[0, \bar{x}]$. Since $\{\cdots\}$ is non-decreasing for $y \geqq \bar{x}$, the minimum in (23) is assumed at $y = \bar{x}$ for $x < \bar{x}$. (17(a)) and (17(b)) yield the same value of $F(\bar{x})$ so (23) holds for $x < \bar{x}$ if and only if

$$F(x) + kx = k\bar{x} + F(\bar{x}), \tag{26}$$

which clearly follows from (17(a)).

In the case $ap \leqq k$, taking $\bar{x} = 0$ in (17) yields an F which is easily seen to satisfy (23), since, as above, $F(y) + ky$ is non-decreasing.

This completes the proof. It is interesting to note that the solution for $0 \leqq x \leqq \bar{x}$, the most important part of the solution, can be found without reference to the form of the solution for $x > \bar{x}$.

7. Proportional Cost—Multi-Dimensional Case

Let us now consider the multi-dimensional version of the problem. Here we have N items whose stock levels will be denoted by $x_1, x_2, \cdots, x_n$, and whose demand $(s_1, s_2, \cdots, s_n)$ at any time is subject to a distribution function

$$\phi(s_1, s_2, \cdots, s_n).$$

In formulating the functional equation for the function $f(s_1, x_2, \cdots, x_n)$, the minimum expected over-all discounted cost, let us, for the sake of simplicity, consider only the two-dimensional case.

The remarkable fact that emerges is that the form of the solution is precisely the same as if $\phi(s_1, s_2, \cdots, s_n)$ had the form $\phi_1(s_1)\phi_2(s_2) \cdots \phi_n(s_n)$; i.e., uncorrelated demands. It is this which yields the important sub-optimalization of the solution which we discuss below.

An enumeration of cases yields the following functional equation for $f(x_1, x_2)$:

$$\begin{aligned} f(x_1, x_2) = \underset{y_i \geqq x_i}{\text{Min}}\ &[k_1(y_1 - x_1) + k_2(y_2 - x_2) \\ &+ a\Bigg[\int_{y_1}^{\infty}\int_{y_2}^{\infty} [p_1(s_1 - y_1) + p_2(s_2 - y_2)]\phi(s_1, s_2)\, ds_1\, ds_2 \\ &+ f(0, 0)\int_{y_1}^{\infty}\int_{y_2}^{\infty} \phi(s_1, s_2)\, ds_1\, ds_2 \\ &+ \int_{y_1}^{\infty}\int_{0}^{y_2} [p_1(s_1 - y_1) + f(0, y_2 - s_2)]\phi(s_1, s_2)\, ds_1\, ds_2 \\ &+ \int_{0}^{y_1}\int_{y_2}^{\infty} [f(y_1 - s_1, 0) + p_2(s_2 - y_2)]\phi(s_1, s_2)\, ds_1\, ds_2 \\ &+ \int_{0}^{y_1}\int_{0}^{y_2} f(y_1 - s_1, y_2 - s_2)\phi(s_1, s_2)\, ds_1\, ds_2\Bigg]\Bigg] \end{aligned} \tag{1}$$

Let us simplify our notation a bit by setting $\phi(s_1, s_2)\, ds_1\, ds_2 = dG(s_1, s_2)$ and call the quantity within the brackets $K(y_1, y_2)$. We then have

$$
(2)\quad
\begin{aligned}
\frac{\partial K}{\partial y_1} = k_1 + a\Bigg[&-p_1 \int_{y_1}^{\infty}\left(\int_{s_2=0}^{\infty} dG(s_1, s_2)\right) \\
&+ \int_0^{y_1} \frac{\partial f}{\partial y_1}(y_1 - s_1, 0)\left(\int_{s_2=y_2}^{\infty} dG(s_1, s_2)\right) \\
&+ \int_0^{y_1}\int_0^{y_2} \frac{\partial f}{\partial y_1}(y_1 - s_1, y_2 - s_2)\, dG(s_1, s_2)\Bigg], \\
\frac{\partial K}{\partial y_2} = k_2 + a\Bigg[&-p_2 \int_{y_2}^{\infty}\left(\int_{s_1=0}^{\infty} dG(s_1, s_2)\right) \\
&+ \int_0^{y_2} \frac{\partial f}{\partial y_2}(0, y_2 - s_2)\left(\int_{s_1=y_1}^{\infty} dG(s_1, s_2)\right) \\
&+ \int_0^{y_1}\int_0^{y_2} \frac{\partial f}{\partial y_2}(y_1 - s_1, y_2 - s_2)\, dG(s_1, s_2)\Bigg].
\end{aligned}
$$

Furthermore, as above, if $y_1 > x_1$, $y_2 > x_2$, we can expect that

$$
(3)\qquad \frac{\partial f}{\partial x_1} = -k_1, \quad \frac{\partial f}{\partial x_2} = -k_2 .
$$

Consequently, if we assume that the solution here has the same form as in the one-dimensional case, the critical levels $\bar{x}_1$ and $\bar{x}_2$ are given as roots of the equations

$$
(4)\quad
\begin{aligned}
&\text{(a)}\ k_1 + a\left[-p_1 \int_{\bar{x}_1}^{\infty}\left(\int_{s_2=0}^{\infty} dG(s_1, s_2)\right) - k_1 \int_0^{\bar{x}_1}\left(\int_{s_2=0}^{\infty} dG(s_1, s_2)\right)\right] = 0 \\
&\text{(b)}\ k_2 + a\left[-p_2 \int_{\bar{x}_2}^{\infty}\left(\int_{s_1=0}^{\infty} dG(s_1, s_2)\right) - k_2 \int_0^{\bar{x}_2}\left(\int_{s_1=0}^{\infty} dG(s_1, s_2)\right)\right] = 0 .
\end{aligned}
$$

These roots exist and are unique provided we make the same assumptions as above, namely $ap_1 > k_1$, $ap_2 > k_2$, and $dG > 0$.

We see that $\bar{x}_1$ depends for its determination only upon the unconditional distribution $\int_{s_2=0}^{\infty} dG(s_1, s_2)$, and similarly to determine $\bar{x}_2$ we require only $\int_{s_1=0}^{\infty} dG(s_1, s_2)$.

This is the important property of suboptimalization mentioned above.

The verification of the solution follows precisely the same lines as that for the one-dimensional case, and hence will be omitted, since the details are, of course, much more tedious.

Let us state our conclusion as

Theorem 3. *Let us impose the following conditions upon the equation* (1):

(a) k_i *and* p_i *are positive constants*

(5) (b) $\phi > 0, \quad \int_0^{\infty}\int_0^{\infty} \phi\, ds_1 ds_2 = 1, \quad \int_0^{\infty}\int_0^{\infty} s_i \phi\, ds_1 ds_2 < \infty$

(c) $0 < a < 1$,

(d) $ap_i > k_i$.

Let $\bar{x}_i$ be the unique root of

$$(6) \qquad k_i = ap_i \int_y^\infty \left(\int_{s_2=0}^\infty \phi(s_1, s_2)\, ds_2 \right) ds_1 + ak_i \int_0^y \left(\int_{s_2=0}^\infty \phi(s_1, s_2)\, ds_2 \right) ds_1 .$$

Then the optimal policy has the form

$$(7) \quad \begin{array}{lll} \text{(a) } \textit{for} & 0 \leqq x_i , & y_i = \bar{x}_i \\ \text{(b) } \textit{for} & x_i \geqq \bar{x}_i , & y_i = x_i \end{array}$$

In other words, the optimal stock level for the i^{th} item is $\bar{x}_i$.

If $ap_i \leqq k_i$ for any i, we set $\bar{x}_i = 0$.

It is clear that this form of the solutions extends immediately to the N-dimensional case.

8. Finite Time Period

Let us now consider the corresponding problem for a finite process where we do not discount future costs. We now wish to minimize the total expected cost over a finite time period.

We define

(1) $f_N(x)$ = expected cost over an N-stage period starting with an initial quantity x and using an optimal N-stage policy.

Then

$$f_1(x) = \operatorname*{Min}_{y \geqq x} \left[k(y - x) + p \int_y^\infty (s - y)\phi(s)\, ds \right]$$

$$(2) \quad f_{n+1}(x) = \operatorname*{Min}_{y \geqq x} \left[k(y - x) + p \int_y^\infty (s - y)\phi(s)\, ds + f_n(0) \int_y^\infty \phi(s)\, ds + \int_0^y f_n(y - s)\phi(s)\, ds \right].$$

We wish to prove

Theorem 4. For each n, the optimal policy has the form

$$(3) \quad \begin{array}{lll} \text{(a) } \textit{for} & x \leqq \bar{x}_n , & y = \bar{x}_n , \\ \text{(b) } \textit{for} & x \geqq \bar{x}_n , & y = x, \end{array}$$

where the sequence $\bar{x}_n$ is monotone increasing.

Proof: The proof will be inductive. We have, with $f_1(x)$ defined as in (2), as our critical stock level the solution of

$$(4) \qquad k = p \int_y^\infty \phi(s)\, ds,$$

which if it exists is unique. This value does exist if we assume that $p > k$, as is reasonable to suppose. Call this value $\bar{x}_1$. It is clear then that for $n = 1$, the

optimal policy is $y = \bar{x}_1$ for $x \leqq \bar{x}_1$, $y = x$ for $x > \bar{x}_1$. When $x < \bar{x}_1$, we have $f_1'(x) = -k$, and for $x \geqq \bar{x}_1$, we have

$$(5)\qquad \begin{aligned} f_1(x) &= p\int_x^\infty (s - x)\phi(s)\,ds,\\ f_1'(x) &= -p\int_x^\infty \phi(s)\,ds \geqq -k,\\ f_1''(x) &= p\phi(x) > 0. \end{aligned}$$

Hence $f_1'(x) + k \geqq 0$ for all $x \geqq 0$.

Consider the case $n = 2$. We have

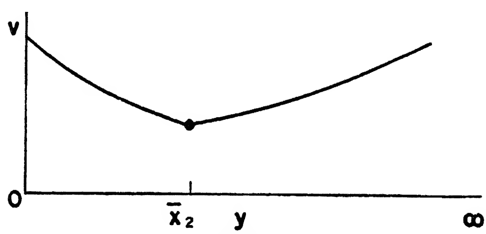
$$(6)\qquad \begin{aligned} f_2(x) = \operatorname*{Min}_{y \geqq x}\Big[k(y - x) + p\int_y^\infty (s - y)\phi(s)\,ds + f_1(0)\int_y^\infty \phi(s)\,ds \\ + \int_0^y f_1(y - s)\phi(s)\,ds\Big]. \end{aligned}$$

The critical value of y is attained by setting the partial derivative with respect to y equal to zero,

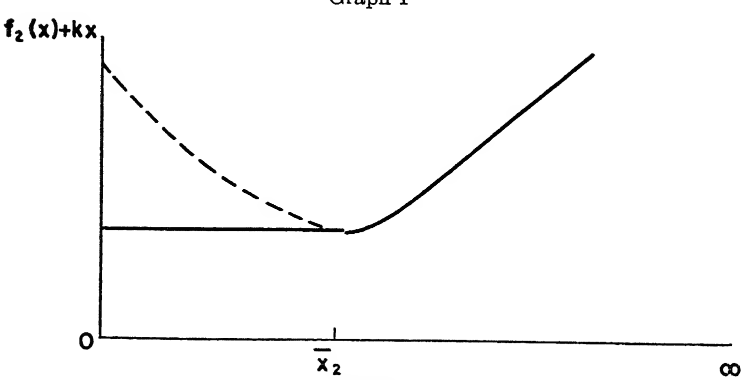
$$(7)\qquad k = p\int_y^\infty \phi(s)\,ds - \int_0^y f_1'(y - s)\phi(s)\,ds = F_1(y).$$

The absolutely continuous function $F_1(y)$ has the derivative

$$(8)\qquad F_1'(y) = -p\phi(y) - f_1'(0)\phi(y) - \int_0^y f_1''(y - s)\phi(s)\,ds.$$



Graph 1



Graph 2

Since $f_1'' > 0$, $p + f_1'(0) > k + f_1'(0) = 0$, we see that $F_1(y)$ is monotone decreasing, and there can be at most one root of (7). However, $F_1(0) = p > k$, $F_1(\infty) = 0$. Hence there is precisely one root. Call this root $\bar{x}_2$.

The policy is then

$$(9) \qquad \begin{aligned} y &= \bar{x}_2, \qquad 0 \leqq x \leqq \bar{x}_2, \\ y &= x, \qquad \bar{x}_2 \leqq x. \end{aligned}$$

The geometric picture is illuminating. Write (6) in the form

$$(10) \qquad f_2(x) + kx = \operatorname*{Min}_{y \geqq x} v(y),$$

where $v(y)$ is a known function. From what we have demonstrated above, $v(y)$ can be shown by graph 1.

The function $f_2(x) + kx$ is obtained by drawing the tangent to $v(y)$ at $y = \bar{x}_2$ and continuing it to the left until it hits the v-axis. The function $f_2(x) + kx$ is now constant for $0 \leqq x \leqq \bar{x}_2$ and equal to $v(x)$ for $x \geqq \bar{x}_2$.

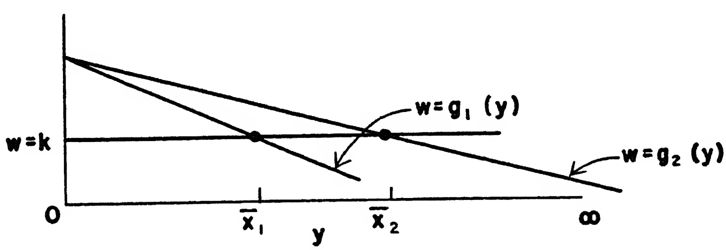
It remains to show that $\bar{x}_2 > \bar{x}_1$. The quantity $\bar{x}_1$ is determined by equation (4), while $\bar{x}_2$ is determined by (7). Since $-f_1' \geqq 0$, it follows that the curve

$$(11) \qquad w = g_2(y) = p \int_y^\infty \phi(s)\, ds - \int_0^y f_1'(y - s)\phi(s)\, ds$$

always lies above the curve

$$(12) \qquad w = g_1(y) = p \int_y^\infty \phi(s)\, ds,$$

for $y > 0$



Graph 3

From this it is clear that $\bar{x}_2 > \bar{x}_1$.

In order to continue this proof inductively, we must show that

$$(13) \qquad -f_2'(x) \geqq -f_1'(x).$$

We have

$$(14) \qquad \begin{aligned} -f_1'(x) &= k, \qquad 0 \leqq x \leqq \bar{x}_1 \\ -f_1'(x) &= p \int_x^\infty \phi(s)\, ds, \qquad x \geqq \bar{x}_1 \end{aligned}$$

and

$$(15)\qquad \begin{aligned} -f_2'(x) &= k, \qquad 0 \leqq x \leqq \bar{x}_2 \\ &= p\int_x^\infty \phi(s)\,ds - \int_0^x f_1'(x-s)\phi(s)\,ds, \qquad x \geqq \bar{x}_2. \end{aligned}$$

In the intervals $[0, \bar{x}_1]$ and $[\bar{x}_2, \infty]$, the inequality is clear. In $[\bar{x}_1, \bar{x}_2]$, the inequality follows from the monotonicity $k - p\int_x^\infty \phi(s)\,ds$, which is zero at $x = \bar{x}_1$.

Finally, we wish to demonstrate the convexity of $f_2(x)$. This is clearly true in $[0, \bar{x}_2]$. In $[\bar{x}_2, \infty]$, we have, using (15),

$$(16)\qquad f_2''(x) = p\phi(x) + f_1'(0)\phi(x) + \int_0^x f_1''(x-s)\phi(s)\,ds$$

Since $f_1'(0) + p > 0, f_1'' \geqq 0$; i.e., we have $f_2''(x) > 0$, and since f_2' is continuous, f_2 is convex. We now have all the ingredients of an inductive proof.

9. Finite Time—Multi-Dimensional Case

The hardy reader may verify that the solution in the multi-dimensional case has precisely the same general character.

10. Non-Proportional Penalty Cost—Red Tape

As soon as we consider the case where the penalty cost is not directly proportional to the excess of demand over supply, we seem to encounter difficulties, and it appears that the simple and elegant solution obtained for the case of proportional cost is no longer valid generally.

There are, however, a number of interesting cases in which we still obtain a solution involving constant stock level. The most interesting of these occur when we take the cost of ordering $(s - y)$ to be $p(s - y) + q$, where q is a fixed administrative cost which appears whenever an excess demand occurs, regardless of the amount of the demand.

In the following part we shall discuss in some detail the case where the initial ordering cost has the same properties.

Let us then consider the equation

$$(1)\qquad \begin{aligned} f(x) = \operatorname*{Min}_{y \geqq x} \Bigg[k(y-x) + a\bigg[\int_y^\infty [p(s-y) + q]\phi(s)\,ds + f(0)\int_y^\infty \phi(s)\,ds \\ + \int_0^y f(y-s)\phi(s)\,ds \bigg]\Bigg], \end{aligned}$$

distinguished from the equation we have considered above by the additional term $aq\int_y^\infty \phi(s)\,ds$. It is surprising how much complication this innocuous appearing expression would seem to introduce.

We shall, to begin with, proceed formally on the assumption that there is a constant stock level solution. The critical level is then determined by the solution of

$$(2)\qquad 0 = k + a\left[-p\int_y^\infty \phi(s)\,ds - q\phi(y) + \int_0^y f'(y-s)\phi(s)\,ds\right],$$

and we have $f'(x) = -k$ when $y > x$.

It follows then that $\bar{x}$ will be a root of

$$(3)\qquad 0 = k + a\left[-p\int_y^\infty \phi(s)\,ds - q\phi(y) - k\int_0^y \phi(s)\,ds\right].$$

Unfortunately, it is not true that this equation has a unique root for all distribution functions $\phi(s)$. This equation may be written in the form

$$(4)\qquad (1-a)k = a(p-k)\int_y^\infty \phi(s)\,ds + aq\phi(y).$$

If $\phi'(y) \leqq 0$, it is true that there is at most one root.

If we assume that this equation has a unique root, the proof is almost exactly as before. There is, however, a more general result where the optimal policy is that of constant stock level which we shall now discuss.

Let us prove

Theorem 5. *Under the above assumptions upon* a, k, p, q *and* $\phi(s)$, *and the additional assumption that the last minimum of*

$$(5)\qquad \psi(y) = ky + a\left[\int_y^\infty [p(s-y)+q]\phi(s)\,ds - k\int_0^y (y-s)\phi(s)\,ds\right]$$

is the absolute minimum in $0 \leqq y \leqq \infty$, *the optimal policy in* (1) *is given by the rule*

$$(7)\ (a)\qquad \begin{aligned} y &= \bar{x}, \quad \text{for} \quad 0 \leqq x \leqq \bar{x}, \\ y &= x, \quad \text{for} \quad x \geqq \bar{x}, \end{aligned}$$

where $\bar{x}$ *is the value of* y *where the absolute minimum is attained.*

Proof: Let $\bar{x}$ be the value of y which yields the last minimum, and the absolute minimum in the interval $[0, \infty]$, of the function $\psi(y)$ above. Then, precisely, as in the case where $q = 0$, we have $f(x) = f(0) - kx$ in $0 \leqq x \leqq \bar{x}$, and $f(0)$ is determined by substituting this result in (1), in the range $0 \leqq x \leqq \bar{x}$. In the interval $[\bar{x}, \infty]$, $f(x)$ is the bracketed term in (1) for $y = x$.

The proof that $f(x)$ actually satisfies the equation now continues in exactly the same way as in the case where $q = 0$.

11. Particular Cases

Some particular cases where the above conditions are satisfied are

$$(1)\qquad \begin{aligned} &(a)\ \phi(x) = e^{-(x-a)^2} \Big/ \int_a^\infty e^{-u^2du} \\ &(b)\ \phi(x) = be^{-bx} \end{aligned}$$

12. The Form of the General Solution

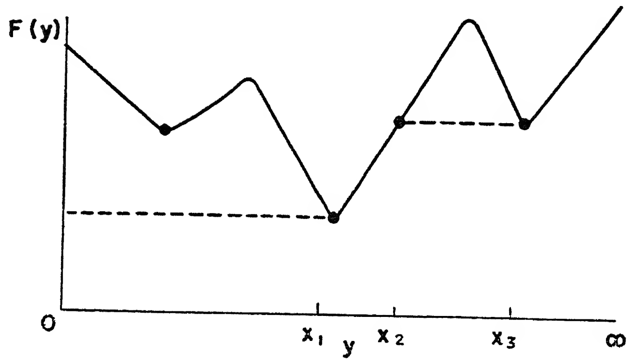
Let $f(x)$ be the solution of (10.1), which is to say

$$f(x) + kx = \operatorname*{Min}_{y \geqq x} F(y), \tag{1}$$

where

$$F(y) = ky + a\left[p \int_y^\infty (s - x)\phi(s)\,ds + (f(0) + q) \int_y^\infty \phi(s)\,ds + \int_0^y f(y - s)\phi(s)\,ds\right]. \tag{2}$$

Let $F(y)$ have graph 4



Graph 4

Then, the optimal policy has the following form

$$(4)\quad \begin{array}{lll} \text{(a)} & y = x_1, & 0 \leqq x \leqq x_1 \\ \text{(b)} & y = x, & x_1 \leqq x \leqq x_2 \\ \text{(c)} & y = x_3, & x_2 < x \leq x_3 \\ \text{(d)} & y = x, & x > x_3. \end{array}$$

Part III—More Complicated Processes

13. Unbounded Process—One Period Time Lag

Let us now state a result for a process of more complicated type. The proof is fairly straightforward, but quite detailed, and depends upon the method of successive approximations.

Theorem. Consider the equation

$$f(x) = \operatorname*{Min}_{z \geqq 0}\left[kz + a\left[\int_x^\infty p(s - x)\phi(s)\,ds + f(z)\int_x^\infty \phi(s)\,ds + \int_0^x f(x - s + z)\phi(s)\,ds\right]\right], \tag{1}$$

under the previous assumptions.

The optimal policy is given by the rule

$$(2)\qquad \begin{aligned} z &= z(x) \text{ for } 0 \leqq x \leqq \bar{x}, \\ z &= 0 \text{ for } x \geqq \bar{x}, \end{aligned}$$

where $z(x) \geqq 0$ *and* $z(\bar{x}) = 0$. *The function* $z(x)$ *is monotone decreasing in* x.

14. Convex Cost Function—Unbounded Process

As another illustration of the type of result which can be obtained, let us consider the case where the cost of ordering is a convex function of the amount ordered. Again applying the method of successive approximations, we can prove

Theorem. Consider the equation

$$(1)\qquad \begin{aligned} f(x) = \operatorname*{Min}_{y \geqq x} \Bigg[g(y - x) + a \Bigg[\int_y^\infty p(s - y)\phi(s)\, ds + f(0) \int_y^\infty \phi(s)\, ds \\ + \int_0^y f(y - s)\phi(s)\, ds \Bigg]\Bigg], \end{aligned}$$

where $g(y)$ *is a convex, monotone increasing function of* y.

There is a function $y(x)$ *and a number* $\bar{x}$ *with the properties*

$$(2)\qquad \begin{aligned} &\text{(a) } y(x) \geqq x,\ y(x) \text{ is monotone decreasing} \\ &\text{(b) } y(x) > x, \text{ for } x \leqq \bar{x},\ y(x) = x,\ x \geqq \bar{x}. \\ &\text{(c) } \bar{x} > 0 \text{ if } ap > g'(0). \end{aligned}$$

This function $y(x)$ *determines the optimal policy for* (1).

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IX-32

DYNAMIC INVENTORY POLICY WITH VARYING STOCHASTIC DEMANDS*†

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A dynamic inventory model is formulated in which the demand distributions may change from period to period. The optimal policy at each stage is characterized by a single critical number which also could vary in successive periods. The dependence of the critical numbers as a function of stochastic ordering amongst distributions is developed under various conditions. Most of the studies are conducted under the assumption of linear purchasing cost. In section 3 the possibility of convex purchasing cost is allowed.

1. Introduction

In this paper we shall consider an extended version of the classical Arrow-Harris-Marschak dynamic inventory model, with emphasis on the varying nature of the demand distributions. For a detailed discussion of this model see Chapters 8, 9, and 10 of [2]. A historical account of the general inventory problem may be found in Chapters 1 and 2 of [2].

Throughout this paper we restrict our attention to the case of a single commodity. A sequence of ordering decisions is to be made at the beginning of a number of periods of equal duration. These decisions result in the building up of inventories. On the other hand, stock is depleted by consumption (demand) in each period. The demand in each period is assumed to be an observation of a random variable with a known distribution function. These random variables are postulated to be independent, but *not necessarily identically distributed.* We also assume that all distribution functions possess continuous densities, and that such distributions as occur belong to non-negative random variables. (The assumption of continuous densities is not an essential restriction, but helps to avoid a tedious consideration of cases. Most of the results developed in this paper remain valid for discrete distributions and to a large extent for the general distribution.)

Several costs are incurred during each period. In general we recognize three types of costs: a purchase or ordering cost $c(z)$, where z is the amount purchased; a holding cost $h(\cdot)$, associated with the cumulative excess of supply over de-

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mand, which is charged at the end of the period; and a shortage or penalty cost $p(\cdot)$, associated with the cumulative excess of demand over supply, which is also charged at the end of the period.[1] We shall also consider a revenue factor, which, of course, should be regarded as a negative cost, and which we shall assume to be linear. Finally, we assume that the cost functions are sufficiently smooth so that all integrals involving these functions exist and that subsequent operations on these integrals are fully justified.

Excess demand in each period is usually handled in one of two ways: either it is considered lost sales, in which case the stock level at the start of the next period has the value zero (the non-backlog model), or it is backlogged and satisfied by subsequent deliveries of stock ordered at the earliest opportunity (the backlog model). We shall deal with both models. For a discussion of the nature of the optimal policy in both cases, see [2], Chapter 10.

Most studies of dynamic inventory models are concerned with determining the characteristics of the optimal policy, i.e., the policy that minimizes the total expected costs, where costs in the future periods are properly discounted. If the cost functions of the model are suitably convex (this includes the case of linear costs and others), and if the demands that arise in successive periods are independent and identically distributed random variables with known distribution functions, then it is clear (see [2], Chapter 9; and [10]) that the optimal policy in each period is characterized by a single critical number, or at most two such numbers, in the following manner: There exist two values s and $S \geqq s$ such that in the event the stock level (including stock on hand and stock ordered) falls below s, the ordering rule calls for replenishing stock to the level S; when the stock level exceeds s, no ordering is done. There is usually a delay (lag) in the delivery of ordered goods.

If the purchase cost is a linear function of the quantity ordered, the optimal policy is characterized by a single critical number: i.e., we have $s = S$ in the policy described above. Moreover, when the demand has the same density $\varphi(\cdot)$ in consecutive periods and delivery is instantaneous, the critical value of the optimal policy can be calculated as the *unique* positive solution of

$$\text{(A)}\quad c + \int_0^x [h'(x-\xi) - \alpha c]\varphi(\xi)\,d\xi - \int_x^\infty [p'(\xi - x) + r]\varphi(\xi)\,d\xi = 0$$

in the non-backlog case, or of

$$\text{(B)}\quad c(1-\alpha) + \int_0^x h'(x-\xi)\varphi(\xi)\,d\xi - \int_x^\infty [p'(\xi - x) + r]\varphi(\xi)\,d\xi = 0$$

in the backlog case. Here, r is the marginal revenue cost and α denotes the effective discount rate. The precise conditions under which these assertions hold are given in [2], Chapter 9. The first result of this kind was obtained by Bellman [3] (see also [4] and [6]). If there are lags in delivery, then there exist corresponding equations from which we may calculate the critical number (see [2], Chapter 10).

The validity of formulas (A) and (B) is based on two factors. The first is

[1] Other ways of charging costs can be dealt with by similar methods.

the assumption that the purchase cost is linear. (In contrast, when the ordering cost is composed of a set-up charge in addition to a linear cost proportional to the quantity of stock ordered, the optimal policy is an (s, S) policy [12] and there is no known way of calculating the critical numbers.) The second is the assumption that demand is stationary over time (identically distributed from period to period). From a practical point of view, it is important to free ourselves from this restriction.

In this paper we shall assume that the demand constitutes a sequence of independent random variables over successive periods which are *not*, in general, identically distributed. We first prove, under the assumption of linear purchase cost (the other cost functions are general convex functions), that the optimal policy again possesses a simple form, i.e., that in each period whether or not to place an order is determined by comparing the stock level with a single critical number. However, this critical number may vary in successive periods, and ordinarily it cannot be explicitly evaluated by solving for the root of a single transcendental equation such as (A), or by any other known means. (In this connection, we note that an explicit algorithm for calculating the critical numbers is available [8] in the special but important case in which the sequence of demand densities varies cyclically.)

The main objective of this paper is to develop qualitative results describing the variation of the critical number over time as a function of the demand densities in all future periods. Since we are primarily concerned with investigating the functional relationship between the optimal policy and the demand distribution, we shall assume, in order to simplify the exposition, that the cost functions are the same in all periods. Moreover, unless there is an explicit statement to the contrary, we shall assume that the purchase cost $c(z) = c \cdot z$ is linear and that $h(\cdot)$ and $p(\cdot)$ are convex, increasing, continuous, and vanishing at the origin. Actually, most of our results remain valid even if the cost functions change in successive time periods, provided we continue to assume the linearity of the purchase cost and the convexity properties of the other cost functions.

Let $\varphi_1, \varphi_2, \varphi_3, \cdots$ represent the demand densities in periods $1, 2, 3, \cdots$, and let $\bar{x}(\varphi_1, \varphi_2, \varphi_3, \cdots)$ denote the optimal critical number in the first period. (That the optimal policy is characterized by a single number was noted above.) From its very definition the optimal critical number in the second period is $\bar{x}(\varphi_2, \varphi_3, \cdots)$, and so forth. In particular, $x(\varphi, \varphi, \varphi, \cdots)$ is the optimal critical number when the demand density φ is the same in each period. This last number can be calculated explicitly from equations (A) and (B).

Ideally, it would be desirable to make comparisons between $\bar{x}(\varphi_1, \varphi_2, \cdots)$ and $\bar{x}(\psi_1, \psi_2, \cdots)$, where $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$ represent two different sequences of demand densities. Usually no such comparison can be made. However, if the respective distributions are *stochastically ordered* (defined below), we can establish certain relationships between the critical numbers.

We say that the density φ is *stochastically smaller* than the density ψ (written $\varphi \subset \psi$) if $\Phi(x) \geqq \Psi(x)$ for all $x \geqq 0$, where

$$\Phi(x) = \int_0^x \varphi(\xi)\, d\xi \quad \text{and} \quad \Psi(x) = \int_0^x \psi(\xi)\, d\xi.$$

In particular, this means that demands based on the density $\varphi(\xi)$ have a larger probability of taking on smaller values than those based on the density $\psi(\xi)$. Not all distributions can be ordered in this manner, but there are important cases in which this ordering relationship applies.

Some specific examples of stochastic ordering are as follows:

(i) One distribution is a translate of the other: i.e., $\varphi \subset \psi$, where $\psi(\xi) = \varphi(\xi - a)$ for $a > 0$.

(ii) If $\Phi(x) = F(x)$ and $\Psi(x) = F^r(x)$, where $r > 1$, then $\varphi \subset \psi$.

(iii) If $\Phi(x)$ is the distribution function of the positive random variable X and $\Psi(x)$ is the distribution function of $X + Y$, where Y is also a positive random variable, then $\varphi \subset \psi$. This example includes convolutions of positive random variables as a special case.

One of the key results of this paper asserts that if $\varphi_i \subset \psi_i$ for all i, then

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\psi_1, \psi_2, \cdots).$$

The result is correct in all situations, regardless of whether the structure of the model permits backlogging and lags in delivery. Thus, we can translate the stochastic order relationships satisfied by the respective demand distributions in successive periods into direct order relationships between the successive critical values of the optimal policy (Corollaries 3 and 4).

A corresponding result is achieved in the case of a convex purchase cost.

In Section 2, we prove that if the purchase cost is linear and the demand distribution varies arbitrarily from period to period, the optimal policy in each period is determined by a single critical number. The method of proof follows closely the method used in [2], Chapter 9 (see also [4]). What is more important here is the proof of the auxiliary statement that the minimum expected discounted cost function is a convex function of stock level. With the characterization of the optimal policy known, we then prove the main theorem pertaining to comparisons of the successive critical numbers, and with the aid of this basic result we develop a series of corollaries that describe $\bar{x}(\varphi_1, \varphi_2, \cdots)$ as a function of the time period.

The corresponding theorems are proved in Section 3 for a convex purchase cost. In Section 4, the theory is developed for a model in which backlogging and lags in delivery are permitted. In Section 5 we discuss the variation of the optimal policy when the demand density has an unknown parameter that must be estimated by statistical methods.

2. Optimal Policy for Linear Purchase Cost

In this section we prove several theorems that relate the critical number of the optimal policy to the nature of future demand distributions. We assume in what follows that the demand distributions are suitably stochastically ordered.

We first characterize the form of the optimal policy under the following conditions:

(i) The purchase cost $c(z)$ is linear $[c(z) = c \cdot z]$.

(ii) The holding and shortage costs $h(\cdot)$ and $p(\cdot)$ are each continuous, convex, increasing functions that vanish at the origin.

(iii) There is a revenue term proportional to the amount sold, with unit revenue factor r.
(iv) There are no time lags in delivery.
(v) Demands in successive periods are described by the sequence of underlying densities $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\varphi_3(\xi)$, $\cdots$, each of which is strictly positive and continuous.
(vi) There is no backlogging of excess demand.

The basic relation analyzed in deriving the optimal ordering rule is the associated functional equation, which expresses the symmetries and renewal properties of the dynamic inventory process. Its explicit form is indicated in equation (2) below. For convenience of notation, we set

$$(1)\quad L(y;\varphi) = \int_0^y [h(y-\xi) - r\xi]\varphi(\xi)\,d\xi + \int_y^\infty [p(\xi - y) - ry]\varphi(\xi)\,d\xi.$$

This represents the expected combined revenue, shortage costs, and holding costs for one period when y units of stock are available. Note that we explicitly exhibit the dependence of L on the demand density φ, since the density will normally vary from period to period.

Let $f(x; \varphi_1, \varphi_2, \cdots)$ denote the discounted expected cost that will be incurred during an infinite sequence of time periods if x is the initial stock level, φ_i is the demand density for period $i (i = 1, 2, \cdots)$, and an optimal ordering rule is used at each purchasing opportunity. If we assume that excess demand cannot be backlogged, we obtain, in the usual fashion,

$$(2)\quad \begin{aligned} f(x;\varphi_1,\varphi_2,\cdots) = \min_{y \geq x}\Big\{ & c(y-x) + L(y;\varphi_1) \\ & + \alpha\Big[f(0;\varphi_2,\varphi_3,\cdots)\int_y^\infty \varphi_1(\xi)\,d\xi + \int_0^y f(y-\xi;\varphi_2,\varphi_3,\cdots)\varphi_1(\xi)\,d\xi\Big]\Big\}, \end{aligned}$$

where α denotes the discount factor and $0 < \alpha < 1$. We shall need to deal with the related function

$$(3)\quad \begin{aligned} G(y,\varphi_1,\varphi_2,\cdots) = {} & cy + L(y;\varphi_1) \\ & + \alpha\Big[f(0;\varphi_2,\varphi_3,\cdots)\int_y^\infty \varphi_1(\xi)\,d\xi + \int_0^y f(y-\xi;\varphi_2,\varphi_3,\cdots)\varphi_1(\xi)\,d\xi\Big]. \end{aligned}$$

In determining the optimal policy, we assume that the following assumptions are satisfied.

Assumption I: $L'(0;\varphi) + c < 0$.

Assumption II: $h'(0) + p'(0) + r - \alpha c > 0$.

Both assumptions will be satisfied if, for example, the marginal revenue is larger than the marginal ordering cost. Assumption I will also be satisfied if the expected marginal shortage cost exceeds the marginal ordering cost. Both assumptions are weak restrictions; they will be satisfied in nearly every sound enterprise. If Assumption II is not satisfied, the optimal policy is never to order (i.e., to fill demands by priority shipments as they arise, thereby suffering a corresponding penalty charge). Under conditions (i)–(vi) and Assumptions I

and II, we shall characterize the optimal ordering rule for the infinite-stage model. The method of proof is a familiar one (see also [4] and [2], Chapter 9), involving induction on the numbers of periods in the model.

We now establish two basic properties of the optimal policy which will be referred to frequently.

Theorem 1. If conditions (i)–(vi) and Assumptions I and II are satisfied, then (a) the optimal ordering rule is characterized by a single critical number $\bar{x}(\varphi_1, \varphi_2, \cdots)$, and (b) $f(x; \varphi_1, \varphi_2, \cdots)$ is a convex function of x. The critical number completely determines the solution in the following manner: if

$$x < \bar{x}(\varphi_1, \varphi_2, \cdots),$$

the optimal policy is to order up to $\bar{x}(\varphi_1, \varphi_2, \cdots)$; if $x > \bar{x}(\varphi_1, \varphi_2, \cdots)$, the optimal policy is not to order.

Proof: The proof proceeds by induction on the number of total periods in the inventory program. Specifically, we shall truncate the model to n periods and subsequently let $n \to \infty$. For the one-period inventory model, we obtain

$$f(x; \varphi_1) = \min_{y \geqq x} \{c(y - x) + L(y; \varphi_1)\} \tag{4}$$

and

$$G(y; \varphi_1) = cy + L(y; \varphi_1). \tag{5}$$

Let $\bar{x}(\varphi_1)$ be defined as the smallest value of y for which

$$G(\bar{x}; \varphi_1) = \min_{y \geqq 0} G(y; \varphi_1).$$

By Assumption I, we have $G'(0; \varphi_1) < 0$, and since $G(y; \varphi_1) \to \infty$ as $y \to \infty$, we infer that $0 < \bar{x}(\varphi_1) < \infty$. Clearly, the value of y that minimizes $G(y; \varphi_1)$ also minimizes $c(y - x) + L(y; \varphi_1)$ with respect to y. If we can show that $G(y; \varphi_1)$ is convex, then $\bar{x}(\varphi_1)$ will be the smallest root of the equation

$$G'(y; \varphi_1) = 0. \tag{6}$$

Differentiating $G(y; \varphi_1)$ twice with respect to y yields

$$G'(y; \varphi_1) = c + \int_0^y h'(y - \xi)\varphi_1(\xi)\, d\xi - \int_0^\infty [p'(\xi - y) + r]\varphi_1(\xi)\, d\xi, \tag{7}$$

$$\begin{aligned} G''(y; \varphi_1) &= \int_0^y h''(y - \xi)\varphi_1(\xi)\, d\xi \\ &\quad + \int_y^\infty p''(\xi - y)\varphi_1(\xi)\, d\xi + [h'(0) + p'(0) + r]\varphi_1(y). \end{aligned} \tag{8}$$

Since $h(\cdot)$ and $p(\cdot)$ are each continuous, convex, increasing functions [condition (ii)] and r is non-negative, obviously $G(y; \varphi_1)$ is convex and $\bar{x}(\varphi_1)$ is the smallest root of (6). In view of (4), it follows that where $x < \bar{x}(\varphi_1)$ the optimal policy calls for ordering to the level $\bar{x}(\varphi_1)$, and where $x > \bar{x}(\varphi_1)$ the optimal policy calls for no ordering. Thus we have

$$f(x;\varphi_1) = \begin{cases} -cx + G[\bar{x}(\varphi_1);\varphi_1] & x < \bar{x}(\varphi_1) \\ -cx + G(x;\varphi_1) & x > \bar{x}(\varphi_1), \end{cases} \tag{9}$$

and from differentiation with respect to x we obtain

$$f'(x;\varphi_1) = \begin{cases} -c & x < \bar{x}(\varphi_1) \\ -c + G'(x;\varphi_1) & x > \bar{x}(\varphi_1). \end{cases} \tag{10}$$

Note that $f'(x;\varphi_1)$ is a continuous function of x, since $G'[\bar{x}(\varphi_1);\varphi_1] = 0$, and that $f'(x;\varphi_1)$ is a non-decreasing function of x, since G is convex. Moreover, the second derivative of $f(x;\varphi_1)$ exists everywhere except possibly at the point $x = \bar{x}(\varphi_1)$, at which, however, left- and right-hand bounded derivatives exist. Thus

$$f''(x;\varphi_1) \geqq 0 \quad \text{except at} \quad x = \bar{x}(\varphi_1), \tag{11}$$

which is enough to show that $f(x;\varphi_1)$ is convex. These considerations prove the theorem for the one-period inventory model.

Assuming now that the theorem has been proved for an $(n-1)$-period model, we shall show that it holds for an n-period model. For the n-period model the discounted expected cost following an optimal policy is

$$\begin{aligned} f(x;\varphi_1,\varphi_2,\cdots,\varphi_n) = \min_{y\geqq x}\Big\{ & c(y-x) + L(y;\varphi_1) \\ & + \alpha\Big[f(0;\varphi_2,\varphi_3,\cdots,\varphi_n)\int_y^\infty \varphi_1(\xi)\,d\xi \\ & + \int_0^y f(y-\xi;\varphi_2,\varphi_3,\cdots,\varphi_n)\varphi_1(\xi)\,d\xi\Big]\Big\}. \end{aligned} \tag{12}$$

Let

$$\begin{aligned} G(y;\varphi_1,\varphi_2,\cdots,\varphi_n) = {} & cy + L(y;\varphi_1) \\ & + \alpha\Big[f(0;\varphi_2,\varphi_3,\cdots,\varphi_n)\int_y^\infty \varphi_1(\xi)\,d\xi \\ & + \int_0^y f(y-\xi;\varphi_2,\varphi_3,\cdots,\varphi_n)\varphi_1(\xi)\,d\xi\Big]. \end{aligned} \tag{13}$$

Differentiating $G(y;\varphi_1,\varphi_2,\cdots,\varphi_n)$ twice with respect to y gives

$$\begin{aligned} & G''(y;\varphi_1,\varphi_2,\cdots,\varphi_n) \\ & \quad = [h'(0) + p'(0) + r + \alpha f'(0;\varphi_2,\varphi_3,\cdots,\varphi_n)]\varphi_1(y) \\ & \qquad + \int_0^y h''(y-\xi)\varphi_1(\xi)\,d\xi + \int_y^\infty p''(\xi-y)\varphi_1(\xi)\,d\xi \\ & \qquad + \alpha\int_0^y f''(y-\xi;\varphi_2,\varphi_3,\cdots,\varphi_n)\varphi_1(\xi)\,d\xi. \end{aligned} \tag{14}$$

It follows from our induction assumption that

$$f'(0; \varphi_2, \varphi_3, \cdots, \varphi_n) = -c. \tag{15}$$

In view of Assumption II, it is now clear that $G(y; \varphi_1, \varphi_2, \cdots, \varphi_n)$ is convex. The argument hereafter is identical to the one used for the one-period model. It thus follows that the theorem holds for the n-period model.

By applying standard limiting arguments (see [7] for the details) we can show that $f(x; \varphi_1, \varphi_2, \cdots, \varphi_n)$ converges to $f(x; \varphi_1, \varphi_2, \cdots)$, and similarly that the critical number $\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$ of the truncated model converges to $\bar{x}(\varphi_1, \varphi_2, \cdots)$, the critical number of the full dynamic model. The proof of the theorem is complete.

If we alter condition (vi) to allow backlogging of excess demand, the associated functional equation takes the form

$$\begin{aligned} &f(x; \varphi_1, \varphi_2, \cdots, \varphi_n) \\ &= \min_{y \geqq x} \left\{ c(y - x) + L(y; \varphi_1) + \alpha \int_0^\infty f(y - \xi; \varphi_2, \cdots, \varphi_n)\varphi_1(\xi)\, d\xi \right\}. \end{aligned} \tag{16}$$

In this case we can prove Theorem 1 without requiring that Assumption II be satisfied. The argument here is identical to that used in the nonbacklog case and will not be repeated.

Assuming now that the successive demands are suitably stochastically ordered (see Section 1), we proceed to derive several qualitative results that describe the variation of the critical number. We assume again that conditions (i)–(vi) and Assumptions I and II are satisfied.

Theorem 2. If we are given two sequences of demand densities $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$, and if $\varphi_i \subset \psi_i$ for $i = 1, 2, \cdots$, then

(a) $\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\psi_1, \psi_2, \cdots)$

and

(b) $f'(x; \varphi_1, \varphi_2, \cdots) \geqq f'(x; \psi_1, \psi_2, \cdots)$ for all x.

Proof: The proof proceeds by induction on the number of periods; we first prove the results corresponding to (a) and (b) for an n period model and then let $n \to \infty$. For the one-period model we consider demand densities φ_1 and ψ_1, where $\varphi_1 \subset \psi_1$. Integrating (7) by parts, we obtain for the first integral

$$\int_0^y h'(y - \xi)\varphi_1(\xi)\, d\xi = h'(0)\Phi_1(y) + \int_0^y h''(y - \xi)\Phi_1(\xi)\, d\xi, \tag{17}$$

where

$$\Phi_1(\xi) = \int_0^\xi \varphi_1(\eta)\, d\eta;$$

and for the second integral

$$\begin{aligned} &-\int_y^\infty [p'(\xi - y) + r]\varphi_1(\xi)\, d\xi \\ &= [p'(0) + r]\Phi_1(y) - p'(0) - r - \int_y^\infty p''(\xi - y)[1 - \Phi_1(\xi)]\, d\xi. \end{aligned} \tag{18}$$

Combining (17) and (18), we may write (7) as

$$
\begin{aligned}
(19)\qquad G'(y; \varphi_1) &= c + [h'(0) + p'(0) + r]\Phi_1(y) - p'(0) - r \\
&\quad + \int_0^y h''(y - \xi)\Phi_1(\xi)\, d\xi - \int_y^\infty p''(\xi - y)[1 - \Phi_1(\xi)]\, d\xi.
\end{aligned}
$$

A similar expression is obtained for $G'(y; \psi_1)$; we simply replace $\Phi_1(\xi)$ in (19) by $\Psi_1(\xi)$, where

$$\Psi_1(\xi) = \int_0^\xi \psi_1(\eta)\, d\eta.$$

Since $\varphi_1 \subset \psi_1$, it follows by definition that $\Phi_1(\xi) \geqq \Psi_1(\xi)$ for all $\xi \geqq 0$; hence

$$(20)\qquad G'(y; \varphi_1) \geqq G'(y; \psi_1) \quad \text{for all} \quad y \geqq 0.$$

But Theorem 1 tells us that $\bar{x}(\varphi_1)$ is the smallest root of the equation

$$G'(y; \varphi_1) = 0.$$

Hence it follows immediately from (20) that

$$(21)\qquad \bar{x}(\varphi_1) \leqq \bar{x}(\psi_1).$$

It is clear from Theorem 1 that the optimal policies for φ_1 and ψ_1 are each characterized by a single critical number. Consequently, if we compare (20) and (21) with (10) and the corresponding equation for ψ_1,

$$(22)\qquad f'(x; \psi_1) = \begin{cases} -c & x < \bar{x}(\psi_1) \\ -c + G'(x; \psi_1) & x > \bar{x}(\psi_1), \end{cases}$$

it is clear that

$$(23)\qquad f'(x; \varphi_1) \geqq f'(x; \psi_1) \quad \text{for all} \quad x \geqq 0.$$

Thus we have proved the theorem for the one-period model.

Assume now that we have proved

$$\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_{n-1}) \leqq \bar{x}(\psi_1, \psi_2, \cdots, \psi_{n-1})$$

and

$$f'(x; \varphi_1, \varphi_2, \cdots, \varphi_{n-1}) \geqq f'(x; \psi_1, \psi_2, \cdots, \psi_{n-1}) \quad \text{for all} \quad x \geqq 0,$$

for any $n - 1$ pairs of demand densities φ_i and ψ_i satisfying $\varphi_i \subset \psi_i$. Differentiating (13) with respect to y yields

$$
\begin{aligned}
(24)\qquad G'(y; \varphi_1, \varphi_2, \cdots, \varphi_n) &= c + L'(y; \varphi_1) \\
&\quad + \alpha \int_0^y f'(y - \xi; \varphi_2, \varphi_3, \cdots, \varphi_n)\varphi_1(\xi)\, d\xi,
\end{aligned}
$$

from which, invoking our induction assumption, we obtain

$$
\begin{aligned}
(25)\qquad G'(y; \varphi_1, \varphi_2, \cdots, \varphi_n) &\geqq c + L'(y; \varphi_1) \\
&\quad + \alpha \int_0^y f'(y - \xi; \psi_2, \psi_3, \cdots, \psi_n)\varphi_1(\xi)\, d\xi.
\end{aligned}
$$

Integrating the last integral in (25) by parts produces

$$\alpha \int_0^y f'(y - \xi; \psi_2, \psi_3, \cdots, \psi_n)\varphi_1(\xi)\, d\xi = \alpha f'(0; \psi_2, \psi_3, \cdots, \psi_n)\Phi_1(y) \\ + \alpha \int_0^y f''(y - \xi; \psi_2, \psi_3, \cdots, \psi_n)\Phi_1(\xi)\, d\xi. \tag{26}$$

Combining (15), (17), (18), (25), and (26), we obtain

$$G'(y; \varphi_1, \varphi_2, \cdots, \varphi_n) \geqq c - p'(0) - r + [h'(0) + p'(0) + r - \alpha c]\Phi_1(y) \\ + \int_0^y h''(y - \xi)\Phi_1(\xi)\, d\xi - \int_y^\infty p''(\xi - y)[1 - \Phi_1(\xi)]\, d\xi \\ + \alpha \int_0^y f''(y - \xi; \psi_2, \psi_3, \cdots, \psi_n)\Phi_1(\xi)\, d\xi. \tag{27}$$

If in the right-hand side of (27) we replace $\Phi_1(\cdot)$ by $\Psi_1(\cdot)$, the inequality will be strengthened. This follows from condition (ii), Assumption II, statement (b) of Theorem 1, and the fact that $\varphi_1 \subset \psi_1$. When we replace $\Phi_1(\xi)$ by $\Psi_1(\xi)$, the right-hand side of (27) becomes identically equal to $G'(y; \psi_1, \psi_2, \cdots, \psi_n)$. Therefore,

$$G'(y; \varphi_1, \varphi_2, \cdots, \varphi_n) \geqq G'(y; \psi_1, \psi_2, \cdots, \psi_n) \quad \text{for all} \quad y \geqq 0. \tag{28}$$

The same argument used for the one-period model now readily yields

$$\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n) \leqq \bar{x}(\psi_1, \psi_2, \cdots, \psi_n) \tag{29}$$

and

$$f'(x; \varphi_1, \varphi_2, \cdots, \varphi_n) \geqq f'(x; \psi_1, \psi_2, \cdots, \psi_n) \quad \text{for all} \quad x \geqq 0. \tag{30}$$

This proves the theorem for the n-period model. The corresponding results for the infinite-stage model may be arrived at by the limiting procedure referred to in the proof of Theorem 1.

If we alter condition (vi) to allow backlogging of excess demand, the conclusions of Theorem 2 (like those of Theorem 1) remain valid regardless of whether or not Assumption II is satisfied.

From Theorem 2 we shall now deduce several corollaries that describe the variation in the critical number when the demand density distribution varies in some definite pattern; for example, Corollary 1 states in effect that the critical number in the first period is a monotone-increasing function of the length of the inventory program. In the corollaries and lemmas to follow, we assume that conditions (i)–(vi) and Assumptions I and II are satisfied.

Corollary 1. If the demand densities for the first $n + 1$ periods are given by $\varphi_1, \varphi_2, \cdots, \varphi_n, \varphi_{n+1}$, then

$$\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n, 0) \leqq \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n, \varphi_{n+1}).$$

The zero following φ_n on the left-hand side of the inequality is to be interpreted as the density of a random variable whose only possible value is 0.

Proof: Since $0 \subset \varphi_{n+1}$ for any φ_{n+1}, the corollary follows immediately from Theorem 2.

We next prove a lemma that we shall require in the corollaries to follow.

Lemma 1. If the demand densities are given by $\varphi_1, \varphi_2, \cdots$, and if

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots),$$

then $\bar{x}(\varphi_1, \varphi_2, \cdots) = \bar{x}(\varphi_1, \varphi_1, \varphi_1, \cdots)$, where $\bar{x}(\varphi_1, \varphi_1, \varphi_1, \cdots)$ can be explicitly computed as the unique positive root of the equation

$$c + \int_0^y [h'(y - \xi) - \alpha c]\varphi_1(\xi)\, d\xi - \int_y^\infty [p'(\xi - y) + r]\varphi_1(\xi)\, d\xi = 0. \tag{31}$$

Proof: We have seen in the proof of Theorem 1 that $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is the smallest root of the equation $G'(y; \varphi_1, \varphi_2, \cdots) = 0$, where

$$G'(y; \varphi_1, \varphi_2, \cdots) = c + L'(y; \varphi_1) + \alpha \int_0^y f'(y - \xi; \varphi_2, \varphi_3, \cdots)\varphi_1(\xi)\, d\xi. \tag{32}$$

Moreover, by Theorem 1,

$$f'(x; \varphi_2, \varphi_3, \cdots) = -c \quad \text{for} \quad x \leqq \bar{x}(\varphi_2, \varphi_3, \cdots). \tag{33}$$

It is clear from our hypothesis that

$$f'(x; \varphi_2, \varphi_3, \cdots) = -c \quad \text{for} \quad x \leqq \bar{x}(\varphi_1, \varphi_2, \cdots). \tag{34}$$

If we set $y = \bar{x}(\varphi_1, \varphi_2, \cdots)$ in (32), it follows that in the integrand $f'(x; \varphi_2, \varphi_3, \cdots)$ is always evaluated at a point $x \leqq \bar{x}(\varphi_1, \varphi_2, \cdots)$. Hence $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is the smallest root of the equation

$$c + L'(y; \varphi_1) - \alpha c \int_0^y \varphi_1(\xi)\, d\xi = 0. \tag{35}$$

However, the smallest root of (35) is also $\bar{x}(\varphi_1, \varphi_1, \cdots)$ by a similar argument involving the range of values of $f'(x; \varphi_1, \varphi_1, \cdots)$ in the equation for

$$G'(y; \varphi_1, \varphi_1, \cdots) = 0.$$

Thus $\bar{x}(\varphi_1, \varphi_2, \cdots) = \bar{x}(\varphi_1, \varphi_1, \cdots)$, and the lemma is proved.

According to the next corollary, if the demand density increases (stochastically) in successive periods, the optimal critical number also increases, and the critical number can be calculated in each period as if the demand density in the future periods were stationary.

Corollary 2. If $\varphi_1 \subset \varphi_2 \subset \varphi_3 \subset \cdots$, then (a) $\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots)$ and (b) $\bar{x}(\varphi_1, \varphi_2, \cdots) = \bar{x}(\varphi_1, \varphi_1, \cdots)$.

Proof: Conclusion (a) follows from Theorem 2 and conclusion (b) from Lemma 1.

Lemma 2. If the demand densities for the future are given by $\varphi_1, \varphi_2, \cdots$, then $\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_1, \varphi_1, \cdots)$ always.

Proof: We have proved before that $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is the smallest root of the

equation $G'(y; \varphi_1, \varphi_2, \cdots) = 0$, where $G'(y; \varphi_1, \varphi_2, \cdots)$ is defined by (32). From Theorem 1 we know that

$$(36) \quad f'(x; \varphi_2, \varphi_3, \cdots) = \begin{cases} -c & x < \bar{x}(\varphi_2, \varphi_3, \cdots) \\ -c + G'(x; \varphi_2, \varphi_3, \cdots) & x > \bar{x}(\varphi_2, \varphi_3, \cdots), \end{cases}$$

and also that $\bar{x}(\varphi_1, \varphi_1, \cdots)$ is the smallest root of (31). Since

$$f'(x; \varphi_2, \varphi_3, \cdots) \geqq -c$$

always for all $x \geqq 0$, it follows that

$$(37) \qquad G'(y; \varphi_1, \varphi_2, \cdots) \geqq G'(y; \varphi_1, \varphi_1, \cdots) \quad \text{for all} \quad y \geqq 0.$$

Hence

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_1, \varphi_1, \cdots).$$

Corollary 3. If the demand densities for the future are given by $\varphi_1, \varphi_2, \cdots$, and if $\varphi_2 \subset \varphi_1$, then

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \geqq \bar{x}(\varphi_2, \varphi_3, \cdots).$$

Proof: Suppose the contrary is true, namely that

$$\bar{x}(\varphi_1, \varphi_2, \cdots) < \bar{x}(\varphi_2, \varphi_3, \cdots).$$

Then by Lemma 1

$$\bar{x}(\varphi_1, \varphi_2, \cdots) = \bar{x}(\varphi_1, \varphi_1, \cdots).$$

By Theorem 2 and the hypothesis,

$$\bar{x}(\varphi_1, \varphi_1, \cdots) \geqq \bar{x}(\varphi_2, \varphi_2, \cdots),$$

and by Lemma 2 we know that

$$\bar{x}(\varphi_2, \varphi_2, \cdots) \geqq \bar{x}(\varphi_2, \varphi_3, \cdots).$$

Hence

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \geqq \bar{x}(\varphi_2, \varphi_3, \cdots),$$

contradicting our assumption.

Corollary 4. If $\bar{x}(\varphi_1, \varphi_2, \cdots) > \bar{x}(\varphi_2, \varphi_3, \cdots)$, where $\varphi_1, \varphi_2, \cdots$, are the demand densities for the future and $\varphi_1 \subset \varphi_2$, then

$$\bar{x}(\varphi_2, \varphi_3, \cdots) > \bar{x}(\varphi_3, \varphi_4, \cdots).$$

Proof: Suppose the contrary is true, namely that

$$\bar{x}(\varphi_2, \varphi_3, \cdots) \leqq \bar{x}(\varphi_3, \varphi_4, \cdots).$$

Then by Lemma 1

$$\bar{x}(\varphi_2, \varphi_3, \cdots) = \bar{x}(\varphi_2, \varphi_2, \cdots),$$

and by Theorem 2

$$\bar{x}(\varphi_1, \varphi_1, \cdots) \leqq \bar{x}(\varphi_2, \varphi_2, \cdots).$$

Applying Lemma 2, we obtain

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_1, \varphi_1, \cdots).$$

Hence

$$\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots),$$

contradicting our hypothesis.

Corollaries 3 and 4 permit a global description of the variation of the critical number in terms of the variation of the demand density. Suppose we plot the stochastic order of the demand densities in successive periods against the periods concerned. This curve is properly a sequence of discrete points, but we shall represent it by a curve (Fig. 1).

If the densities in successive periods take the form shown in Fig. 1, then Corollaries 3 and 4 assert that the critical numbers in the respective periods vary essentially in the manner indicated in Fig. 2.

The following general conclusions are to be drawn from this analysis. In any set of consecutive periods, when the demand density stochastically decreases, the critical number necessarily decreases. On the other hand, when the demand density stochastically increases, the critical number may or may not increase. But if in two successive periods the demand density stochastically increases while the critical number decreases, then the critical number in the following period is still smaller.

The results of Corollaries 1–4 and Lemmas 1 and 2 remain valid when the backlogging of excess demand is permitted. The proofs require only slight modifications.

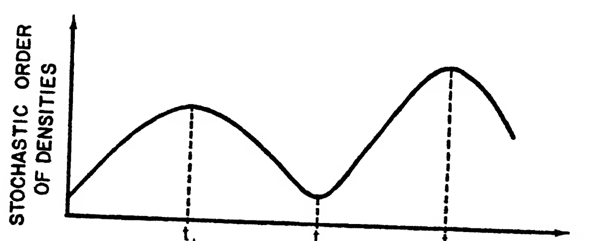

FIG. 1

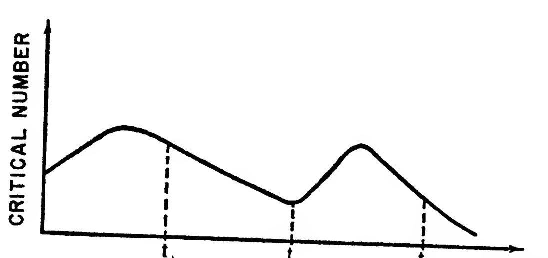

FIG. 2

3. Optimal Policy for Convex Purchase Cost

In this section we characterize the form of the optimal policy when we have a strictly convex, increasing purchase cost and a linear shortage cost. The other assumptions stipulated in conditions (i)–(vi), p. 9, are assumed to be satisfied. We assume in addition that $p + r > c'(0)$, where $c(z)$ is the cost of purchasing an amount z. This assumption is necessary to ensure positive ordering when the stock level is zero; if it is not satisfied, we simply do not order, or—what amounts to the same thing—no inventories are kept on hand. The theorems, corollaries, and lemmas in this section are numbered with primes to emphasize their relation to those of Section 2.

Theorem 1′. A. *Infinite-Stage Model.*

If the demand densities for the future are given by $\varphi_1, \varphi_2, \cdots$, if the purchase cost $c(z)$ is strictly convex, the shortage cost linear with marginal unit penalty cost p, and the holding cost convex, and if $p + r > c'(0)$, then there exists a number $\bar{x}(\varphi_1, \varphi_2, \cdots)[0 < \bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \infty]$ and a function $y(x; \varphi_1, \varphi_2, \cdots)$ such that

(a) $1 > dy/dx > 0$,
(b) $-f'(x; \varphi_1, \varphi_2, \cdots) < p + r$,
(c) $f''(x; \varphi_1, \varphi_2, \cdots) > 0$,
(d) $y(x; \varphi_1, \varphi_2, \cdots) > x$ for $x < \bar{x}(\varphi_1, \varphi_2, \cdots)$.

The number $\bar{x}(\varphi_1, \varphi_2, \cdots)$ and the function $y(x; \varphi_1, \varphi_2, \cdots)$ determine the optimal policy in the following manner: if $x < \bar{x}(\varphi_1, \varphi_2, \cdots)$, the optimal policy is to order up to $y(x; \varphi_1, \varphi_2, \cdots)$; if $x > \bar{x}(\varphi_1, \varphi_2, \cdots)$, the optimal policy is not to order.

B. *n-Stage Model.*

For the truncated dynamic model of n periods $(n \geqq 1)$, $\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$ and $y(x; \varphi_1, \varphi_2, \cdots, \varphi_n)$ express the optimal policy for the first ordering opportunity in the same sense as above, and these quantities satisfy the properties

(a) $0 < \dfrac{dy(x; \varphi_1, \varphi_2, \cdots, \varphi_n)}{dx} < 1$,
(b) $-f'(x; \varphi_1, \varphi_2, \cdots, \varphi_{n-1}) < -f'(x; \varphi_1, \varphi_2, \cdots, \varphi_n)$ for $x < \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$,
(c) $-f'(x; \varphi_1, \varphi_2, \cdots, \varphi_n) < p + r$,
(d) $f''(x; \varphi_1, \varphi_2, \cdots, \varphi_n) > 0$,
(e) $y(x; \varphi_1, \varphi_2, \cdots, \varphi_n) > y(x; \varphi_1, \varphi_2, \cdots, \varphi_{n-1})$ for all $x < \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$,
(f) $\bar{x}(\varphi_1, \varphi_2, \cdots) > \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n) > \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_{n-1})$,

provided $\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$ is finite; if $\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$ is infinite for any n, all $\bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$ are infinite.

Proof: For the case of a stationary demand distribution, this theorem is proved in [2], p. 150. If the demand density varies from period to period, the proof is the same and will not be repeated here. We point out a slight error in the proof given in [2]: property (ii) of part B—property (b) of part B above—is valid only for x in the range $x < \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$, and not for all $x > 0$. This

fact is readily demonstrated by an induction argument. The rest of the proof in [2] is unaltered, since property (b) is required only in carrying out the induction step for $x < \bar{x}(\varphi_1, \varphi_2, \cdots, \varphi_n)$.

Theorem 2 is now stated and proved for the case of strictly convex, increasing purchase cost, convex increasing holding cost, linear shortage cost, and the condition $p + r > c'(0)$.

Theorem 2$'$. If we are given two sequences of demand densities $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$, and if $\varphi_i \subset \psi_i$ for $i = 1, 2, \cdots$, then

$$\text{(38)}\quad \begin{aligned} &\text{(a)}\ y(x;\ \varphi_1,\ \varphi_2, \cdots) \leqq y(x;\ \psi_1,\ \psi_2, \cdots) \quad \text{for all} \quad x \geqq 0, \\ &\text{(b)}\ f'(x;\ \varphi_1,\ \varphi_2, \cdots) \geqq f'(x;\ \psi_1,\ \psi_2, \cdots) \quad \text{for all} \quad x \geqq 0. \end{aligned}$$

Proof: The proof is by induction on the number of periods. For the one-period model consider demands φ_1 and ψ_1, where $\varphi_1 \subset \psi_1$. The expected cost if we follow an optimal policy is

$$\text{(39)}\quad \begin{aligned} f(x;\varphi_1) = \min_{y \geqq x} \Big\{ c(y - x) &+ \int_0^y [h(y - \xi) - r\xi]\varphi_1(\xi)\, d\xi \\ &+ \int_y^\infty [p(\xi - y) - ry]\varphi_1(\xi)\, d\xi \Big\}. \end{aligned}$$

Set

$$\text{(40)}\quad \begin{aligned} G(y, x; \varphi_1) = c(y - x) &+ \int_0^y [h(y - \xi) - r\xi]\varphi_1(\xi)\, d\xi \\ &+ \int_y^\infty [p(\xi - y) - ry]\varphi_1(\xi)\, d\xi, \end{aligned}$$

and define

$$\text{(41)}\quad H(y;\varphi_1) = \int_0^y [h(y - \xi) - r\xi]\varphi_1(\xi)\, d\xi + \int_y^\infty [p(\xi - y) - ry]\varphi_1(\xi)\, d\xi.$$

A similar expression is obtained for the demand ψ_1. $H(y;\varphi_1)$ is precisely $L(y;\varphi_1)$, which is a convex function of y; hence $-H'(y;\varphi_1)$ is a decreasing function of y. By assumption, $c'(y - x)$ is a strictly increasing function of y. Thus

$$\text{(42)}\quad \frac{\partial G(y, x; \varphi_1)}{\partial y} = c'(y - x) + H'(y;\varphi_1) = 0$$

has at most one root, call it $y(x;\varphi_1)$, for $x < \bar{x}(\varphi_1)$, where $\bar{x}(\varphi_1)$ is the root of

$$\text{(43)}\quad c'(0) + H'(y;\varphi_1) = 0,$$

if it exists. In our case we are assured that (43) has one root, since by assumption $p + r > c'(0)$. From (20) we have

$$\text{(44)}\quad H'(y;\varphi_1) \geqq H'(y;\psi_1) \quad \text{for all} \quad y \geqq 0.$$

Since both $H'(y; \varphi_1)$ and $H'(y; \psi_1)$ are increasing functions of y, it follows, in view of (43), that

$$\bar{x}(\varphi_1) \leqq \bar{x}(\psi_1). \tag{45}$$

Similarly, we see from (42) that

$$y(x; \varphi_1) \leqq y(x; \psi_1) \quad \text{for} \quad x < \bar{x}(\psi_1). \tag{46}$$

Recall from Theorem 1′ that

$$\begin{aligned} y(x; \varphi_1) &> x \quad \text{for} \quad x < \bar{x}(\varphi_1) \\ y(x; \varphi_1) &= x \quad \text{for} \quad \bar{x}(\varphi_1) < x. \end{aligned} \tag{47}$$

Combining (45), (46) and (47), we obtain assertion (a).

To obtain (b) we observe that

$$f'(x; \varphi_1) = \begin{cases} -c'[y(x; \varphi_1) - x] & x < \bar{x}(\varphi_1) \\ H'(x; \varphi_1) & \bar{x}(\varphi_1) < x. \end{cases} \tag{48}$$

We consider three cases. For $x < \bar{x}(\varphi_1) < \bar{x}(\psi_1)$,

$$f'(x; \varphi_1) = -c'[y(x; \varphi_1) - x] \geqq -c'[y(x; \psi_1) - x] = f'(x; \psi_1). \tag{49}$$

For $\bar{x}(\varphi_1) < x < \bar{x}(\psi_1)$,

$$f'(x; \varphi_1) = H'(x; \varphi_1) > -c'(0) > -c'[y(x; \psi_1) - x] = f'(x; \psi_1). \tag{50}$$

For $\bar{x}(\varphi_1) < \bar{x}(\psi_1) < x$,

$$f'(x; \varphi_1) = H'(x; \varphi_1) \geqq H'(x; \psi_1) = f'(x; \psi_1),$$

and (b) is verified.

Assume now that we have proved the following: For any $n - 1$ pairs of demand densities φ_i and ψ_i satisfying $\varphi_i \subset \psi_i$, $y(x; \varphi_1, \varphi_2, \cdots, \varphi_{n-1}) \leqq y(x; \psi_1, \psi_2, \cdots, \psi_{n-1})$ for all $x \geqq 0$, and

$$f'(x; \varphi_1, \varphi_2, \cdots, \varphi_{n-1}) \geqq f'(x; \psi_1, \psi_2, \cdots, \psi_{n-1}) \text{ for all } x \geqq 0.$$

We now prove the theorem for the n-period model. The functional equation for the n-period model becomes

$$\begin{aligned} f(x; \varphi_1, \varphi_2, \cdots, \varphi_n) = \min_{y \geqq x} \Bigg\{ & c(y - x) + L(y, \varphi_1) + \alpha \Bigg[f(0, \varphi_2, \cdots, \varphi_n) \\ & \cdot \int_y^\infty \varphi_1(\xi)\, d\xi + \int_0^y f(y - \xi; \varphi_2, \cdots, \varphi_n) \varphi_1(\xi)\, d\xi \Bigg] \Bigg\}. \end{aligned} \tag{51}$$

As in (40) and (41), we define

$$\begin{aligned} G(y, x; \varphi_1, \varphi_2, \cdots, \varphi_n) = {} & c(y - x) + L(y; \varphi_1) \\ & + \alpha \Bigg[f(0; \varphi_2, \varphi_3, \cdots, \varphi_n) \int_y^\infty \varphi_1(\xi)\, d\xi \\ & + \int_0^y f(y - \xi; \varphi_2, \varphi_3, \cdots, \varphi_n) \varphi_1(\xi)\, d\xi \Bigg] \end{aligned}$$

and

$$H(y;\varphi_1,\varphi_2,\cdots\varphi_n) = L(y;\varphi_1) + \alpha\Big[f(0;\varphi_2,\varphi_3,\cdots,\varphi_n)\int_y^\infty \varphi_1(\xi)\,d\xi \tag{52}$$
$$+ \int_0^y f(y-\xi;\varphi_2,\varphi_3,\cdots,\varphi_n)\varphi_1(\xi)\,d\xi\Big].$$

We also define $y(x;\varphi_1,\varphi_2,\cdots,\varphi_n)$ to be the unique root of

$$\frac{\partial G(y,x;\varphi_1,\cdots,\varphi_n)}{\partial y} = 0 \quad \text{for} \quad x < \bar{x}(\varphi_1,\varphi_2,\cdots,\varphi_n), \tag{53}$$

where $\bar{x}(\varphi_1,\varphi_2,\cdots,\varphi_n)$ is the unique root of

$$c'(0) + H'(x;\varphi_1,\varphi_2,\cdots,\varphi_n) = 0. \tag{54}$$

The existence of these roots is guaranteed by the convexity of $c(\cdot)$ and $H(x;\varphi_1,\varphi_2,\cdots,\varphi_n)$ and the assumption that $p + r > c'(0)$; the convexity of $H(x;\varphi_1,\varphi_2,\cdots,\varphi_n)$ follows from Theorem 1′, part B, property (d). Using our induction assumption, we obtain

$$H'(y;\varphi_1,\varphi_2,\cdots,\varphi_n) \geqq L'(y;\varphi_1) + \int_0^y f'(y-\xi;\psi_2,\cdots,\psi_n)\varphi_1(\xi)\,d\xi. \tag{55}$$

Integrating the integral on the right-hand side of (55) by parts gives

$$H'(y;\varphi_1,\varphi_2,\cdots,\varphi_n) \geqq [h'(0) + p + r + \alpha f'(0;\psi_2,\psi_3,\cdots,\psi_n)]\Phi_1(y)$$
$$+ \int_0^y h''(y-\xi)\Phi_1(\xi)\,d\xi - \int_y^\infty (p+r)[1-\Phi_1(\xi)]\,d\xi \tag{56}$$
$$+ \int_0^y f''(y-\xi;\psi_2,\psi_3,\cdots,\psi_n)\Phi_1(\xi)\,d\xi,$$

where $\Phi_1(\xi)$ is defined in (17). Recalling Theorem 1′, part B, property (c), we have

$$h'(0) + p + r + \alpha f'(0;\psi_2,\psi_3,\cdots,\psi_n) > 0. \tag{57}$$

Since $\varphi_1 \subset \psi_1$, we can replace $\Phi_1(\xi)$ by $\psi_1(\xi)$ in (56) and retain the inequality; hence

$$H'(y;\varphi_1,\varphi_2,\cdots,\varphi_n) \geqq H'(y;\psi_1,\psi_2,\cdots,\psi_n) \quad \text{for all} \quad y \geqq 0. \tag{58}$$

Arguing from (53), (54), and (58) as we did from the analogous relations for the one-period model, we obtain

$$y(x;\varphi_1,\varphi_2,\cdots,\varphi_n) \leqq y(x;\psi_1,\psi_2,\cdots,\psi_n)$$

and

$$f'(x;\varphi_1,\varphi_2,\cdots,\varphi_n) \geqq f'(x;\psi_1,\psi_2,\cdots,\psi_n).$$

The result of the theorem for the infinite-stage model is obtained by the usual limiting argument.

From Theorem 2′ we proceed to deduce a series of corollaries comparable to those for linear ordering costs. In the corollaries and lemmas to follow, our assumptions are the same as in Theorem 2′. The proofs entail adaptations of the reasoning employed in establishing Corollaries 1–4 of Section 2. We omit the formal details.

Corollary 1′: If the demand densities for the first $n + 1$ periods are given by $\varphi_1, \varphi_2, \cdots, \varphi_{n+1}$, then

$$y(x; \varphi_1, \varphi_2, \cdots, \varphi_n, 0) \leqq y(x; \varphi_1, \varphi_2, \cdots, \varphi_n, \varphi_{n+1}) \quad \text{for all} \quad x \geqq 0.$$

Corollary 2′: If $\varphi_1 \subset \varphi_2 \subset \varphi_3 \subset \cdots$, then

$$y(x; \varphi_1, \varphi_2, \cdots) \leqq y(x; \varphi_2, \varphi_3, \cdots).$$

Corollary 3′: If the demand densities are $\varphi_1, \varphi_2, \cdots$, and if $\varphi_2 \subset \varphi_1$, then

$$y(x; \varphi_1, \varphi_2, \cdots) \geqq y(x; \varphi_2, \varphi_3, \cdots) \quad \text{for all} \quad x.$$

Corollary 4′: If $y(x; \varphi_1, \varphi_2, \cdots) \geqq y(x; \varphi_2, \varphi_3, \cdots)$ for all $x \geqq 0$, where $\varphi_1, \varphi_2, \cdots$ are the demand densities and $\varphi_1 \subset \varphi_2$, then

$$y(x; \varphi_2, \varphi_3, \cdots) \geqq y(x; \varphi_3, \varphi_4, \cdots) \quad \text{for all} \quad x \geqq 0.$$

The remarks made in Section 2 concerning patterns of demand and corresponding patterns of variation in the form of the optimal policy apply also in this case.

4. Optimal Policy for Linear Purchase Cost, Where There Are Lags in Delivery

In this section we characterize the form of the optimal policy when $h(\cdot)$ and $p(\cdot)$ are convex increasing, $c(z) = c \cdot z$, excess demand is backlogged, and there are time lags in delivery. We consider a lag of λ periods between ordering and delivery. Delivery occurs only at the beginning of a period. Let x represent the current stock size. Let $y_1, y_2, \cdots, y_{\lambda-1}$ represent the outstanding orders, where y_1 is due in at the start of the next period, y_2 is to be delivered two periods hence, etc. We define z as the amount of stock to be ordered at the start of the present period. Finally, let $f(x, y_1, y_2, \cdots, y_{\lambda-1}; \varphi_1, \varphi_2, \cdots)$ denote the minimum expected loss following an optimal policy, where $(x, y_1, y_2, \cdots, y_{\lambda-1})$ takes into account the current stock level and quantities of goods to be delivered during the following $\lambda - 1$ periods, and $\varphi_1, \varphi_2, \cdots$ are the demand densities starting from the present period. The functional equation in this case becomes

$$(59) \qquad \begin{aligned} f(x, y_1, y_2, \cdots, y_{\lambda-1}; \varphi_1, \varphi_2, \cdots) = \min_{z \geqq 0} \Big\{ & c \cdot z + L(x; \varphi_1) \\ & + \alpha \int_0^\infty f(x + y_1 - \xi, y_2, \cdots, y_{\lambda-1}, z; \varphi_2, \varphi_3, \cdots) \varphi_1(\xi)\, d\xi \Big\}, \end{aligned}$$

where

$$(60)\qquad L(x;\varphi_1) = \begin{cases} \int_0^x [h(x-\xi)]\varphi_1(\xi)\,d\xi + \int_x^\infty p(\xi - x)\varphi_1(\xi)\,d\xi & x > 0 \\ \int_0^\infty p(\xi - x)\varphi_1(\xi)\,d\xi & x \leqq 0. \end{cases}$$

In the remainder of this section we consider only the case of a one-period lag, since in the general case the notation becomes quite complex, and the arguments more tedious than the matter warrants. In what follows we assume

$$\lim_{x\to\infty} p'(x) > \frac{1-\alpha}{\alpha}\, c.$$

Double-prime numbering is used to relate the results of this section to those of the previous sections.

Theorem 1″: If $h(\cdot)$ and $p(\cdot)$ are convex increasing, if $c(z) = c\cdot z$, and if there is a one-period lag, then the optimal policy is of the form

$$z^*(x) = \begin{cases} \bar{x}(\varphi_1, \varphi_2, \cdots) - x & x < \bar{x}(\varphi_1, \varphi_2, \cdots) \\ 0 & \text{otherwise}, \end{cases}$$

where $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is the unique root of

$$(61)\qquad c + \alpha \int_0^\infty f'(y - \xi, \varphi_2 \cdots)\, \varphi_1(\xi)\, d\xi = 0.$$

Moreover, $f(x;\, \varphi_1, \varphi_2, \cdots)$ is convex and twice continuously differentiable, except possibly at $\bar{x}(\varphi_1, \varphi_2, \cdots)$.

Proof: This proof for stationary demands is given in [2], p. 162. The proof for varying demands is the same and will not be repeated here. Equation (iv) on p. 163 of [2] should be corrected to read

$$-f'_n(x) \geqq -f_{n-1}(x) \quad \text{for} \quad x < \bar{x}_n\,.$$

Theorem 2″: If we are given two sequences of demand densities $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$, if $\varphi_i \subset \psi_i$ for $i = 1, 2, \cdots$, and if the hypotheses of Theorem 1″ are satisfied, then

(a) $z^*(x;\, \varphi_1, \varphi_2, \cdots) \leqq z^*(x;\, \psi_1, \psi_2, \cdots)$ for all x,

and

(b) $f'(x;\, \varphi_1, \varphi_2, \cdots) \geqq f'(x;\, \psi_1, \psi_2, \cdots)$ for all x.

The proof proceeds along the same lines as the proofs of Theorems 2 and 2′, and therefore will not be repeated.

We turn to the corollaries for the case of time lags.

Corollary 1″: If the demand densities for the first $n + 1$ periods are given by $\varphi_1, \varphi_2, \cdots, \varphi_n, \varphi_{n+1}$, then

$$z^*(x;\varphi_1, \varphi_2, \cdots, \varphi_n, 0) \leqq z^*(x;\varphi_1, \varphi_2, \cdots, \varphi_n, \varphi_{n+1}).$$

Proof: The proof is analogous to that of Corollary 1.

Lemma 1″: If the demand densities are $\varphi_1, \varphi_2, \cdots$, and if $\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots)$, then $\bar{x}(\varphi_1, \varphi_2, \cdots)$ may be computed as the root of

$$c[1 - \alpha] + \alpha \int_0^\infty L'(w - \xi; \varphi_2)\varphi_1(\xi)\, d\xi = 0. \tag{62}$$

Proof: By Theorem 1″ $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is the smallest root of

$$c + \alpha \int_0^\infty f'(w - \xi; \varphi_2, \varphi_3, \cdots)\varphi_1(\xi)\, d\xi = 0. \tag{63}$$

In view of the form of the optimal policy, we obtain the explicit formula

$$f'(x; \varphi_2, \varphi_3, \cdots) = \begin{cases} -c + L'(x; \varphi_2) & x < \bar{x}(\varphi_2, \varphi_3, \cdots) \\ L'(x; \varphi_2) + \alpha \displaystyle\int_0^\infty f'(x - \xi; \varphi_3, \varphi_4, \cdots)\varphi_2(\xi)\, d\xi & \\ & x > \bar{x}(\varphi_2, \varphi_3, \cdots). \end{cases} \tag{64}$$

In searching (63) for its first root, we note that $f'(x; \varphi_2, \varphi_3, \cdots)$ is evaluated only for $x < \bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots)$. But on this range $f'(x; \varphi_2, \varphi_3, \cdots) = -c + L'(x; \varphi_2)$, which when substituted in (63) gives the desired result.

Corollary 2″: If $\varphi_1 \subset \varphi_2 \subset \varphi_3 \subset \cdots$, then (a) $z^*(x; \varphi_1, \varphi_2, \cdots) \leqq z^*(x; \varphi_2, \varphi_3, \cdots)$ for all x, and (b) $\bar{x}(\varphi_1, \varphi_2, \cdots)$ is computed by (62).

Proof: Conclusion (a) follows from Theorem 2″ and conclusion (b) from Lemma 1″.

Lemma 2″: If the demand densities are $\varphi_1, \varphi_2, \cdots$, then

$$f'(x; \varphi_1, \varphi_2, \cdots) \geqq -c + L'(x; \varphi_1) \quad \text{for all} \quad x. \tag{65}$$

Proof: In view of the form of the optimal policy, we have

$$f'(x; \varphi_1, \varphi_2, \cdots) = \begin{cases} -c + L'(x; \varphi_1) & x < \bar{x}(\varphi_1, \varphi_2, \cdots) \\ L'(x; \varphi_1) + \alpha \displaystyle\int_0^\infty f'(x - \xi; \varphi_2, \varphi_3, \cdots)\varphi_1(\xi)\, d\xi & \\ & x > \bar{x}(\varphi_1, \varphi_2, \cdots). \end{cases} \tag{66}$$

But for $x > \bar{x}(\varphi_1, \varphi_2, \cdots)$

$$c + \alpha \int_0^\infty f'(x - \xi; \varphi_2, \varphi_3, \cdots)\varphi_1(\xi)\, d\xi > 0. \tag{67}$$

It is clear from a comparison of (66) and (67) that (65) obtains.

Corollary 3″: If the demand densities are $\varphi_1, \varphi_2, \cdots$, and if $\varphi_3 \subset \varphi_2 \subset \varphi_1$, then

$$z^*(x; \varphi_1, \varphi_2, \cdots) \geqq z^*(x; \varphi_2, \varphi_3, \cdots) \quad \text{for all} \quad x.$$

(In the case of a λ-period lag in delivery, the hypothesis should read $\varphi_{\lambda+2} \subset \varphi_{\lambda+1} \cdots \subset \varphi_1$.)

Proof: Assume the contrary, i.e.,

$$z^*(x; \varphi_1, \varphi_2, \cdots) < z^*(x; \varphi_2, \varphi_3, \cdots) \quad \text{for all} \quad x.$$

Then we have

$$\bar{x}(\varphi_1, \varphi_2, \cdots) < \bar{x}(\varphi_2, \varphi_3, \cdots), \tag{68}$$

and by Lemma 1″ $\bar{x}(\varphi_1, \varphi_2, \cdots)$, is the smallest root of

$$c[1 - \alpha] + \alpha \int_0^\infty L'(w - \xi; \varphi_2)\varphi_1(\xi)\, d\xi = 0. \tag{69}$$

Since $\varphi_3 < \varphi_2$, we deduce that

$$L'(x; \varphi_2) \leqq L'(x; \varphi_3) \quad \text{for all} \quad x. \tag{70}$$

Also, by hypothesis,

$$\Phi_1(x) \leqq \Phi_2(x) \quad \text{for all} \quad x \geqq 0. \tag{71}$$

From (69), (70), (71), and the fact that $L(x; \varphi)$ is convex, we obtain

$$c[1 - \alpha] + \alpha \int_0^\infty L'(w - \xi; \varphi_3)\varphi_2(\xi)\, d\xi \geqq 0. \tag{72}$$

Now Lemma 2″ states that

$$f'(x; \varphi_3, \varphi_4, \cdots) \geqq -c + L'(x; \varphi_3) \quad \text{for all} \quad x. \tag{73}$$

Hence

$$c + \alpha \int_0^\infty f'(w - \xi; \varphi_3, \varphi_4, \cdots)\varphi_2(\xi)\, d\xi \geqq 0. \tag{74}$$

But (74) implies $\bar{x}(\varphi_2, \varphi_3, \cdots) \leqq \bar{x}(\varphi_1, \varphi_2, \cdots)$, contradicting (68).

Corollary 4″: If the demand densities are $\varphi_1, \varphi_2, \varphi_3, \cdots$, and if $\varphi_1 \subset \varphi_2 \subset \varphi_3$ and $\bar{x}(\varphi_1, \varphi_2, \cdots) > \bar{x}(\varphi_2, \varphi_3, \cdots)$, then $\bar{x}(\varphi_2, \varphi_3, \cdots) \geqq \bar{x}(\varphi_3, \varphi_4, \cdots)$. (In the case of a λ-period lag, $\varphi_1 \subset \varphi_2 \subset \varphi_3$ is replaced by $\varphi_1 \subset \varphi_2 \subset \varphi_3 \subset \cdots \subset \varphi_{\lambda+2}$.)

Proof: Assume the contrary, i.e., $\bar{x}(\varphi_2, \varphi_3, \cdots) < \bar{x}(\varphi_3, \varphi_4, \cdots)$. Then, by Lemma 1″, $\bar{x}(\varphi_2, \varphi_3, \cdots)$ is the smallest root of

$$c[1 - \alpha] + \alpha \int_0^\infty L'(w - \xi; \varphi_3)\varphi_2(\xi)\, d\xi = 0. \tag{75}$$

Since $\varphi_1 \subset \varphi_2 \subset \varphi_3$, we have

$$c[1 - \alpha] + \alpha \int_0^\infty L'(w - \xi; \varphi_2)\varphi_1(\xi)\, d\xi \geqq 0 \tag{76}$$

as in the proof of Corollary 3″, and by Lemma 2″

$$c + \alpha \int_0^\infty f'(w - \xi; \varphi_2, \varphi_3, \cdots)\varphi_1(\xi)\, d\xi \geqq 0. \tag{77}$$

But (77) implies $\bar{x}(\varphi_1, \varphi_2, \cdots) \leqq \bar{x}(\varphi_2, \varphi_3, \cdots)$, which contradicts the hypothesis.

This completes our discussion of optimal policies when there is a time lag in delivery. The remarks at the close of Section 2 also apply in the case of time lags.

5. Some Statistical Examples

In this section we study the behavior of the optimal critical number for the dynamic inventory model where the demand distribution is stationary over time but is unknown. More exactly, we shall assume that the demand distribution has a known functional form but involves an unknown parameter w, which is assumed initially to be estimable in terms of an a priori distribution. The demands that occur in each period provide additional observations on the demand density. These samples of the demand density are cumulatively used to improve our estimate of the unknown parameter, until ultimately our modifications of the a priori distribution of w yield an a posteriori distribution of w.

The expected costs for any policy are computed by averaging out with respect to the distribution function of the parameter w.

We confine our attention to distribution functions that admit a single sufficient statistic under repeated independent observations. There are two classes of distributions in this category: the exponential family and the so-called range family [5].

We shall develop several results on stochastic ordering for the a posteriori demand densities that are induced by successive realizations of demand. Scarf [13] obtained some of these results for the exponential family; it was his analysis that suggested our more unified approach. At the close of the section we shall apply these theorems on stochastic orderings to describing the variation of the critical number as a function of the observations.

Consider densities of the exponential family

$$\varphi(\xi \mid w)\, d\xi = \beta(w) e^{w\xi} r(\xi)\, d\xi, \qquad \frac{1}{\beta(w)} = \int_0^\infty e^{w\xi} r(\xi)\, d\xi, \tag{78}$$

where w is an unknown parameter that is assumed to be estimable in terms of an a priori density function $f(w)$. To simplify the notation, we have written these densities in continuous form only. There is a discrete version that occurs where $r(\xi)\, d\xi$ is replaced by a general regular σ-finite measure. The exponential family includes the Gamma distributions in the continuous case and the Poisson and negative binomial distributions in the discrete case.

If we take n independent observations, the joint density is

$$\begin{aligned}\varphi(\xi_1, \xi_2, \cdots, \xi_n \mid w)\, d\xi_1\, d\xi_2 \cdots d\xi_n &= \prod_{i=1}^n \varphi(\xi_i \mid w)\, d\xi_1\, d\xi_2 \cdots d\xi_n \\ &= \beta^n(w) \exp\left[w \sum_{i=1}^n \xi_i\right] r(\xi_1) \cdots r(\xi_n)\, d\xi_1 \cdots d\xi_n.\end{aligned} \tag{79}$$

Let

$$\sum_{i=1}^{n} \xi_i = S_n ;$$

then S_n is clearly a sufficient statistic for the parameter w. By Bayes' rule the a posteriori density of w given S_n is

$$g(w \mid S_n)\, dw = \frac{\beta^n(w) e^{wS_n} f(w)\, dw}{\int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S_n} f(\theta)\, d\theta} . \tag{80}$$

Hence the density of ξ given S_n is

$$\begin{aligned} \psi(\xi \mid S_n)\, d\xi &= \left[\int_{-\infty}^{\infty} \beta(\theta) e^{\theta\xi} g(\theta \mid S_n)\, d\theta \right] r(\xi)\, d\xi \\ &= \frac{\left[\int_{-\infty}^{\infty} \beta^{n+1}(\theta) e^{\theta\xi} e^{\theta S_n} f(\theta)\, d\theta \right] r(\xi)\, d\xi}{\int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S_n} f(\theta)\, d\theta} . \end{aligned} \tag{81}$$

Let us introduce the notation $\psi(\xi \mid S_n) = \psi_n(\xi \mid S)$. We may then state the following theorem on stochastic ordering.

Theorem 3: If $S' > S$, then $\psi_n(\cdot \mid S) \subset \psi_n(\cdot \mid S')$.

Proof: We are required to prove that

$$\int_0^y [\psi_n(\xi \mid S) - \psi_n(\xi \mid S')]\, d\xi \geqq 0 \quad \text{for all} \quad y.$$

Since $\psi_n(\xi \mid S)$ and $\psi_n(\xi \mid S')$ are densities, we have

$$\int_0^{\infty} [\psi_n(\xi \mid S) - \psi_n(\xi \mid S')]\, d\xi = 0. \tag{82}$$

To complete the proof, it is sufficient to show that $\psi_n(\xi \mid S) - \psi_n(\xi \mid S')$ changes sign once from $+$ values to $-$ values as ξ traverses the positive axis from 0 to ∞. Consider

$$\begin{aligned} \psi_n(\xi \mid S) - \psi_n(\xi \mid S') &= r(\xi) \\ &\quad \cdot \left[\frac{\int_{-\infty}^{\infty} \beta^{n+1}(\theta) e^{\theta\xi} e^{\theta S} f(\theta)\, d\theta}{\int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S} f(\theta)\, d\theta} - \frac{\int_{-\infty}^{\infty} \beta^{n+1}(\theta) e^{\theta\xi} e^{\theta S'} f(\theta)\, d\theta}{\int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S'} f(\theta)\, d\theta} \right] \\ &= r(\xi) \left\{ \int_{-\infty}^{\infty} e^{\theta\xi} \left[\frac{\beta^{n+1}(\theta) e^{\theta S} f(\theta)}{A} - \frac{\beta^{n+1}(\theta) e^{\theta S'} f(\theta)}{B} \right] d\theta \right\}, \end{aligned} \tag{83}$$

where

$$A = \int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S} f(\theta)\, d\theta \quad \text{and} \quad B = \int_{-\infty}^{\infty} \beta^n(\theta) e^{\theta S'} f(\theta)\, d\theta.$$

We digress for a moment to discuss the concept of variation-diminishing transformations, which plays a part in the analysis of (83). The kernel $e^{\theta\xi}$ belongs to a Pólya-type distribution and is therefore variation-diminishing. This means that the transformed function

$$g(\xi) = \int e^{\theta\xi} h(\theta)\, d\mu(\theta)\ [d\mu(\theta) \geq 0] \tag{84}$$

cannot change signs more frequently than the function $h(\theta)$. Furthermore, if $g(\cdot)$ have the same number of sign changes, then they change sign in the same order or equivalently have the same sign for large values of the arguments. (For a detailed discussion of Pólya-type distributions and their properties, we refer the reader to [9] and [12].)

We see immediately that

$$\frac{\beta^{n+1}(\theta)e^{\theta S}f(\theta)}{A} - \frac{\beta^{n+1}(\theta)e^{\theta S'}f(\theta)}{B} = \frac{\beta^{n+1}(\theta)f(\theta)e^{\theta S'}}{A}\left[e^{\theta(S-S')} - \frac{A}{B}\right]$$

has a single sign change from + values to − values as θ traverses its natural range. Comparing (83) and (84), we deduce that $\psi_n(\xi \mid S) - \psi_n(\xi \mid S')$ has at most one sign change from + to − values. On the other hand, it is clear from (82) that $\psi_n(\xi \mid S) - \psi_n(\xi \mid S')$ must change sign at least once. The proof of the theorem is thus complete.

Theorem 4: $\psi_{n+1}(\cdot \mid S) < \psi_n(\cdot \mid S)$.

The argument is similar to that of the preceding theorem. We show that $\psi_{n+1}(\xi \mid S) - \psi_n(\xi \mid S)$ has one sign change from + to − values as ξ increases from 0 to ∞. To this end, consider

$$\psi_{n+1}(\xi \mid S) - \psi_n(\xi \mid S) = r(\xi)\left\{\int_{-\infty}^{\infty} e^{\theta\xi}\,\frac{\beta^{n+1}(\theta)e^{\theta S}f(\theta)}{A_1}\left[\beta(\theta) - \frac{A_1}{B_1}\right] d\theta\right\}, \tag{85}$$

where

$$A_1 = \int_{-\infty}^{\infty} \beta^{n+1}(\theta)e^{\theta S}f(\theta)\, d\theta, \qquad B_1 = \int_{-\infty}^{\infty} \beta^{n}(\theta)e^{\theta S}f(\theta)\, d\theta,$$

and

$$\frac{1}{\beta(\theta)} = \int_0^{\infty} e^{\theta\xi} r(\xi)\, d\xi.$$

Clearly $\beta(\theta) = +\infty$ for $\theta = -\infty$. Also $\beta(\theta)$ is a decreasing function of θ on its natural domain of definition. Now appealing to the variation-diminishing properties of the Pólya-type kernel $e^{\theta\xi}$, we find by comparing (84) and (85) that $\psi_{n+1}(\xi \mid S) - \psi_n(\xi \mid S)$ changes sign once from + to − values as ξ increases from 0 to ∞. The result now follows as in the preceding theorem.

We now prove the analogous theorems for the range family of distributions. Consider

(86) $$p(\xi \mid w) = q(\xi)r(w)\psi(\xi, w)\, d\xi, \quad \text{where} \quad \psi(\xi, w) = \begin{cases} 1 & \xi \leqq w \\ 0 & \xi > w; \end{cases}$$

here w is again an unknown parameter estimable in terms of an a priori density function $f(w)(w > 0)$. The joint density of an n-tuple of observations is

(87) $$\begin{aligned} p(\xi_1, \xi_2, \cdots, \xi_n \mid w)d\xi_1 \cdots d\xi_n &= \prod_{i=1}^{n} p(\xi_i \mid w)d\xi_i \cdots d\xi_n \\ &= r^n(w) \prod_{i=1}^{n} q(\xi_i) \prod_{i=1}^{n} \psi(\xi_i, w) \prod_{i=1}^{n} d\xi_i . \end{aligned}$$

But

$$\prod_{i=1}^{n} \psi(\xi_i, w) = \psi(\max_{1 \leqq i \leqq n} \xi_i, w),$$

and thus

$$v_n = \max_{1 \leqq i \leqq n} \xi_i$$

defines a sufficient statistic for w. By Bayes' rule the a posteriori density of w given v_n reduces to

$$h(w \mid v_n)\, dw = \frac{\psi(v_n, w)r^n(w)f(w)\, dw}{\int_{-\infty}^{\infty} r^n(\theta)\psi(v_n, \theta)f(\theta)\, d\theta}.$$

Hence the a posteriori density of ξ given v_n is

(88) $$\begin{aligned} p(\xi \mid v_n)\, d\xi &= \left[\int_{-\infty}^{\infty} r(\theta)q(\xi)\psi(\xi, \theta)h(\theta \mid v_n)\, d\theta\right] d\xi \\ &= \frac{q(\xi)\, d\xi \int_{-\infty}^{\infty} r^{n+1}(\theta)f(\theta)\psi(\xi, \theta)\psi(v_n, \theta)\, d\theta}{\int_{-\infty}^{\infty} r^n(\theta)\psi(v_n, \theta)f(\theta)\, d\theta} \end{aligned}$$

We introduce the notation $p(\xi \mid v_n) = p_n(\xi \mid v)$.
Theorem 5: If $v' > v$, then $p_n(\cdot \mid v) \subset p_n(\cdot \mid v')$.
Proof: We have to prove that

$$\int_0^y [p_n(\xi \mid v) - p_n(\xi \mid v')]\, d\xi \geqq 0 \quad \text{for all} \quad y.$$

Note first that

(89) $$\int_0^{\infty} [p_n(\xi \mid v) - p_n(\xi \mid v')]\, d\xi = 0.$$

Thus it is enough to verify that $p_n(\xi\,|\,v) - p_n(\xi\,|\,v')$ changes sign once from + to − values, and the theorem will be proved. To this end, we have

$$p_n(\xi\,|\,v) - p_n(\xi\,|\,v') = q(\xi)\left\{\int_{-\infty}^{\infty} r^{n+1}(\theta)\psi(\xi,\theta)f(\theta)\right.$$

$$\left.\cdot\left[\frac{\psi(v,\theta)}{\int_0^\infty r^n(\theta)\psi(v,\theta)f(\theta)\,d\theta} - \frac{\psi(v',\theta)}{\int_0^\infty r^n(\theta)\psi(v',\theta)f(\theta)\,d\theta}\right]d\theta\right\} \tag{90}$$

$$= q(\xi)\int_{-\infty}^{\infty}\frac{r^{n+1}(\theta)\psi(\xi,\theta)f(\theta)}{A}\left[\psi(v,\theta) - \frac{A}{B}\psi(v',\theta)\right]d\theta,$$

where

$$A = \int_0^\infty r^n(\theta)\psi(v,\theta)f(\theta)\,d\theta, \qquad B = \int_0^\infty r^n(\theta)\psi(v',\theta)f(\theta)\,d\theta$$

and $A > 0$, $B > 0$. Now

$$r(\theta) = \frac{1}{\int_0^\infty q(\xi)\psi(\xi,\theta)\,d\xi} = \frac{1}{\int_0^\theta q(\xi)\,d\xi} > 0 \quad \text{for} \quad \theta > 0.$$

For $v \leqq \theta < v'$, we have

$$\left[\psi(v,\theta) - \frac{A}{B}\psi(v',\theta)\right] = 1;$$

for $v < v' \leqq \theta$, we have

$$\left[\psi(v,\theta) - \frac{A}{B}\psi(v',\theta)\right] = 1 - \frac{A}{B}.$$

Since $v < v'$, it readily follows that $A > B$ and

$$\left[\psi(v,\theta) - \frac{A}{B}\psi(v',\theta)\right] < 0$$

for $v < v' \leqq \theta$. Thus,

$$\left[\psi(v,\theta) - \frac{A}{B}\psi(v'\,\theta)\right]$$

has one sign change from + to − values. But the kernel $\psi(\xi,\theta)$ is the density of a Pólya-type distribution. Invoking the variation-diminishing properties of this kernel, we infer that $p_n(\xi\,|\,v) - p_n(\xi\,|\,v')$ changes sign once from + to − values. This completes the proof of the theorem.

Theorem 6: $p_{n+1}(\cdot\,|\,v) \subset p_n(\cdot\,|\,v)$.

Proof: As in the preceding theorem, we must show that $p_{n+1}(\cdot\,|\,v) - p_n(\cdot\,|\,v)$

changes sign once from + to − values. This is verified by the same kind of arguments as before: i.e., we exploit the identity

$$p_{n+1}(\xi \mid v) - p_n(\xi \mid v) = q(\xi)\left\{\int_0^\infty \frac{r^{n+1}(\theta)f(\theta)\psi(\xi, \theta)\psi(v, \theta)}{A_1}\left[r(\theta) - \frac{A_1}{B_1}\right] d\theta\right\},$$

where

$$A_1 = \int_0^\infty r^{n+1}(\theta)\psi(v, \theta)f(\theta)\, d\theta \quad \text{and} \quad B_1 \int_0^\infty r^n(\theta)\psi(v, \theta)f(\theta)\, d\theta.$$

We omit the remaining details.

We now return to our discussion of the inventory problem in which the distribution of demands has an unknown parameter. Let $\varphi(\xi \mid S, n)$ be the a posteriori density of ξ, where the value of the sufficient statistic is S based on n observations and where $\varphi(\xi \mid w)$ is a density belonging to either the exponential or the range family. Furthermore, let $C^N(x \mid S, n)$ denote the expected loss for an N-period inventory model when the initial stock is of size x, the demand density for the first period is $\varphi(\xi \mid S, n)$, and an optimal purchasing policy is followed. With this definition, $C^N(x \mid S, n)$ satisfies

$$\begin{aligned} C^N(x \mid S, n) = \min_{y \geqq x}\Big\{c(y - x) &+ \int_0^y h(y - \xi)\varphi(\xi \mid S, n)\, d\xi \\ + \int_y^\infty p(\xi - y)\varphi(\xi \mid S, n)\, d\xi &+ \alpha\Big[\int_y^\infty C^{N-1}(0 \mid S\cdot\xi, n + 1)\varphi(\xi \mid S, n)\, d\xi \\ &+ \int_0^y C^{N-1}(y - \xi \mid S\cdot\xi, n + 1)\varphi(\xi \mid S, n)\, d\xi\Big]\Big\}, \end{aligned} \tag{91}$$

where $S\cdot\xi$ is to be interpreted as $S + \xi$ if $\varphi(\xi \mid w)$ is a member of the exponential family, and as $\max(S, \xi)$ if $\varphi(\xi \mid w)$ is a member of the range family. For the one-period model in which the value of the sufficient statistic is either S or $S'(S < S')$, it follows from Theorems 2, 2′, 3, and 5 that $\bar{x}_1(S) \leqq \bar{x}_1(S')$ and $y_1(x \mid S) \leqq y_1(x \mid S')$ for all $x \geqq 0$. By virtue of these same theorems we also have

$$C^{(1)'}(x \mid S, n) \geqq C^{(1)'}(x \mid S', n) \quad \text{for all} \quad x \geqq 0. \tag{92}$$

Assume now that we have proved for the $(N - 1)$-period model the relations

$$C^{(N-1)'}(x \mid S, n) \geqq C^{(N-1)'}(x \mid S', n) \quad \text{for all} \quad x \geqq 0,$$

and

$$\bar{x}_{N-1}(S) < \bar{x}_{N-1}(S') \quad \text{or} \quad y_{N-1}(x \mid S) \leqq y_{N-1}(x \mid S') \quad \text{for all} \quad x \geqq 0.$$

Employing an induction argument parallel to that of Theorems 2 and 2′, we obtain

$$\bar{x}_N(S) \leqq \bar{x}_N(S') \quad \text{or}$$
$$y_N(x \mid S) \leqq y_N(x \mid S') \quad \text{for all} \quad x \geqq 0.$$

The corresponding propositions can be established in the case of time lags. Finally, by referring to Theorems 4 and 6 we achieve similar results pertaining to the behavior of the critical numbers for the case of $n + 1$ observations versus n observations.

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IX–33

COST HORIZONS AND CERTAINTY EQUIVALENTS: AN APPROACH TO STOCHASTIC PROGRAMMING OF HEATING OIL*[1]

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1. Scope of the Study

Scheduling heating oil production is an important management problem. It is also a complex one. Weather and demand uncertainties, allocation of production between different refineries, joint- and by-product relations, storage limitations, maintenance of minimal supplies and many other factors need to be considered.

This paper is concerned with one of an integrated series of operations research studies directed toward improvement in such scheduling methods. Emphasis is on essentials of the mathematical model. Institutional features and other phases of the OR studies are brought in only as required.

The decision to focus on this study phase as the first of a series of releases was dictated by a variety of reasons including (it is hoped) provision of a convenient frame of reference for subsequent discussions. The scheduling model played a central role in these studies. It should not be inferred, however, that it was the most important (or valuable) portion. Other OR techniques were equally critical

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from a methodological standpoint.[2] Some were readily available; others had to be adapted or developed by cooperating company personnel and study participants. Finally, completion and perfection of all phases of the study is attendant on still further such developments, some of which will be indicated in the text of this article.

Substantive considerations were at least as important as OR methodology. These were supplied from the accumulated experience and judgment of company officials who made their advice systematically available to the study groups.[3] By drawing on this source for guidance in formulation, testing and validation, it was possible, in an expeditious manner, to develop the desired improvements in guides for setting the company's heating oil schedules.

2. General Considerations

Only those portions of the model which are of general interest will be presented in this paper. It may be helpful, therefore, to commence with a sketch of the problem along with relevant background materials. Production is to be planned over a time interval specified in advance. This interval is called the horizon. The "objective"[4] is to maximize profits subject to a series of constraining relations over the horizon. Two such series of constraints will be considered: (1) "marketing" and (2) "storage."[5]

The marketing constraints require production to be planned in a way which will meet customer demands as they materialize. Minimal inventory levels (deemed to be necessary for efficient functioning of the distribution system) are also incorporated in them. Another series of constraints refer to storage capacity. Schedules should honor maximum permissible inventory levels established relative to this capacity.

Demand for this product is heavily weather dependent. The improved methods for forecasting weather components and consequent demands undertaken during the course of these studies need only be dealt with formally in this paper. For, as was recognized from the outset, only probability forecasts of demands and related statistical distributions may (at best) be secured. Scheduling models must therefore be formulated accordingly and "exact" objectives and constraints replaced by their probabilistic counterparts. Thus, the marketing constraints become probability relations with a stipulated high degree of reliability (confidence coefficient) for each scheduling period. Similar remarks apply to the storage constraints. Also, the exact functional (for the objective) is replaced by its expected value.

Maximization of expected profits is, in these terms, to be undertaken over the

[2] See, e.g., G. H. Symonds [22] for a brief sketch of some of the statistical considerations.

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[4] This term is used in a technical sense. See [7] for further discussion.

[5] Other constraints such as non-negativity conditions on refinery schedules must also be observed.

entire horizon. A deeper consideration of the constraints, however, suggests that the maximizing objective may be replaced by one of minimization. These constraints, which are stated in physical terms, are interpreted to mean that demands are to be taken as given (i.e., stochastically determined).[6] To the indicated (high) degrees of reliability customer wants are to be met as they occur due to weather (or related) considerations. The planning objective is thus to supply whatever demands may emerge at minimum total cost.

Programs are to be determined stochastically. As events materialize refinery rates for the following period are to be determined in conditional fashion taking account of accumulating experience and future possibilities. The meaning, and quantitative consequences, to be assigned to each such event is to be determined beforehand so that plans can be formulated (and evaluated) in advance for each possible contingency. Finally, it is desirable to be able to assess potential future changes in structure or requirements.[7]

A decision rule was devised for the purpose of comprehending these elements of the problem. By means of detailed statistical studies the relevant densities were isolated and tested. In some cases suitable transformations were required, as well as means for converting from weather data to demand forecasts. The tests that were conducted indicated that statistical independence of the weather dependent portions of the demands could safely be assumed. This assumption will therefore constitute one portion of what will hereafter be called the "null hypothesis". More fully, the null hypothesis, as thus validated, assumes that (a) the sales densities are known and (b) the relevant variables are statistically independent. Continuing test procedures may then be introduced in a manner analogous to usages in statistical quality control, in order to make sure that significant (and relevant) changes in the underlying universes do not go undetected.

By reference to these densities (all of them) suitable weights and adjustment factors are calculated for incorporation in the decision rule.[8] First, it is necessary to determine limits for these values relative to the constraints. Within these limits the weights and adjustment factors required for applying the decision rule are then determined with reference to the cost minimizing objective.

This part of the calculations requires recourse to suitable approximating routines[9] since the relations are, in general, nonlinear. The decision rule is characterized as linear. Confusion will be avoided, however, if it is remembered that the rule is linear only after the weights (and adjustment factors) are determined. As the random events materialize—in the form of actual sales demands—the

[6] *Vide* [9] for discussion of cases in which price (and monetary) considerations are incorporated in the constraints.

[7] This topic is closely related to hypothesis testing aspects of the study which are reserved for separate treatment. Sufficient additional work is required in this area so that it does not appear worthwhile to complicate the present paper by introducing detailed consideration of these points.

[8] These are really a series of decision rules with the same general form.

[9] By means of suitable controlled "Monte Carlo" calculations. No attempt will be made in the present paper to describe these routines in detail.

weights and adjustment factors thus calculated (and incorporated in the rule) are applied to determine the production schedules in succeeding periods.[10]

The decision rule is applied in a manner which transforms the originally stated problem into one which deals with "certainty equivalents".[11] This kind of transformation represents one way of dealing with probabilistic constraints and functionals. It is useful, therefore, to distinguish between this general approach and the particular rule employed. The latter may require modification in other contexts even when the general approach is applicable.

Also, for the problem—at least as thus far described—the rule which was selected is not necessarily the best one against all possible contenders.[12] Factors in addition to those already mentioned entered into the choice of a suitable rule. Examination of past procedures indicated that mathematical reformulations would yield decision rules which were essentially linear. It was deemed desirable to utilize the experience incorporated in these procedures and to select a more general rule which would encompass them as special cases. Some of the other desiderata were as follows: (1) Computational expediency and flexibility in application. (2) A desire to secure assistance for locating "check dates" or sub-horizons, in reviewing schedules. (3) Some degree of stability in these schedules was also deemed to be desirable, even though it was not practically possible to state constraining relations in advance, or to adopt an alternative approach and impute specific costs for association with fluctuating production beyond certain limits.[13] Finally, methods were to be devised which would be amenable to other problems not directly involved in setting heating oil schedules. This was important since these studies were to provide a test for the value of OR work to the company.

After some experimentation a rule was adopted which seemed to give the best promise of yielding a combination of the desired results. It is expected that still further improvement will be made as experience develops.[14]

3. Cost Functions

Additional considerations which also entered into model designs may best be introduced as the discussion proceeds.

Carefully conducted cost studies were undertaken at relevant refineries. From these results a total company cost function was synthesized in the form

[10] By suitable adaptations it should be possible to extend these procedures for studies extending beyond the short-term scheduling horizon. Some of the problems associated with such extensions are discussed in the concluding section of this paper.

[11] See footnote 21, *infra*.

[12] This problem will be dealt with in a separate paper. It should perhaps be noted that certain differences in the heating oil problem, including the character of the constraints, precluded the possibility of direct access to the results (now classic) in [1], [11], and [12].

[13] An analysis of such costs for ordinary manufacturing (*not* refining) operations may be found in [5] and [14].

[14] Such experience may also bring to light additional qualities which are desired such as a rule which ends the scheduling horizon with a relatively low inventory level.

$$c_j\,(R_j)R_j = \frac{a_j\,R_j}{b_j - R_j} + k_j\,R_j = \frac{a_j\,b_j}{b_j - R_j} - a_j + k_j\,R_j$$

(1) where

$c_j \;=\; c_j(R_j)$ represents average variable unit cost (in \$/bbl) at period $j = 1, 2, \cdots, N$.
$R_j \;=$ scheduled production (M bbls/day)

and a_j, b_j and k_j are positive constants, or parameters, which are applicable in the j^{th} period.

A series of (inverse) hyperbolas was thus obtained so that average variable costs are convex and increasing functions of R_j over the relevant range of production rates.

Because of joint- and common-cost features it was necessary to estimate costs of producing heating oil on the basis of incremental production above standard levels. The methods of estimation were designed to insure, so far as possible, that variations in demand for related products were supplied in a manner which would not distort the estimates of cost for heating oil production. This was done for each relevant refinery as a preliminary to synthesizing the total company function shown in (1). For this purpose incremental production was assigned to the refineries in a manner designed to achieve lowest total costs for the company at each output level. This also had to be done for each period $j = 1, 2, \cdots, N$ because of time variations which were found to be present in the cost functions.

It would have been possible to render the model in such form as to achieve an inter-refinery allocation. This would have caused complications which did not appear to be warranted by the benefits that were apparent.

There are additional cost features which required detailed attention in the actual study, although they are not of major interest for this presentation. Improved methods for forecasting material and transportation costs, and ascertaining proper interest rates and inventory carrying charges, provide representative specimens.[15] For present purposes these can be regarded as included (at least formally) in the expression

$$C_j(R_j)R_j + K_j\bar{I}_j = (c_j + T_j)R_j + K_j\bar{I}_j$$

(2) where,

$T_j \;=$ transportation cost in period j (in \$/bbl.)
$\bar{I}_j \;=\; (I_{j-1} + I_j)/2 \;=$ average inventory (M bbls.) in period j.
$K_j \;=$ inventory carrying cost in (\$/bbl. per day) for period j.

[15] In some cases—e.g., transportation costs—it was necessary to adjust the model to allow for different horizons due to differing contract periods and other such institutional features.

For simplicity, it is here assumed that produced crudes are transported to the refinery at cost T_j in period j. They are converted to heating oil and, if necessary, stored in inventory for anticipated demands.[16]

4. Model Details

It is easier to enter the model through the constraints. First, there is a set of sales constraints

$$\Pr\left\{I_0 + \sum_{l=1}^{j} R_l \geqq \sum_{l=1}^{j} S_l + I_{\min}\right\} \geqq \alpha_j, \tag{3.1}$$

where α_j, $j = 1, 2, \cdots, N$, is a suitably high "confidence coefficient" prescribed for the j^{th} scheduling interval. Second, there is a set of storage constraints,

$$\Pr\left\{I_0 + \sum_{l=1}^{j} R_l \leqq \sum_{l=1}^{j} S_l + I_{\max}\right\} \geqq \mu_j. \tag{3.2}$$

The conditions (3.1) and (3.2) together requires that sales and storage conditions be met with probabilities at least α_j and μ_j, respectively, where

(3.3)
- I_0 = initial inventory (M bbls)
- R_j $\geqq 0$ is the production rate (stated in M bbls./day) to be scheduled in period j.
- S_j $\geqq 0$ is the anticipated sales for period j (stated in M bbls./day).
- $I_{\min}$ = a minimum inventory level (in M bbls.) which is to be maintained.
- $I_{\max}$ = maximum inventory level (M bbls.) set by storage capacity.

Here S_j is a random variable with known density

$$f_j(S_j), \qquad j = 1, 2, \cdots, N, \tag{3.4}$$

so that the magnitude of sales which will materialize in period j is known only in probability. This forms part of what is here called the null hypothesis.[17]

To simplify the exposition and center attention on main principles, it will be assumed that

$$\mu_j = 1, \qquad j = 1, 2, \cdots, N. \tag{3.5}$$

Refinery schedules must then never violate[18] the storage limits. Similarly, the non-negativity constraint

$$\Pr\{R_j \geqq 0\} = 1, \qquad j = 1, 2, \cdots, N. \tag{3.6}$$

is also stipulated with probability one. The value of α_j in each sales constraint may be arbitrary. It is, however, usually prescribed at a high level.

[16] More detailed treatment was needed in the actual study to allow for different locations, etc. Cf., also, footnote 10, *supra*. Only the simplest case will be examined here.

[17] See remarks in section 2, *supra*.

[18] I.e., *logically* and on the null hypothesis.

The functional, to be maximized, may be formulated in terms of expected profits—viz.,

(4)
$$\max. E\pi = \max_{R} \int_{D} \pi(S, R) f(S)\, dS$$

where,

$$\pi(S, R) = \frac{1}{N} \sum_{j=1}^{N} [p_j S_j - (c_j + T_j)R_j - K_j \bar{I}_j]$$

$$\int_{D} \pi(S, R) f(S)\, dS = \int_{D_1} \cdots \int_{D_N} \pi(S, R) f_1(S_1) \cdots f_N(S_N)\, dS_1 \cdots dS_N$$

and

p_j = price (\$/bbl.) expected to prevail in period j.
D_j = range of sales variation to be considered in period j.
R, D = domains of refinery and sales variations to be considered.

The other terms have been previously defined in conjunction with expressions (1)–(3.6).[19]

To the measures of reliability stipulated in the constraints, maximization is to be undertaken with respect to the anticipated sales developments. Since the sales variables and their associated densities lie outside company control, it is useful to decompose the profits functional into its receipts and cost components. Specifically, the terms involving $\sum p_j S_j$ are taken as constant with respect to variation in the R_j's. The problem is then converted from one of maximizing expected profits to one of minimizing expected costs—i.e.,

(5)
$$\min. \frac{1}{N} \sum_{j=1}^{N} \int_{D} [C_j(R_j)R_j + K_j \bar{I}_j] f(S)\, dS$$

subject to (3.1), (3.2) and the associated non-negativity requirements (3.6).

5. Decision Rule and Certainty Equivalents

Numerous methods may be devised for handling problems of this kind. Of the variants tried the one which appeared to be most successful involved the use of a decision rule molded along the following lines:[20]

[19] In some instances it is desirable to allow for varying lengths in each period $j = 1, 2, \ldots, N$. This may easily be done but is here omitted to avoid extra complications.

[20] Associated with this rule is a method of application (to be described in due course) and the implicit stipulation

$$\sum_{l=1}^{k-1} \beta_{kl} S_l + \sum_{l=1}^{k} \gamma_l \geqq \sum_{l=1}^{k-2} \beta_{k-1l} S_l + \sum_{l=1}^{k-1} \gamma_l \geqq 0$$

since $R_j \geqq 0$. Some of the values γ_l are thus allowed to be negative if better performance can be secured by doing so. Because (6) is stated as a cumulant, inventory position in each period $j = 1, 2, \cdots, N$ is also considered implicitly.

$$\sum_{l=1}^{j} R_l = \sum_{l=1}^{j-1} \beta_{jl} S_l + \sum_{l=1}^{j} \gamma_l \tag{6}$$

$$R_1 = \gamma_1 \geqq 0.$$

This rule is linear, but, as has already been noted, only after the β_{jl} and γ_l are available. These values are jointly determined via the relevant densities to suitable degrees of approximation.

The weights, β and γ, are determined in advance for the entire horizon. Applied to past sales they yield what may be called "certainty equivalents"[21] for the production schedules needed to satisfy emerging demands at desired levels or reliability. Thus, suppose that R_1, the refinery rate for the first period is wanted. The value γ_1, known in advance, gives $R_1 = \gamma_1$. *After* S_1 has materialized, period 2 production is ascertained from $R_2 = \beta_{21}S_1 + \gamma_2$.[22] Continuing in this fashion each R_j, $j = 1, 2, \cdots, N$, is determined in conditional stochastic fashion, taking account of actual sales and production, forecasts of the future (via the weights) and forecast corrections whenever accumulating information and experience indicates that adjustments are required.

Although the weights are calculated by reference to the originally assumed (and tested) densities, it does not necessarily follow that they will have to be recalculated for each new situation. Only at certain critical points (closely associated with the "cost subhorizons")[23] are these weights sensitive to variations in

21 I.e., the known weights are to be applied to past sales (also known) in order to determine the refinery rates. When these adjustments are effected and the refinery rates set as though these (adjusted) "sales" are each to be met with perfect certainty then future (actual) sales demand will (on the null hypothesis) be satisfied at prescribed reliability levels. Note that the values of β and γ are determined for the entire horizon. Hence, the refinery rate scheduled for the next period applies not only to sales of that period but to other periods as well, allowing for inventory carrying costs and other charges and constraining relations which need to be considered in the overall optimization.

For reference to the general problem of certainty equivalents and its relation to price and cost regimes which produce equivalent resources allocations under certainty and uncertainty, see [17], [18], [20] and [21]. Notice, however, that the present approach differs from these standard versions in at least the following respects: (1) In place of the usually employed parameters—e.g., the standard deviation as a measure of risk—the entire densities are used here in order to allow for such phenomena as oppositely skewed distributions which represent different business situations. (2) The usual exact pairing of points for income subject to risk and the corresponding certainty equivalents is here replaced by inequalities. (3) These inequalities refer to *relations* rather than the usual number pairings. Notice also that the question of existence of certainty equivalents does not arise in the present context because the ranges of the sales densities are finite and below capacity limits.

Confusion will be avoided if the above points are kept in mind while reviewing this literature. It will then also become apparent that, when viewed in terms of risk analysis, still another interpretation is possible for the objective in the present problem. By undertaking to minimize its cost relative to the constraining relations, the company may be viewed as attempting to maximize the surplus relative to the implied risk premiums.

22 Note that the β_{jl} for the same sales variable are (in general) different for each period.

23 This term is explained in the immediately following paragraphs and elaborated in detail in later sections of this paper.

the data. Over other intervals considerable "slack" is (in general) present in the system. The stability arising from this slack may be further reenforced by the possibility of offsetting variations in the densities along with differences in the cost functions which are applicable in different periods. Also, most of the relevant statistical tests are one-sided so that it is necessary to consider only those variations which affect the size of the critical regions in the tails of the resulting distributions.

Two stages of analysis are used to secure the values β and γ. The decision rule (6) is first substituted in the constraints. This yields (among other things) lower bounds for ascertaining these components of the sales certainty equivalents. Optimization is then undertaken relative to these constraints. For this purpose the decision rule (6) is substituted in the functional and values β and γ are secured which minimize the cost of meeting the (inequality) levels prescribed in stage one.

By this two-stage process the original problem is transformed to one which involves minimizing a convex functional subject to linear constraints. As such, the problem is amenable to solution by available methods which extend the power of linear programming techniques to embrace this class of nonlinear problems.[24]

Because of the need for systematizing the continuing test procedure and because advance "check dates" were desired for committee review of schedules an alternate approach was employed to bring these factors to the fore. The concept of "cost horizons" was introduced and developed for these purposes. This term is used to distinguish the approach used here from more standard versions of subhorizon analysis.[25] The latter proceed directly from production or sales data as follows. Relevant "peaks" in the production requirements are ascertained in a way which makes it possible to locate subhorizons intervals and to determine schedules within each such interval which minimize the total cost of meeting anticipated requirements. These "production horizon" approaches may be contrasted with "cost horizons" in which the cost functions themselves provide the point of departure. Relevant "peaks" in incremental (or marginal) costs are ascertained for demarcating subhorizon dates. Setting these peaks as low as the constraints allow while increasing incremental costs for preceding periods as far as may be necessary yields a minimum total cost schedule which fulfills the requirements at the critical subhorizon dates and (in general) overfulfills them in between these dates.[26] In principle, therefore, the cost horizon approach determines physical schedules by a series of cost inequalities. The physical "check dates" are also thereby ascertained. Finally, computational advantages may be secured, relative to the burden which would otherwise be incurred in a straightforward linear programming approach.

[24] *Vide* [2] and [4].

[25] *Vide* [8] and [19]. See [3], however, for an earlier version in which the approach is also from the cost side.

[26] Cf. (3.1) and (3.2) which are stated as double inequalities.

6. Procedure for Determining Certainty Equivalents

The range over which integration is to be taken for the probability integrals furnishes a basis for the first stage of analysis in securing the required certainty equivalents. When (6) is inserted into (3.1) the sales constraints are re-expressed as

$$\Pr\left\{I_0 - I_{\min} + \sum_{l=1}^{j} \gamma_l \geqq \sum_{l=1}^{j-1} (1 - \beta_{jl})S_l + S_j\right\} \geqq \alpha_j \tag{7.1}$$

for $j = 1, 2, \cdots, N$. Similarly, (3.2) becomes

$$\Pr\left\{I_0 - I_{\max} + \sum_{l=1}^{j} \gamma_l \leqq \sum_{l=1}^{j-1} (1 - \beta_{jl})S_l + S_j\right\} = 1 \tag{7.2}$$

for the case in which all $\mu_j = 1$.

This last expression is directly translatable as

$$\sum_{l=1}^{j} \gamma_l \leqq I_{\max} - I_0 + \min_{S}\left[\sum_{l=1}^{j-1} (1 - \beta_{jl})S_l + S_j\right], \quad j = 1, 2, \cdots, N. \tag{8.1}$$

Its predecessor, the sales contraint, may be translated into analytical expressions of the form

$$\int_{-\infty}^{A_j(\gamma)} \int_{-\infty\varepsilon_{j,j-1}}^{B_{j,j-1}} \cdots \int_{-\infty\varepsilon_{jk}}^{B_{jk}} \cdots \int_{-\infty\varepsilon_{j1}}^{B_{j1}} f_j(S_j)\, dS_j \prod_{k=1}^{j-1} \varepsilon_{jk} f_k(\hat{S}_k)\, d\hat{S}_k \geqq \alpha_j$$

(8.2) where

$$B_{jk} = \frac{A_j(\gamma) - S_j - \sum_{l=k+1}^{j-1} |1 - \beta_{jl}|\, \hat{S}_l}{|1 - \beta_{jk}|}$$

and

$$A_j(\gamma) = I_0 - I_{\min} + \sum_{l=1}^{j} \gamma_l$$

$$\hat{S}_k = \varepsilon_{jk} S_k$$

$\varepsilon_{jk} = \pm 1$ according to whether $\beta_{jk} \gtrless 1$[27] for $j \geqq 2$.

At $j = 1$ the condition reduces to

$$F_1[A_1(\gamma)] \equiv \int_{-\infty}^{A_1(\gamma)} f_1(S_1)\, dS_1 \geqq \alpha_1 \tag{8.3}$$

or,

$$A_1(\gamma) \geqq F_1^{-1}(\alpha_1). \qquad \text{I.e., } \gamma_1 \geqq F_1^{-1}(\alpha_1) + I_{\min} - I_0 .$$

To satisfy the period one sales constraint it is therefore necessary to determine a value γ_1 which satisfies this inequality. It is similarly necessary to secure limits of integration which satisfy (8.2) in each period $j = 1, 2, \cdots, N$.

[27] When $\beta_{jk} = 1$ the integral involving this term is omitted from the iterated integration.

The values of α_j are all close to unity. Indeed, it may be argued that this will be typical of most situations by recognizing the business significance of constraints such as (3.1) from which (8.2) was derived.[28] The upper limits of the integrals will usually have to be quite large to insure these results—i.e., each integral will have to include most of the area under its density. Recognition of these factors makes it practicable to develop computer routines which require exploration of only relatively small regions at the tails of the densities in order to obtain approximate solutions of sufficient validity.

When large numbers of time periods are involved, recourse to high-speed computation facilities will normally be required.[29] If, however, the β_{jl} are set equal to unity for $j > k \geqq 1$ a considerable simplification is attained.[30] The calculations then proceed in a relatively simple and straightforward manner.

This simplifying assumption will be made here. With all $\beta_{jl} = 1$ the decision rule (6) may be rewritten as

(9.1)
$$\sum_{l=1}^{j} R_l = \sum_{l=1}^{j-1} S_l + \sum_{l=1}^{j} \gamma_l$$
$$R_1 = \gamma_1$$

or,

(9.2)
$$\begin{aligned} R_1 &= \gamma_1 \\ R_2 &= S_1 + \gamma_2 \\ &\cdots \\ R_j &= S_{j-1} + \gamma_j \end{aligned}$$

with the additional conditions (see footnote 20) that $S_{j-1} + \gamma_j \geqq 0$.

The constraints can now be reduced to linear inequalities on the γ_j. Since $\beta_{jl} = 1$, (7.1) becomes

(10.1)
$$\Pr \left\{ I_0 - I_{\min} + \sum_{l=1}^{j} \gamma_l \geqq S_j \right\} \geqq \alpha_j .$$

An equivalent statement is

$$I_0 - I_{\min} + \sum_{l=1}^{j} \gamma_l \geqq F_j^{-1}(\alpha_j),$$

(11) where

$$F_j(x) = \int_{-\infty}^{x} f_j(S_j)\, dS_j , \qquad j = 1, 2, \cdots , N.$$

It should be noted that $F_j^{-1}(\alpha_j)$ is a specific number prescribed by α_j and the frequency distribution of sales in the j^{th} period. Since I_0 , $I_{\min}$ and F^{-1} (α_j) are

[28] Cf. section 13.

[29] An approximate method utilizing so-called Monte Carlo routines has been developed for these calculations.

[30] Normally these values have been found to be close to unity.

known, they provide appropriate stipulations[31] for constraints on the γ values which are determined for (11). After optimization is undertaken the resulting values may be incorporated in (6). In conjunction with past sales (as they materialize) they supply the certainty equivalents by which refinery rates may be programmed.

By a similar development, (7.2) becomes [see (8.1)]:

$$\sum_{l=1}^{j} \gamma_l \leqq I_{\max} - I_0 + \min S_j, \qquad j = 1, 2, \cdots, N. \tag{10.2}$$

and the condition $R_j \geqq 0$ becomes

$$\min. S_{j-1} + \gamma_j \geqq 0, \tag{9.3}$$

since non-negativity is prescribed with probability one.

The expected costs in (5), which are to be minimized, i.e.,

$$\frac{1}{N} \sum_{j=1}^{N} \int_D \{[c_j(R_j) + T_j]R_j + K_j \bar{I}_j\} f(S)\, dS, \tag{12}$$

reduces to a sum of integrals, each over a specified domain, D_i, since, by (9.2) $R_j = S_{j-1} + \gamma_j$. Moreover, only S_{j-1} (and hence only γ_j) is involved in the c_j—i.e., the cost elements which involve R_j nonlinearly.[32] A separable convex function in the γ_j is thus obtained.

Minimization of such functionals, subject to linear inequalities, is (as Charnes and Lemke [2] have shown)[33] amenable to usual procedures employing adjacent extreme point techniques in linear programming to any desired degree of approximation. In this sense the problem may formally be regarded as solved.

Because of other features—e.g., location of subhorizons—it is desirable, however, to explore additional approaches. One such additional approach can be developed by utilizing the necessary and sufficient conditions for minimization of non-linear convex functionals developed by H. W. Kuhn and A. W. Tucker [16]. The resulting analyses and interpretations can be used to simplify and guide the programming calculations. They can also be used to throw additional light on matters of general theoretical interest. It is this latter aspect which will be given primary emphasis in the discussion which follows.

7. A Method for Scheduling by Cost Horizons

The problem of minimizing (12) subject to (11) and (9.3) is a special case of the problem of finding $\hat{\gamma}_j$ that

$$\text{(13)}\quad \text{minimize} \qquad \mathfrak{C} = \sum_{j=1}^{N} \mathfrak{C}_j(\hat{\gamma}_j)$$

[31] See [7].

[32] For purposes of this discussion the transportation costs are taken as linear functions of the scheduled rates of production with the relevant constants known (with certainty) in advance.

[33] See also [4] for further development.

subject to

(14) $$\sum_{i=1}^{j} \hat{\gamma}_i \geqq \sigma_j \qquad j = 1, \cdots, N$$

(15) $$\hat{\gamma}_j \geqq 0 \qquad j = 1, \cdots, N.$$

To see this result, one simply lets $\hat{\gamma}_j = \gamma_j + \min S_{j-1}$ for $j > 1$, $\hat{\gamma}_1 = \gamma_1$, and defines $\mathcal{C}_j$ and σ_j in the obvious way.

In this section we develop a procedure for solving (13)–(15) under the assumptions that for each j the derivative $\mathcal{C}_j'(\hat{\gamma}_j)$ of $\mathcal{C}_j$ exists, that $\mathcal{C}_j'(0) = 0$,[34]

[34] This hypothesis is not essential but simplifies the ensuing discussion.

and that $\mathcal{C}_j'(\hat{\gamma}_j)$ is monotone increasing. The monotonicity of $\mathcal{C}_j'(\hat{\gamma}_j)$ implies that its inverse function $\hat{\gamma}_j(\mathcal{C}_j')$ is also monotone increasing. A final assumption is that $0 \leq \sigma_1 \leq \cdots \leq \sigma_N$. The procedure we are about to describe can be generalized to incorporate restrictions like (10.2). However, in order to simplify the argument, we will not do this.

The necessary and sufficient conditions of optimization described by Kuhn and Tucker [16] are equivalent, for the present problem, to (14), (15), and

(16)
$$\text{(a)} \quad \mathcal{C}_j'(\hat{\gamma}_j) \geqq \mathcal{C}_{j+1}'(\hat{\gamma}_{j+1})$$
$$\text{(b)} \quad \mathcal{C}_j'(\hat{\gamma}_j) = \mathcal{C}_{j+1}'(\hat{\gamma}_{j+1}) \quad \text{whenever} \quad \sum_{i=1}^{j} \hat{\gamma}_i > \sigma_j$$
$$\text{for} \quad j = 1, \ldots, N \quad \text{where} \quad \mathcal{C}_{N+1} \equiv 0.$$

A general characterization of the procedure to be employed is as follows. At each stage a new subhorizon is determined and old ones (possibly) removed. An optimum is thereby obtained for a problem which is more restricted than its predecessor, but less restricted than the problem which is to be solved *in toto*. In at most N stages the solution to the original problem is obtained.

To commence the process, determine c_0 by

(17) $$\sum_{l=1}^{N} \hat{\gamma}_l(c_0) = \sigma_N.$$

The values secured for $\hat{\gamma}_l$ are then optimal for the less restricted problem in which the only production condition is (17). If the $\hat{\gamma}_l(c_0)$ also satisfy the other constraints as well an overall optimum is then achieved.

Suppose that this is not the case. Let

(18) $$j_1 = \text{the first } j \text{ such that } \sum_{l=1}^{j} \hat{\gamma}_l\,(c_0) < \sigma_j.$$

This value, j, is then selected as the new subhorizon corresponding to this stage. New values c_1 and c'_1 are now determined by the respective conditions

$$\sum_{l=1}^{j_1} \hat{\gamma}_l(c_1) = \sigma_{j_1}$$
(19)
$$\sum_{l=j_1+1}^{N} \hat{\gamma}_l(c'_1) = \sigma_N - \sigma_{j_1}.$$

Evidently $c_1 > c_0 > c'_1$. The values $\hat{\gamma}_j(c_1)$ and $\hat{\gamma}_j(c'_1)$ now furnish an optimum for a problem which includes at least one more production condition than its predecessor. It is also less restricted than the total problem so that if these new values of $\hat{\gamma}_l$ satisfy all of the originally stated constraints the overall optimum is achieved.

It will suffice to carry the indicated procedure only one more stage. Suppose, therefore, that one or more of the originally stated constraints are still not satisfied. Let

j_2 be the first $j > j_1$ such that

$$\sum_{l=j_1+1}^{j} \hat{\gamma}_l(c'_1) < \sigma_j - \sigma_{j_1} \tag{20}$$

Determine c_2 such that

$$\sum_{l=j_1+1}^{j_2} \hat{\gamma}_l(c_2) = \sigma_{j_2} - \sigma_{j_1}. \tag{21}$$

Evidently $c_2 > c'_1$. Now if

(i) $c_1 \geqq c_2$ then determine c'_2 by

$$\sum_{j_2+1}^{N} \hat{\gamma}_l(c'_2) = \sigma_N - \sigma_{j_2} \tag{22}$$

Evidently $c'_1 > c'_2$ so that an optimum to a more restricted problem than its predecessor is achieved using c_1, c_2, c'_2 as the "marginal costs" in the ranges indicated.

If

(ii) $c_1 < c_2$

then the horizon at j_1 is to be eliminated. Determine $\hat{c}_2$ such that

$$\sum_{l=1}^{j_2} \hat{\gamma}_l(\hat{c}_2) = \sigma_{j_2} \tag{23}$$

Evidently $c_2 > \hat{c}_2 > c_1$. Also determine $\hat{c}'_2$ such that

$$\sum_{l=j_2+1}^{N} \hat{\gamma}_l(\hat{c}'_2) = \sigma_N - \sigma_{j_2} \tag{24}$$

Evidently $c'_1 > \hat{c}'_2$ so that in either case (i) or (ii) an optimum to a problem which is more restricted than its predecessor is secured.

In passing to further stages the only novel element is the slight extension of case (ii) whereby, say, c_k is greater than several of its predecessor c_j's. The sub-horizons corresponding to the latter indices are then to be eliminated and values $\hat{c}_k$, $\hat{c}'_k$ determined in the fashion made evident by the foregoing discussion. Clearly in at most N stages the overall optimum is achieved.

The lower case variables, c, thus determined are, of course, appropriate values of the $\mathfrak{C}'_j$ satisfying the optimality conditions (16). The inequalities in the text, introduced by the term "evidently", are consequences of the fact that the $\hat{\gamma}_j(\mathfrak{C}'_j)$ are monotone increasing functions.

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IX–34

AN INVENTORY MODEL FOR ARBITRARY INTERVAL AND QUANTITY DISTRIBUTIONS OF DEMAND*

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The inventory problem for continuous time is studied under the following assumptions about the demand process 1) an arbitrary distribution of the length of intervals between successive demands; 2) a distribution of the quantity demanded which is independent of the last quantity demanded and any previous events but may depend on the time elapsed since the last demand; 3) unfilled orders are backlogged. The delivery time is fixed. Costs considered are fixed ordering costs and proportional costs of purchase, storage and shortage. The loss function and the equations for reordering point and minimal ordering quantity are derived. Formulae are calculated for the Poisson, stuttering Poisson, geometric, negative binomial, Gamma and compounded distributions.

1. Introduction[1]

Demand is the most important and difficult element to consider in making good stock control decisions [1]. It is also perhaps the most difficult element to analyze theoretically. The stochastic processes which characterize demand, even in simple realistic situations, are of a discomforting complexity. For this reason, it has been customary to make the very simplest assumptions about demand distributions but to hopefully apply the results of simple models under more general conditions. Thus it is usually assumed that demand is independently and identically distributed in different periods. In the continuous case, this is equivalent to assuming that the intervals between demands are subject to an exponential distribution, and that the quantity demanded at a given trial is either fixed or is identically and independently distributed. Only recently has the case of arbitrary interval distributions been analyzed (Reference 2, Chapters 16 and 17).

In this paper, the continuous inventory problem is studied for arbitrary interval distributions and distributions of the quantity demanded which may depend on the time elapsed since the last demand but which are independent of the last quantity demanded or of anything preceding this. In other words, a stochastic process is considered which is independent of the past at those moments which follow immediately upon a demand.

The approach will yield convenient expressions for the loss function and a set of two equations for the reorder points and the minimum order quantity.

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2. Structure of the Model

A. Demand Distribution

The S, s policy—with constant values of s and S—can be optimal only when the stochastic process by which demand is generated has sufficient stationarity properties. In the following, it will be assumed that this process is purely discontinuous[2] and that after every event the process is again independent of the past.

One way of describing the demand distribution in terms of this process is as follows. The distribution of the length of intervals between successive events is given. Different intervals are independently and identically distributed. Also given is the probability distribution of the quantity demanded at an event. This distribution may depend on the length of the time interval preceding it, but it is independent of any previous (or subsequent) demands.

An alternative description is in terms of the probability that a quantity i is demanded during an interval t following an event. The two descriptions will be related to each other.

B. Supply

Delivery of an order is assumed to require a fixed time, T. No restriction is placed on the size or number of orders that may be outstanding at any time. There is a fixed cost of ordering, k, which is incurred when the order is made.

Per item and per unit item, a carrying cost of h dollars is incurred. Per unit shortage and per unit time a penalty cost of g dollars is incurred. Let y denote the stock level when positive, the shortage level when negative. Then

$$f(y) = \begin{matrix} hy & y \geqq 0 \\ -gy & y < 0 \end{matrix}$$

denotes the storage and shortage cost function. Future costs are discounted by a discount rate of $\hat{a} = e^{-\alpha}$ per unit time. The unavoidable outlay of the purchasing price per item need not be considered: the effect that the timing of this expense has on total cost may be contained in the storage and shortage cost function.

C. Decisions

Decisions concerning whether to order stock replacements and how much to order may be made immediately after a demand has arisen. While this is ob-

[2] For a definition see [3]. Continuous processes may occur when the commodity under consideration is infinitely divisible, such as a liquid, and its rate of consumption is a random variable. The examination of some stationary continuous processes has shown that the S, s policy is always optimal and that the principal formula (8) to be derived below remains valid provided the sum is replaced by an integral. Consideration of the more typical discontinuous stochastic processes is therefore no real restriction of generality.

viously less general than the case in which decisions may be made at any time, This restriction does not seem to be of practical importance.

D. Translation in Time

Since delivery requires a time of exactly T, it is natural to associate with the present time not the inventory cost which is incurred now but that which is incurred T units of time later.

E. Horizon

The planning horizon is infinite, i.e., no definite duration of the program is envisaged. However, by adjusting the discount rate and carrying cost, one can allow for the possibility that with the constant probability $\mu\, dt$, the system will be terminated in the next dt units of time, and that the salvage value of the inventory (both stock-on-hand and on-order) is a fraction of the unit cost of an item, so that a loss of c dollars is incurred upon termination. Of course, c may be zero when the stock can be liquidated at its full value.

F. Object

The decision objective is to minimize the expected value of present and discounted future avoidable cost. Unavoidable costs are, for instance, those incurred between now and time T (i.e., during the delivery time) because no decisions we now make can take effect before a time span T has elapsed.

It will be shown that this is equivalent to a first degree of approximation to minimizing cost averaged over time.

This expected cost—or loss function—depends on two "state variables" of the system: the inventory on-hand plus on-order, and the time that has elapsed since the last demand. However, we shall restrict our consideration of the loss function to times immediately after demand. Our first task is to formulate a recursive or "dynamic programming" equation for the loss function $L(y)$, i.e., for the expected value of discounted avoidable cost at a time immediately after a demand, conditional on inventory y and an optimum policy.

G. Probability Concepts

The following probabilities are used

$p(i, t)$ the probability that i units are demanded during an interval of length t immediately following a demand

$q(t)\, dt$ the probability that the interval between successive demands lies between t and $t\, dt$

$\pi_j(t)$ the conditional probability that the quantity demanded is j given that the time interval since the last demand is t

$\rho(n, t)$ the probability that there will be n occasions of demand during an interval of length t following a demand

The following identities will be used

$$p(0, t) = \int_t^\infty q(\tau)\, d\tau$$

$$p(i, t) = \sum_{n=1}^{\infty} \int_0^t [\pi_i(\tau)q(\tau)]^{(n)}[1 - Q(t - \tau)]\, d\tau \qquad (i > 0),$$

where $[\quad]^{(n)}$ denotes the nth convolution of the probability in brackets, defined recursively as

$$[\pi_i(t)q(t)]^{(n)} = \sum_{j=0}^{i} \int_0^t [\pi_{i-j}(t - \tau)q(t - \tau)]^{(n-1)}\pi_j(\tau)q(\tau)\, d\tau$$

$$[\pi_i(t)q(t)]^{(1)} = \pi_i(t)q(t),$$

and

$$Q(t) = \int_0^t q(\tau)\, d\tau$$

is the probability that a demand occurs in time t or less; hence $1 - Q(t - \tau)$ is the probability of no occasion for demand during an interval of length $t - \tau$ following after a demand.

When $\pi_i(t) = \pi_i$ is independent of t, this formula reduces to

$$p(i, t) = \sum_{n=0}^{\infty} \rho(n, t)\pi_i^{(n)},$$

where $\rho(n, t)$ is the probability of n occasions of demand during an interval of length t following a demand (see Section 6, B, 3).

3. The Inventory Equation

Consider the system immediately after a demand has occurred and after we have taken the appropriate action (whether to order and if so, how much). Let the stock (on-hand and on-order) be y. With probability $q(t)\, dt$ the next demand occurs t units of time later. Consider first the cost of storage or shortage during an interval of length t but placed T units of time later. What is the expected value of this cost conditional on y and t?

Suppose first that $t < T$. With probability $\pi_j(t)$, j units are then demanded. In the interval $\tau - t$, $i - j$ further units are demanded with probability $p(i - j, \tau - t)$. Thus the probability of physical stock being $y - i$ at a time τ, $T \leqq \tau \leqq T + t$, is

$$\sum_{j=0}^{i} \int_0^t q(t)\pi_j(t)p(i - j, \tau - t)\, dt$$

Discounting and integrating over the interval $T \leqq \tau \leqq T + t$, we have the expected cost

$$\sum_{j=0}^{i} \int_0^T q(t)\pi_j(t) \int_T^{T+t} p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \cdot f(y - i).$$

Second, let $t \geqq T$. Then physical stock equals y during $T \leqq \tau \leqq t$. For any time τ in the interval $t \leqq \tau \leqq t + T$, the probability of a stock level $y - i$ is, as before,

$$\sum_{j=0}^{i} \pi_j(t)p(i - j, \tau - t).$$

Discounting and integrating we have

$$\int_T^\infty q(t)\left[\int_T^t e^{-\alpha\tau}\, d\tau f(y) + \int_t^{t+T} \sum_{j=0}^{i} \pi_j(t)p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau f(y - i)\right] dt$$

The total cost is therefore

$$\begin{aligned} F(y) = \sum_{i=0}^{\infty}\Bigg[\int_0^T q(t) \sum_{j=0}^{i} \pi_j(t) \int_T^{T+t} p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \\ + \int_T^\infty q(t) \sum_{j=0}^{i} \pi_j(t) \int_t^{T+t} p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau\, dt\Bigg] f(y - i) \\ + \int_T^\infty q(t)\, \frac{1}{\alpha}\, [e^{-\alpha T} - e^{-\alpha t}]\, dt f(y) \qquad (1) \\ = e^{-\alpha T} \sum_{j=0}^{\infty} br_j f(y - j) \end{aligned}$$

where

$$b = \frac{1}{\alpha} \int_0^\infty q(t)[1 - e^{-\alpha t}]\, dt = \int_0^\infty q(t) \int_0^t e^{-\alpha\tau}\, d\tau\, dt$$

and

$$\begin{aligned} r_0 = \frac{e^{\alpha T}}{b} \int_0^T q(t)\pi_0(t) \int_T^{T+t} p(0, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \\ + \frac{e^{\alpha T}}{b} \int_T^\infty q(t)\pi_0(t) \int_t^{T+t} p(0, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \\ + \frac{e^{\alpha T}}{b} \int_T^\infty q(t)\, \frac{1}{\alpha}\, [e^{-\alpha T} - e^{-\alpha t}]\, dt \qquad (2) \\ r_i = \frac{e^{\alpha T}}{b} \int_0^T q(t) \sum_{j=0}^{i} \pi_j(t) \int_T^{T+t} p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \\ + \frac{e^{\alpha T}}{b} \int_T^\infty q(t) \sum_{j=0}^{i} \pi_j(t) \int_t^{T+t} p(i - j, \tau - t)e^{-\alpha\tau}\, d\tau\, dt \end{aligned}$$

Here, b is so defined as to make

$$\sum_{i=0}^{\infty} r_i = 1$$

Since also $r_i \geqq 0$, the r_i may be regarded as probabilities. When demands are for one unit at a time, $\pi_1(t) \equiv 1$ and $\pi_j(t) = 0 (j \neq 1)$, and we have the simpler

expressions

$$r'_0 = \frac{e^{\alpha T}}{b} \frac{1}{\alpha} \int_T^\infty q(t)[e^{-\alpha T} - e^{-\alpha t}]\, dt$$

$$r'_i = \frac{e^{\alpha T}}{b} \int_0^T q(t) \int_T^{T+t} p(i-1, \tau - t) e^{-\alpha\tau}\, d\tau\, dt \qquad (2)$$

$$+ \frac{e^{\alpha T}}{b} \int_T^\infty q(t) \int_t^{T+t} p(i-1, \tau - t) e^{-\alpha\tau}\, d\tau\, dt$$

Observe that since $F(y)$ is a weighted sum, with non-negative weights of convex functions $f(y - i)$, $F(y)$ is convex.

Let $L(x)$ denote the total expected discounted cost over the infinite time horizon when an optimal policy is followed and x is the amount of stock on hand and on order immediately after a demand has occurred but before a decision whether or not to order is made. Denote by y the amount of stock on hand and on order after an order is placed. Clearly $y(\geqq x)$ should be chosen so as to minimize

$$k\delta(y - x) + F(y) + \sum_{i=0}^{\infty} \left\{ \int_0^\infty \pi_i(t) q(t) e^{-\alpha t}\, dt \right\} L(y - i) \qquad (3)$$

where

$$\delta(z) = \begin{cases} 0, & z = 0 \\ 1, & z > 0. \end{cases}$$

Hence

$$L(x) = \min_{y \geqq x} \left[k\delta(y - x) + F(y) + \sum_{i=0}^{\infty} \left\{ \int_0^\infty \pi_i(t) q(t) e^{-\alpha t}\, dt \right\} L(y - i) \right].$$

Define

$$\int_0^\infty \pi_i(t) q(t) e^{-\alpha t}\, dt = a p_i$$

where

$$a = \int_0^\infty q(t) e^{-\alpha t}\, dt$$

Since

$$\sum_{i=0}^{\infty} p_i = \frac{1}{a} \int_0^\infty \sum_{i=0}^{\infty} \pi_i(t) q(t) e^{-\alpha t}\, dt$$

$$= \frac{1}{a} \int_0^\infty q(t) e^{-\alpha t}\, dt = 1$$

and $p_i \geq 0$, p_i may be regarded as a probability. Now

$$L(x) = \min_{y \geqq x} \left[k\delta(y - x) + F(y) + a \sum_{i=0}^{\infty} p_i L(y - i) \right] \tag{4}$$

In the special case of unit demands

$$L(x) = \min_{y \geqq x} [k\delta(y - x) + F(y) + aL(y - 1)] \tag{5}$$

If maintaining an inventory is at all optimal, there will be integers s, $S (S > s)$ such that if $x \leqq s$, the minimizing value of y in (5) is S while if $x > s$, the minimizing value of y in (5) is x [6]. We then obtain the following difference equation from (5):

$$\begin{aligned} L(x) &= F(x) + aL(x - 1) & x &> s \\ L(x) &= k + L(S) & x &\leqq s \end{aligned}$$

Thus

$$L(x) = \sum_{i=0}^{x-s-1} F(x - i)a^i + a^{x-s}[k + L(S)] \qquad x > s.$$

In particular,

$$L(S) = \sum_{i=0}^{D-1} F(y - i)a^i + a^D[k + L(S)] \quad \text{where} \quad D = S - s$$

or

$$L(S) = \frac{a^D k + \sum_{i=0}^{D-1} F(y - i)a^i}{1 - a^D}.$$

When $s + 1 > 0$ as we may assume in all practical cases

$$L(0) = k + L(S)$$

$$L(0) = \frac{k + \sum_{i=0}^{D-1} F(y - i)a^i}{1 - a^D} \tag{6}$$

This formula was derived in [4] for the special case that demand is described by a Poisson process.

Returning to the general equation (4), observe that it is an Arrow-Harris-Marschak equation [5]. Since $F(y)$ is convex, Scarf's theorem [6] applies, stating that the optimal policy is an s,S policy

$$y = \begin{cases} S \text{ if } x \leqq s \\ x \text{ if } x > s. \end{cases}$$

Let $p_i^{(n)}$ denote the nth convolution of p_i and let $w_n(i)$ be the probability that n

trials are required to yield a total demand of i units or more. (This is of course based on p_i). Then the solution of (4) is

$$L(x) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{x-s-1} a^n p_i^{(n)} F(x-i) + a^n w_n(x-s)[k + L(S)]\right), x \geqq s$$

In particular,

$$L(S) = \frac{k\sum_{n=0}^{\infty} a^n w_n(D) + \sum_{n=0}^{\infty}\sum_{i=0}^{D-1} a^n p_i^{(n)} F(S-i)}{1 - \sum_{n=0}^{\infty} a^n w_n(D)}$$

and if $s > 0$ $\quad L(0) = L(S) + k$

$$L(0) = \frac{k + \sum_{n=0}^{\infty}\sum_{i=0}^{D-1} a^n p_i^{(n)} F(S-i)}{1 - \sum_{n=0}^{\infty} a^n w_n(D)} \tag{7}$$

4. Discussion of the Loss Function

A. Alternative Formulation

In Equation (7), the timing and accumulation of demand appear separately. The first is expressed by a and its powers, the second by the probabilities $p_i^{(n)}$ and $w_n(D)$. An alternative expression can be given which refers to the basic probabilities $p(i, t)$, where demand equals i in an interval of length t (which starts immediately after a demand), which describe the stochastic process.[3] We will show in the appendix that (7) is equivalent to:

$$L(0) = \frac{k + \sum_{i=0}^{D-1} \frac{F(S-i)}{b} \int_0^{\infty} p(i, t) e^{-at}\, dt}{a \int_0^{\infty} \sum_{i=0}^{D-1} p(i, t) e^{-at}\, dt} \tag{8}$$

The relationship of $p(i, t)$ to the probabilities $q(t)$, $\pi_i(t)$ of the previous analysis is exhibited by the definition of $p(i, t)$ on p. 425.

Formula (8) is convenient when the detailed structure of the process that generates demand is not known, but the demand probabilities for time intervals (after demands) of any length are known.

The numerator of this formula is intuitively obvious; it represents the expected value of storage and shortage cost, properly discounted, through an inventory cycle. To interpret the denominator, note the identity

$$\int_{t+0}^{\infty} q(t, D)\, dt = \sum_{i=0}^{D-1} p(i, t),$$

[3] A general discussion of the relationship between the spacing of events and the cumulative number of events that occur during an interval of time may be found in [7].

where $q(t, i)$ is the probability that demand first equals i at time t. Differentiating with respect to t,

$$-q(t, D) = \sum_{i=0}^{D-1} \frac{dp(i, t)}{dt}.$$

Consider

$$1 - \int_0^\infty q(t, D)e^{-\alpha t}\, dt = 1 + \int_0^\infty e^{-\alpha t} \frac{d}{dt} \sum_{i=0}^{D-1} p(i, t)\, dt$$

$$= \alpha \int_0^\infty \sum_{i=0}^{\infty} p(i, t)e^{-\alpha t}\, dt,$$

after integration by parts. Thus,

$$L(0) = \frac{k + \sum_{i=0}^{D-1} \frac{1}{b} F(S - i) \int_0^\infty p(i, t)e^{-\alpha t}\, dt}{1 - \int_0^\infty q(t, D)e^{-\alpha t}\, dt}. \tag{8'}$$

The meaning of this denominator becomes clear when we use the approximation $e^{-\alpha dt} = 1 - \alpha\, dt$, which is legitimate since α is small and $q(t, D)$ is negligible for large t:

$$\int_0^\infty q(t, D)e^{-\alpha t}\, dt \doteq 1 - \alpha \int_0^\infty tq(t, D)\, dt = 1 - \alpha\bar{t},$$

say, where $\bar{t}$ denotes the average length of an inventory cycle. Thus,

$$L(0) \doteq \frac{k + \sum_{i=0}^{D-1} 1/b\, F(S - i) \int_0^\infty p(i, t)e^{-\alpha t}\, dt}{\alpha\bar{t}} \tag{8''}$$

Up to a factor of proportionality, Equation (8″) and hence, (8), is approximately the average cost of the system per unit of time. The same can be shown with Equation (6) or (7). In terms of $f(y)$, Equation (8) is given by

$$L(0) = \frac{k + e^{-\alpha T} \sum_{i=0}^{D-1} \sum_{j=0}^{\infty} f(S - i - j) r_j \int_0^\infty p(i, t)e^{-\alpha t}\, dt}{\alpha \sum_{i=0}^{D-1} \int_0^\infty p(i, t)e^{-\alpha t}\, dt}.$$

Finally, for convenience, let

$$e^{-\alpha T} \int_0^\infty p(i, t)e^{-\alpha t}\, dt = e^{-\alpha T} b \sum_{n=0}^{\infty} a^n p_i^{(n)} \qquad \text{by (7), (8)}$$

$$= u_i,$$

whereupon,

$$L(0) = \frac{ke^{\alpha T} + \sum_{i=0}^{D-1} \sum_{j=0}^{\infty} u_i r_j f(S - i - j)}{\alpha \sum_{i=0}^{D-1} u_i}. \tag{8'''}$$

5. Reorder Point and Reorder Quantity Determination

A. Determination of s and D for Unit Demand

Before deriving general formulas for the s and D which minimize the loss function, it is convenient to consider these parameters in the case where demands are for one unit at a time. For simplicity, α is set to zero so that average cost is minimized per unit of time (see page 430, above).

The definition of u_i shows that since $p_i^{(n)} = \delta_{ni}$, $\alpha = 0$, and $a = 1$, then $u_i = b$. Therefore,

$$\lim_{\alpha \to 0} \alpha L(0) = \frac{k + b \sum_{i=0}^{D-1} \sum_{j=0}^{\infty} r_j f(s + D - i - j)}{bD} = \bar{L}_{s,D}, \tag{9}$$

represents indeed average cost per unit time, b being the average time between successive demands. A simpler expression for (9) is

$$\bar{L}_{s,D} = \frac{k + \sum_{i=0}^{D-1} F(s + D - i)}{bD} \tag{10}$$

The optimal s is a smallest integer for which

$$\bar{L}_{s+1,D} - \bar{L}_{s,D} \geq 0;$$

i.e.,

$$\sum_{i=0}^{D-1} \Delta_s F(s + D - i) = F(s + D - 1) - F(s + 1) \geq 0,$$

or

$$\sum_{j=0}^{\infty} r_j [f(S + 1 - j) - f(s + 1 - j)] \geq 0. \tag{11}$$

The storage and shortage costs will now be specified as follows:

$$f(y) = \begin{cases} hy & y \geq 0 \\ -gy & y < 0. \end{cases} \tag{12}$$

The inequality (11) assumes the form:

$$\sum_{j=0}^{S} h \cdot (S + 1 - j) r_j + \sum_{j=S+1}^{\infty} g \cdot (j - S - 1) r_j - \sum_{j=0}^{s} h \cdot (s + 1 - j) r_j - \sum_{j=s+1}^{\infty} g \cdot (j - s - 1) r_j \geq 0.$$

Since $\sum_{j=0}^{\infty} r_j = 1$, this becomes

$$(h + g) D \sum_{j=0}^{s} r_j + (h + g) \sum_{j=s+1}^{S} (S + 1 - j) r_j - gD \geq 0.$$

Writing

$$R_y = \sum_{j=0}^{y} r_j ,$$

$$R_s + \frac{1}{D} \sum_{i=0}^{D-1} i r_{s+i+1} \geq \frac{g}{g+h} \tag{13}$$

This may be interpreted as follows: The left-hand side represents a weighted average of R_y, say $R_{s+\theta}$, where $0 < \theta < D$. Expression (13) says that the probability of demand exceeding $s + \theta$ should be kept below $h/(h + g)$, a small fraction since generally g is large compared to h. A conservative solution is to set $\theta = 0$, and have $R_s \geq g/(g + h)$.

Similarly, the optimal D is the smallest integer for which $\bar{L}_{s,D+1} - \bar{L}_{s,D} \geq 0$; i.e.,

$$\Delta_D \left[\sum_{i=0}^{D-1} F(s + D - i) - \left(k + \sum_{i=0}^{D-1} F(s + D - i) \right) \right] \geq 0,$$

which implies

$$\sum_{i=0}^{D-1} [F(S + 1) - F(S - i)] \geq k.$$

Substituting for F,

$$\sum_{i=0}^{D-1} \sum_{j=0}^{\infty} b r_j [f(S + 1 - j) - f(S - i - j)] \geq \frac{k}{b},$$

and substituting for f,

$$\sum_{i=0}^{D-1} \left\{ \sum_{j=0}^{S-i-1} h \cdot (1 + i) r_j + \sum_{j=S-i}^{S} [h \cdot (S + 1 - j) - g \cdot (i + j - S)] r_j \right.$$
$$\left. - \sum_{j=S+1}^{\infty} g \cdot (1 + i) r_j \right\} \geq \frac{k}{b} \tag{14}$$

By definition of r_j, r_j is small for large j. If, as a rough approximation, r_j is set to zero for $j > s$, then (14) becomes simply

$$\sum_{i=0}^{D-1} h(1 + i) \geq \frac{k}{b},$$

or

$$\frac{hD(D + 1)}{2} \geq km,$$

where $m = 1/b$ is the average demand per unit time. This may be written

$$D(D + 1) \geq \frac{2km}{h},$$

or

$$D \doteq \sqrt{\frac{2km}{h}}, \tag{14'}$$

which is the Wilson lot-size formula. Without this simplification, (14) leads to

$$\sum_{i=0}^{D-1} \left\{ \left[(i+1) \sum_{j=0}^{S-i-1} r_j + \sum_{j=S-i}^{S} (S+1-j) r_j \right] (g+h) - (i+1) g \right\} \geq \frac{k}{b} \tag{15}$$

B. Determination of s and D in General

Substituting (12) for $f(y)$ in (8‴),

$$L = \frac{ke^{\alpha T} + \sum_{i=0}^{D-1} u_i \left[\sum_{j=0}^{S-i} h \cdot (S-i-j) r_j + \sum_{j=S-i+1}^{\infty} g \cdot (i+j-S) r_j \right]}{\alpha \sum_{i=0}^{D-1} u_i} \tag{16}$$

Adding and subtracting $\sum_{j=0}^{S-i} g \cdot (i+j-S) r_j$,

$$L = \frac{k + \sum_{i=0}^{D-1} u_i \left[\sum_{j=0}^{S-i} (h + g(S-i-j) r_j + g \cdot (i + \bar{j} - S) \right]}{\alpha \sum_{i=0}^{D-1} u_i},$$

where $\bar{j} = \sum_{j=0}^{\infty} j r_j$. Setting $S = s + D$ and $R_y = \sum_{i=0}^{y} r_i$,

$$L = \frac{k + \sum_{i=0}^{D-1} u_i [(h+g) R_{s+D-i-1} + g \cdot (i + \bar{j} - s - D)]}{\alpha \sum_{i=0}^{D-1} u_i}$$

Differencing with respect to s yields the following condition of optimality: The optimal s is the smallest integer for which

$$\frac{\sum_{i=0}^{D-1} u_i R_{s+D-i}}{\sum_{i=0}^{D-1} u_i} \geq \frac{g}{g+h} \tag{17}$$

This formula is reminiscent of the "newsboy" equation,

$$P_s = \frac{h}{g+h} \tag{17'}$$

Differencing (16) with respect to D, the following condition for optimality results:

$$\alpha \sum_{i=0}^{D-1} u_i \left\{ u_D[(h+g)R_s + g(\bar{\jmath} - 1 - s)] + \sum_{i=0}^{D-1} u_i[(h+g)R_{s+D-i-1} - g] \right\}$$
$$- \alpha u_D \left\{ k + \sum_{i=0}^{D-1} u_i[(h+g)R_{s+D-i} + g(i + \bar{\jmath} - s - D)] \right\} \geq 0.$$

From this, the optimal D is a smallest integer for which

$$\sum_{i=0}^{D-1} \left[g(D-i-1) - (h+g) \sum_{j=s+1}^{s+D-i-1} R_j \right] u_i u_D + \sum_{i=0}^{D-1} \left[(h+g)R_{s+D-i} - g \right] u_i \sum_{j=0}^{D-1} u_j \geq k u_D e^{\alpha T}. \tag{18}$$

Notice that by (17), the second term on the left-hand side of (18) is nearly zero but non-negative. A good approximation for (18) is therefore

$$\sum_{i=0}^{D-1} \left[g(D-i-1) - (h+g) \sum_{j=1}^{D-i-1} R_{s+j} \right] u_i \geq k e^{\alpha T}. \tag{18'}$$

Of course s and D are the joint solutions of the equation system (17) and (18). The numbers r_j and u_i are known constants whose properties and values for special distributions will be discussed in Section 6 below. At this stage, no particular recommendations can be made concerning efficient ways of solving (17) and (18). Presumably, some iteration method is appropriate, starting with trial values obtained from the Wilson lot-size formula (14′) and the newsboy equation (17′).

C. Special Case of High Value, Low Demand

High value items are sufficiently costly, compared to the fixed ordering cost, that they are ordered each time a demand has occurred, implying that $D = 1$. Conditions (17) and (18) specify the precise conditions under which this is the optimum policy. For this case,

$$R_{s+1} \geq \frac{g}{g+h} \text{ and } R_{s+1} \geq \frac{g + ke^{\alpha T}(u_1/u_0^2)}{g+h} \tag{19}$$

The former condition is the definition of s. If k is sufficiently small, the second condition follows.

Low demand may be taken to mean that stocking one unit is sufficient. A necessary condition for this to be the optimal policy is that (19) be satisfied with $s = 0$.

6. Special Distributions

A. Demand for One Unit at a Time

Generally, if demand occurs one unit at a time,

$$u_i = ba^i$$

$$br'_i = \int_0^T q(t)e^{-\alpha t} \int_{T-t}^T p(i-1,\tau)e^{-\alpha\tau}\,d\tau\,dt$$
$$+ \int_T^\infty q(t)e^{-\alpha t} \int_0^T p(i-1,\tau)e^{-\alpha\tau}\,d\tau\,dt.$$

Integrating the first integral by parts yields

$$br'_i = e^{-\alpha T} \int_0^T p(i-1, T-t)e^{\alpha t} \int_t^\infty q(\tau)e^{-\alpha\tau}\,d\tau\,dt.$$

Define $Q(t) = \int_0^t q(\tau)\,d\tau$ and observe that since demand is for one unit at a time,

$$p(i, t) = Q^{(i)}(t) - Q^{(i+1)}(t),$$

where $Q^{(i)}$ is the ith convolution of Q. Disregarding discount factors,

$$br'_i = \int_0^T [Q^{(i-1)}(T-t) - Q^{(i)}(T-t)][1 - Q(t)]\,dt.$$

Since

$$\int_0^T Q(t)Q^{(n)}(T-t)\,dt = \int_0^T Q^{(n+1)}(t)\,dt,$$

the following results:

$$br'_i = \int_0^T [Q^{(i+1)}(t) - 2Q^{(i)}(t) + Q^{(i-1)}(t)]\,dt$$

1. *Gamma Distribution of Interval*

Consider the gamma distribution

$$q(t) = \frac{(\lambda t)^{m-1}}{\Gamma(m)} \lambda e^{-\lambda t},$$

with integral m. Since

$$q^{(i)}(t) = \frac{(\lambda t)^{mi-1}}{\Gamma(mi)} \lambda e^{-\lambda t}$$

and

$$Q^{(i)}(t) = 1 - \sum_{j=0}^{mi-1} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$
$$= 1 - \sum_{j=0}^{mi-1} p_j(\lambda t),$$

say, then

$$br'_i = \int_0^T \left[- \sum_{j=0}^{m(i+1)-1} + 2 \sum_{j=0}^{mi-1} - \sum_{j=0}^{m(i-1)-1} \right] p_j(\lambda t)\, dt$$
$$= \int_0^T \left[\sum_{j=m(i-1)}^{mi-1} - \sum_{j=mi}^{m(i+1)-1} \right] p_j(\lambda t)\, dt.$$

Now,

$$\int_0^T p_j(\lambda t)\, dt = \frac{1}{\lambda}\left[1 - \sum_{n=0}^{j} p_n(\lambda T)\right]$$
$$= \frac{1}{\lambda}\left[1 - P_j(\lambda T)\right],$$

and

$$br'_i = \frac{1}{\lambda} \sum_{j=mi}^{mi+m-1} P_j(\lambda T) - \frac{1}{\lambda} \sum_{j=mi-m}^{mi-1} P_j(\lambda T)$$
$$= \frac{1}{\lambda} \sum_{j=mi-m}^{mi-1} \sum_{n=j+1}^{j+m} p_n(\lambda T).$$

Since $b = m/\lambda$,

$$r'_i = \frac{1}{m} \sum_{j=m(i-1)}^{mi-1} \sum_{n=j+1}^{j+m} p_n(\lambda T).$$

In particular, for exponential interval distributions, $m = 1$ and

$$r'_i = p_i(\lambda T).$$

B. Multiple Demand

1. *Fixed-Length Intervals*

Let θ denote the length of the period between successive demands.

Case a: $\theta > T$. Disregarding discounting,

$$br_i = \int_0^T \sum_{j=0}^{i} \pi_j\, p(i - j, t)\, dt \qquad (i > 0)$$
$$(\pi_j = \pi_j(\theta))$$
$$= \int_0^T p(i, t + \theta)\, dt.$$

But since $\theta > T$, there is exactly one occasion for demand in the interval $T \leq t \leq T + \theta$, namely at $t = \theta$. Hence, $br_i = \pi_i T$. Now

$$b = \int_0^\infty t q(t)\, dt = \theta.$$

Thus

$$r_i = \pi_i \frac{T}{\theta}.$$

$$br_0 = \pi_0 T + \int_T^\infty (t - T)q(t)\,dt = \pi_0 T + \theta - T$$

$$= \theta + (\pi_0 - 1)\,T$$

$$r_0 = 1 - (1 - \pi_0)\frac{T}{\theta}.$$

Case b: $\theta < T$.

$$br_i = \sum_{j=0}^{i} \pi_j \int_{T-\theta}^{T} p(i - j, t)\,dt$$

$$= \int_T^{T-\theta} p(i, t)\,dt$$

$$= \left(\left[\frac{T}{\theta}\right]\theta + \theta - T\right)\pi_i\left(\left[\frac{T}{\theta}\right]\right) + \left(T - \left[\frac{T}{\theta}\right]\theta\right)\pi_i\left(\left[\frac{T}{\theta}\right] + 1\right),$$

where $[x]$ denotes the largest integer not exceeding x. If θ divides T, say $T = N\theta$, then

$$br_i = \theta\pi_i^{(N)}$$

$$r_i = \pi_i^{(N)}.$$

In any case,

$$u_i = \int_0^\infty p(i, t)\,e^{-at}\,dt = b\sum_{n=0}^{\infty} a^n\pi_i^{(n)} = b\sum_{n=0}^{\infty} \pi_i^{(n)}$$

when discounting is disregarded.

When demand occurs at regularly spaced points in time, essentially a period model is obtained. The present model, although formulated in continuous terms, contains the period model as a special case.

The following distributions for π_i are of particular interest in this connection.

a. *Geometric Distribution*

$$\pi_i = qp^i \quad (i = 0, 1, 2, \cdots)$$

$$\pi_i^{(n)} = \binom{n + i - 1}{i} q^n p^i \qquad \text{(a negative binominal distribution).}$$

$$u_i = \frac{1}{1 - aq}\left(\frac{p}{1 - aq}\right)^i$$

$$= \frac{1}{p} \qquad \text{when discounts are disregarded.}$$

b. *Geometric Distribution With* $\pi_0 = 0$

This distribution is obtained when a replacement part will fail with probability p upon installation. The probability that i parts are needed to make a successful replacement is given by the distribution

$$\pi_i = qp^{i-1} \qquad (i = 1, 2, 3, \cdots)$$

The generating function is $g(x) = qx/(1 - px)$,

$$g^n(x) = q^n x^n (1 - px)^{-n}$$

$$\pi_i^{(n)} = \binom{i-1}{i-n} q^n p^{1-n} = \binom{i-1}{n-1} \left(\frac{q}{p}\right)^n p^i.$$

This is a negative binomial in terms of the variable $i - n$.

$$u_i = aq(p + aq)^{i-1}$$

$$= q \qquad \text{if discounts are disregarded.}$$

c. *Poisson Distribution*

$$\pi_i = \frac{\rho^i e^{-\rho}}{i!}$$

$$\pi_i^{(n)} = \frac{(n\rho)^i e^{-n\rho}}{i!}$$

$$u_i = \frac{\rho^i}{i!} \sum_{n=0}^{\infty} n^i c^n, \qquad \text{where } c = ae^{-\rho}.$$

This may be evaluated by successive application of the operator $c(d/dc)$ to the series

$$\frac{1}{1-c} = \sum_{n=0}^{\infty} c^n \qquad (|c| < 1).$$

It can be shown that

$$\sum_{n=0}^{\infty} n^i c^n = \frac{M_i(c)}{(1-c)^{i+1}},$$

where $M_i(c)$ is a polynomial in c of degree i, whose coefficients are all positive and add up to $i!$

2. *Poisson Process with Independent Increments*

Let intervals between demand be exponentially distributed, $q(t) = \lambda e^{-\lambda t}$, and let the quantity demanded be independently and identically distributed; $\pi_j(t) = \pi_j$. It is easily shown that both mean and variance of this process are strictly proportional to time.

$$b = \int_0^\infty \lambda e^{-\lambda t} \int_0^t e^{-\alpha\tau}\, d\tau\, dt$$

$$= \frac{1}{\alpha + \lambda}$$

$$br_i = \sum_{j=0}^{i} \pi_j \left[\int_0^T \lambda e^{-\lambda t} \int_T^{T+t} p(i-j, T-t)\, e^{-\alpha\tau}\, d\tau\, dt \right.$$

$$\left. + \int_T^\infty \lambda e^{-\lambda t} \int_t^{T+t} p(i-j, T-\tau)\, e^{-\alpha\tau}\, d\tau\, dt \right],$$

which, upon integration by parts, becomes

$$br_i = \sum_{j=0}^{i} \pi_j \frac{1}{\alpha+\lambda} \int_0^T \lambda e^{-\lambda t}\, p(i-j, T-t)\, dt$$

$$= \frac{1}{\alpha+\lambda}\, p(i, T),$$

so that

$$r_i = p(i, T).$$

The generating function for this process is

$$G(x, t) = e^{-\lambda t + \lambda t g(x)}.$$

Hence,

$$\sum_{i=0}^{\infty} u_i x^i = \int_0^\infty G(x, t)\, e^{-\alpha t}\, dt$$

$$= \frac{1}{\lambda - (\alpha+\lambda) g(x)}$$

$$= \frac{1}{\alpha+\lambda} \frac{1}{1 - ag(x)}$$

$$= b \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} a^n \pi_i^{(n)} x^i,$$

so that again as in the fixed interval case,

$$u_i = \sum_{n=0}^{\infty} a^n \pi_i^{(n)}.$$

The expressions for u_i obtained in the fixed interval case are therefore applicable. New calculations are needed, however, for the compound distributions

$$r_i = p(i, T) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \pi_i^{(n)}.$$

a. *Geometric Distribution: Stuttering Poisson Process*

$$\pi_i = qp^i$$

$$p(i, T) = p^i e^{-\lambda pT} \sum_{j=1}^{i} \binom{i-1}{j-1} \frac{(\lambda qT)^j}{j!}.$$

b. *Geometric Distribution With* $\pi_0 = 0$

$$\pi_i = qp^{i-1} \qquad (i = 1, 2, \cdots)$$

$$p(i, T) = p^i e^{-\lambda T} \sum_{j=0}^{i+1} \frac{(\lambda qT)^j}{p^j j!}.$$

c. *Poisson Distribution*

$$\pi_i = \frac{\rho^i e^{-\rho}}{i!}$$

$$p(i, T) = \frac{\rho^i}{i!} e^{-\lambda T} \sum_{n=0}^{\infty} \frac{n^i}{n!} c^n.$$

where $c = \lambda T e^{-\rho}$.

3. *Gamma Distributed Interval*

Let

$$q(t) \doteq \frac{(\lambda t)^{m-1} \lambda e^{-\lambda t}}{\Gamma(m)} \text{ and } \pi_j(t) = \pi_j .$$

Omitting discounts,

$$b = \int_0^\infty tq(t) = \bar{t} = \frac{m}{\lambda}$$

$$r_i = \frac{\lambda}{m} \int_0^T q(t) \sum_{j=0}^{i} \pi_j \int_T^{T+t} p(i-j, t-\tau)\, d\tau\, dt$$

$$+ \frac{\lambda}{m} \int_T^\infty q(t) \sum_{j=0}^{i} \pi_j \int_t^{T+t} p(i-j, \tau - t)\, d\tau\, dt.$$

Since $[q(t)\pi_i]^{(n)} = q^{(n)}(t)\pi_i^{(n)}$,

$$p(i, t) = \sum_{n=0}^{\infty} \int_0^t q^{(n)}(\tau)\pi_i^{(n)} [1 - Q(t-\tau)]\, d\tau$$

$$= \sum_{n=0}^{\infty} \pi_i^{(n)} \rho(n, t) \qquad \text{(see Section 3, A)}.$$

Thus, whenever $\pi_i(t) = \pi_i$,

$$r_i = \frac{1}{b}\int_0^T q(t) \sum_{n=0}^{\infty} \left[\int_{T-t}^{T} q(n, \tau)\, d\tau \sum_{j=0}^{i} \pi_{i-j}^{(n)} \pi_j\right] dt + \frac{1}{b}\int_T^{\infty} q(t) \sum_{n=0}^{\infty} \left[\int_0^T q(n, \tau)\, d\tau \sum_{j=0}^{i} \pi_{i-j}^{(n)} \pi_j\right] dt$$

$$= \frac{1}{b}\int_0^T q(t) \int_{T-t}^{T} \sum_{n=0}^{\infty} q(n, \tau)\pi_i^{(n+1)}\, d\tau\, dt + \frac{1}{b}\int_T^{\infty} q(t) \int_0^T \sum_{n=0}^{\infty} q(n, \tau)\pi_i^{(n+1)}\, d\tau\, dt$$

For gamma distributions of demand intervals,

$$\rho(n, t) = \sum_{j=mn}^{mn+m-1} p_j(\lambda T)$$

$$= P_{(m+1)n-1}(\lambda t) - P_{mn-1}(\lambda t)$$

$$q(t) = \frac{(\lambda t)^{m-1}}{(m-1)!}\lambda e^{-\lambda t} = \lambda p_{m-1}(\lambda t)$$

$$r_i = \frac{1}{b}\int_0^T \lambda p_{m-1}(\lambda t) \int_{T-t}^{T} \sum_{n=0}^{\infty} [P_{(m+1)n-1}(\lambda\tau) - P_{mn-1}(\lambda\tau)]\pi_i^{(n+1)}\, d\tau\, dt + \frac{1}{b}\int_T^{\infty} \lambda p_{m-1}(\lambda t) \int_0^T \sum_{n=0}^{\infty} [P_{(m+1)n-1}(\lambda\tau) - P_{mn-1}(\lambda\tau)]\pi_i^{(n+1)}\, d\tau\, dt$$

The integrations can be carried out explicitly leading to expressions composed of Poisson terms. Unfortunately, they appear to be too complex to be of much interest.

7. Conclusions

Although this model carries the analysis of optimal inventory policies to more general distributions than have been considered before, it is still inadequate to cope with the situation where demand is generated by a fixed number of customers, each using a non-trivial S, s policy. For then, the distribution of demand is not independent of the timing (and quantities) of demands that preceded the last demand. Even when the decision-maker chooses to disregard the additional information contained in the timing of past demands, thus restricting himself to an S, s policy, the analysis cannot disregard it in constructing, say, what corresponds to the present function $F(y)$ during a stock cycle. The present dynamic programming approach, which is crucially dependent on the irrelevance of the past beyond the last demand, cannot be extended to these non-Markovian processes in any obvious way.

Appendix

Note first that

$$\int_0^{\infty} q(t, D)e^{-\alpha t}\, dt = \sum_{n=1}^{\infty} w_D(n) \int_0^{\infty} q^{(n)}(t)e^{-\alpha t}\, dt. \tag{A.1}$$

Thus

$$\int_0^\infty q^{(n)}(t)e^{-\alpha t}\,dt = \int_0^\infty e^{-\alpha t}\int_0^t q^{(n-1)}(\tau)q(t-\tau)\,d\tau\,dt$$

$$= \int_0^\infty \int_0^t e^{-\alpha t}q^{(n-1)}(\tau)q(t-\tau)e^{-\alpha(t-\tau)}\,d\tau\,dt$$

The variable transformation, $t - \tau = t'$, yields

$$\begin{aligned}\int_0^\infty q^{(n)}(t)e^{-\alpha t}\,dt &= \int_0^\infty q(t')e^{-\alpha t'}\,dt' \int_0^\infty e^{-\alpha\tau}q^{(n-1)}(\tau)\,d\tau \\ &= a\int_0^\infty e^{-\alpha t}q^{(n-1)}(t)\,dt.\end{aligned} \tag{A.2}$$

Now,

$$\int_0^\infty e^{-\alpha t}q^{(1)}(t)\,dt = \int_0^\infty e^{-\alpha t}q(t)\,dt = a.$$

Suppose that

$$\int_0^\infty e^{-\alpha t}q^{(n-1)}(t)\,dt = a^{n-1}.$$

Then (A.2) shows that $\int_0^\infty e^{-\alpha t}q^{(n)}(t)\,dt = a^n$ is true for all n by induction.

Hence, by (A.1),

$$\sum_{n=1}^\infty w_D(n)a^n = \sum_{n=1}^\infty w_D(n)\int_0^\infty q^{(n)}(t)e^{-\alpha t}\,dt = \int_0^\infty q(t, D)e^{-\alpha t}\,dt.$$

Consider next,

$$p(i, t) = \sum_{n=0}^\infty \int_0^t [\pi_i(\tau)q(\tau)]^{(n)}[1 - Q(t-\tau)]\,d\tau.$$

The probability that, at time t, (past) demand equals i units is the probability that by some previous time τ, n trials were completed which resulted in a total demand for i units and that, since then, no trial has occurred. In order to evaluate

$$\int_0^\infty e^{-\alpha t}p(i, t)\,dt = \sum_{n=1}^\infty \int_0^\infty e^{-\alpha t}\int_0^t [q(\tau)\pi_i(\tau)]^{(n)}[1 - Q(t-\tau)]\,d\tau\,dt,$$

consider

$$\int_0^\infty e^{-\alpha t}\int_0^t [q(\tau)\pi_i(\tau)]^{(n)}[1 - Q(t-\tau)]\,d\tau\,dt$$

$$= \int_0^\infty \int_0^t [q(\tau)\pi_i(\tau)]^{(n)}e^{-\alpha\tau}e^{-\alpha(t-\tau)}[1 - Q(t-\tau)]\,d\tau\,dt.$$

With the variable transformation, $t' = t - \tau$, the above becomes

$$= \int_0^\infty [q(\tau)\pi_i(\tau)]^{(n)} e^{-\alpha\tau}\, d\tau \int_0^\infty e^{-\alpha t'}[1 - Q(t')]\, dt'$$

$$= b \int_0^\infty [q(\tau)\pi_i(\tau)]^{(n)} e^{-\alpha\tau}\, d\tau,$$

by definition of b. It may now be shown by induction that

$$\int_0^\infty [q(t)\tau_i(t)]^{(n)} e^{-\alpha t}\, dt = a^n p_j^{(n)}. \tag{A.3}$$

For $n = 1$,

$$\int_0^\infty q(t)\pi_i(t)e^{-\alpha t}\, dt = \int_0^\infty p(i, t)q(t)e^{-\alpha t}\, dt$$

$$= ap_i$$

by definition of a and p_i. Suppose (A.3) is true for $n - 1$. Substituting,

$$[\pi_i(t)q(t)]^{(n)} = \sum_{j=0}^{i} \int_0^t [q(\tau)\pi_j(\tau)]^{(n-1)} q(t - \tau)\pi_{i-j}(t - \tau)\, d\tau,$$

then

$$\int_0^\infty [\pi_i(t)q(t)]^{(n)} e^{-\alpha t}\, dt = \sum_{j=0}^{i} \int_0^\infty [q(\tau)\pi_j(\tau)]^{(n-1)} e^{-\alpha\tau}\, dt \int_0^\infty \pi_{i-j}(t')q(t')e^{-\alpha t'}\, dt'.$$

$$= \sum_{j=0}^{i} a^{n-1} p_j^{(n-1)} a p_{i-j}$$

$$= a^n p_i^{(n)}.$$

Thus, finally,

$$\int_0^\infty p(i, t)e^{-\alpha t}\, dt = b \sum_{n=1}^{\infty} a^n p_i^{(n)}.$$

This completes the proof of Equation (8).

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[Section 3 above has been revised slightly in order to correct errors in the original paper. In addition section 4B of the original paper has been omitted. —The Editor.]

OPTIMAL POLICIES FOR A MULTI-ECHELON INVENTORY PROBLEM*

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1. Introduction[3]

In the last several years there have been a number of papers (Reference 1) discussing optimal policies for the inventory problem. Almost without exception these papers are devoted to the determination of optimal purchasing quantities at a single installation faced with some pattern of demand. It has been customary to make the assumption that when the installation in question requests a shipment of stock, this shipment will be delivered in a fixed or perhaps random length of time, but at any rate with a time lag which is independent of the size of the order placed. There are, however, a number of situations met in practice in which this assumption is not a tenable one. An important example arises when there are several installations, say $1, 2, \cdots, N$, with installation 1 receiving stock from 2, with 2 receiving stock from 3, etc. In this example, if an order is placed by installation 1 for stock from installation 2, the length of time for delivery of this stock is determined not only by the natural lead time between these two sites, but also by the availability of stock at the second installation.

In this paper we shall consider the problem of determining optimal purchasing quantities in a multi-installation model of this type. First of all, let us remark that once the parameters of the model have been specified (lead times, purchase costs, demand distributions, holding and shortage costs, etc.), the optimal purchasing quantities may, in theory at least, be determined. The obvious way to proceed would be to define a cost function for each configuration of stock at the various installations, and in transit from one installation to another. We then remark that this cost function satisfies the type of functional equation which always appears in inventory theory, and from which the optimal provisioning policies may be determined by a recursive computation. It is clear, however, that this procedure is in general completely impractical since it necessitates the recursive computation of a sequence of functions of at least N variables.

The question is, therefore, whether the obvious recursive computation of optimal policies may be simplified for our multi-installation problem without compromising the optimality of the solution. The answer is that such a simplification may be obtained if several very plausible assumptions are incorporated in the model. With these assumptions, it will be demonstrated in this paper that the solution suggested by Clark in Reference 3 is indeed optimal. The solution will be described in detail below. It should be remarked here, however, that the virtue

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of the solution given by Clark is that it permits the optimal levels to be computed separately by precisely those techniques which have been used in the past for the computation of optimal policies at a single installation.

In Section IV we shall discuss various applications of the multiple-installation technique to problems in which several installations have the same supplier. The type of complex discussed in Section III may be described by the scheme:

$$\underset{N}{[\ ]} \to [\ ] \to [\ ] \cdots \underset{2}{[\ ]} \to \underset{1}{[\ ]},$$

whereas the complex in Section IV has the scheme

$$\begin{tikzcd}
 & & {[\ ]\cdots} \\
 & {[\ ]} \arrow[ur] \arrow[dr] & \\
{[\ ]} \arrow[ur] \arrow[dr] & & {[\ ]\cdots\,.} \\
 & {[\ ]} \arrow[r] & {[\ ]\cdots}
\end{tikzcd}$$

Unfortunately, the results for the latter type of complex are not as satisfactory as those for the former.

2. The Multiple-Installation Model and a Description of the Solution

Let us begin with a review of the model to be used for a single installation. An extensive discussion of this model is given in Reference 2, and we shall summarize here that material which will be of use to us.

A sequence of purchasing decisions is made at the beginning of a number of regularly spaced intervals. The cost of purchasing an amount z will initially be a general function $c(z)$, though we shall subsequently restrict ourselves to certain special cases. Delivery of an order occurs, say, λ periods after the order is placed, at which time the stock on hand is augmented by the amount of the order. During each period the stock on hand is depleted by an amount equal to the demand during the period, which is an observation from a distribution with density function $\phi(t)$, the demands being independent from period to period. (The demand distributions may actually differ from period to period.)

In addition to the purchase cost, it is customary to charge several other costs during each period. The first of these costs is a holding cost, proportional to the stock on hand at the beginning of the period if it is positive; and the second, a shortage cost proportional to the deficit of available stock at the end of the period if there is such a deficit. If the stock on hand at the beginning of the period is x, then the cost during the period, exclusive of purchasing costs, is given by

$$(1) \qquad L(x) = \left\{ \begin{array}{ll} hx + p \displaystyle\int_x^\infty (t - x)\phi(t)\, dt; & x > 0 \\ p \displaystyle\int_0^\infty (t - x)\phi(t)\, dt; & x \leqq 0 \end{array} \right\}$$

where h and p are the marginal holding and shortage costs, respectively. It is useful for us to introduce occasionally more general holding and shortage functions than the linear ones described in Equation (1), and for these functions there will be an analogous form for the one-period cost $L(x)$.

Any policy (sequence of purchasing decisions) produces a sequence of costs. Costs which occur n periods in the future are discounted by an amount α^n, so that we may form a total discounted cost as the result of any policy. The optimal purchasing policy is that one which minimizes the total discounted cost.

Let us consider an inventory problem in which there are n periods remaining, with x_1 units of stock on hand, w_1 units to be delivered one period in the future, and generally w_j units to be delivered j periods in the future, where $j = 0, 1, 2, \cdots, \lambda - 1$. Let $C_n(x_1, w_1, \cdots, w_{\lambda-1})$ represent the expectation of the discounted costs, beginning with such a configuration of stock and following an optimal provisioning scheme. Using the type of reasoning employed in Reference 2, this sequence of functions may be shown to satisfy the following functional equation:

$$
\begin{aligned}
C_n(x_1, w_1, \cdots, w_{\lambda-1}) &= \operatorname*{Min}_{z \geqq 0} \Big\{ c(z) + L(x_1) \\
&\quad + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, w_2, \cdots, w_{\lambda-1}, z)\phi(t)\, dt \Big\},
\end{aligned} \tag{2}
$$

where the minimizing value of z is the optimal purchase quantity for the given stock configuration. In the writing of this equation we are explicitly assuming that all excess demand is backlogged until the necessary stock becomes available. This equation has been analyzed in considerable detail and we shall quote for future use those facts of relevance to us.

1. The optimal policy (i.e., the minimizing value of z) is a function of the total stock on hand plus on order, regardless of the dates of delivery. This property depends crucially on the assumption that excess demand is backlogged. Moreover, it may be shown that

$$
\begin{aligned}
C_n(x_1, w_1, \cdots, w_{\lambda-1}) &= L(x_1) + \alpha \int_0^\infty L(x_1 + w_1 - t)\phi(t)\, dt + \cdots \\
&\quad + \alpha^{\lambda-1} \int_0^\infty \cdots \int_0^\infty L(x_1 + w_1 + \cdots + w_{\lambda-1} - t_1 - \cdots - t_{\lambda-1})\phi(t_1) \\
&\qquad \cdots dt_1 \cdots + f_n(x_1 + \cdots + w_{\lambda-1}),
\end{aligned} \tag{3}
$$

and that f_n satisfies the functional equation

$$
\begin{aligned}
f_n(u) &= \operatorname*{Min}_{y \geqq u} \Big\{ c(y - u) + \alpha^\lambda \int_0^\infty \cdots \int_0^\infty L(y - t_1 - \cdots - t_\lambda)\phi(t_1) \\
&\qquad \cdots \phi(t_\lambda)\, dt_1 \cdots dt_\lambda + \alpha \int_0^\infty f_{n-1}(y - t)\phi(t)\, dt \Big\}.
\end{aligned} \tag{4}
$$

If y^* is the minimizing value in Equation (4), then $y^* - u$ is the optimal purchase quantity, where $x_1 + w_1 + \cdots + w_{\lambda-1} = u$. (Obvious modifications in Equations (3) and (4) are required when n is less than the time lag.) These results (Reference 2) permit us to reduce the inventory problem with a time lag to one in which essentially no lag exists.

2. The results mentioned above are valid for any ordering function $c(z)$, whenever excess demand is backlogged. Now let us restrict our attention to the cost function

$$c(z) = \begin{Bmatrix} K + c \cdot z; & z > 0 \\ 0 \quad ; & z = 0 \end{Bmatrix}. \tag{5}$$

(K is the setup cost and c the unit cost.) Let us also assume that the one-period costs

$$\alpha^{\lambda-1} \int_0^\infty \cdots \int_0^\infty L(y - t_1 - \cdots - t_\lambda)\phi(t_1) \cdots \phi(t_\lambda)\, dt_1 \cdots dt_\lambda$$

are convex. (This is certainly correct if the holding and shortage costs are linear, and in other cases also.) Then there exists a sequence of critical numbers (S_n, s_n) so that in period n it is optimal to order only if $x_1 + \cdots + w_{\lambda-1} < s_n$ and if we do order, we order an amount $S_n - (x_1 + \cdots + w_{\lambda-1})$ (Reference 4). The specific form of the one-period costs is irrelevant; we can require only that they be convex.

3. An additional simplification occurs if $K = 0$. The upper and lower critical numbers become the same and it is customary to denote their common value by $\bar{x}_n$. The optimal purchase quantity is given by

$$\text{Max}\ (0, \bar{x}_n - (x_1 + \cdots + w_{\lambda-1})).$$

In this case, somewhat more is known about the properties of the functions $f_n(u)$. First of all, $f_n(u)$ is always convex, and in addition, $f'_n(u) = -c$ for $u \leqq \bar{x}_n$ (Reference 2).

Now let us turn our attention to the description of the multiple-installation model. We shall make the following assumptions:

Assumption 1: Demand originates in the system at the lowest installation (installation 1), and at no other point in the system.

Assumption 2: The cost of purchasing and shipping an item from any installation to the next will be linear, without any setup cost. The only exception to this assumption will be at the highest installation, at which point a setup cost will be permitted.

Assumption 3: At the lowest installation (installation 1), a linear holding and shortage cost will be operative, in the same manner as the single-installation problem described above. We make the assumption that holding and shortage costs for the second installation do not depend only on the stock on hand at the second installation, but are functions of this stock, plus stock in transit to the first installation plus stock on hand at the first installation. Generally speaking, the holding and shortage costs at any level will be assumed to be functions of the stock at that level plus all other stock in the system which is actually at a lower

level or in transit to a lower level. We shall call these costs the *natural* one-period costs at the level. They may, of course, be equal to zero.

Clark in Reference 3 has given the name "echelon" to the system consisting of the stock at any given installation plus stock in transit to or on hand at a lower installation. The echelons will be numbered according to the highest installation in the echelon. Our assumption may be stated as requiring that the one-period costs be functions of the echelon stock rather than installation stock. The simplifications described in this paper are very crucially tied to this assumption and assumption 2.

Assumption 4: Each echelon backlogs excess demand.

With these specifications in mind let us turn our attention to the determination of the optimal provisioning levels. The solution suggested by Clark is best described by means of an example. We consider the case of two installations. The natural lead time from installation 2 to installation 1 will be two periods in this example. Let us denote the stock on hand at installation 1 by x_1 ; the stock to be delivered one period in the future by w_1 ; and the stock on hand at installation 1, plus on hand at installation 2, plus in transit from 2 to 1, by x_2 (i.e., x_2 is echelon 2 stock). The one-period costs at installation 1 will be denoted by $L(x_1)$, and those at echelon 2 by $\tilde{L}(x_2)$. The unit shipping cost from 2 to 1 will be denoted by c_1 .

We begin by solving the problem (that is, determining the single critical numbers $\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3, \bar{x}_4, \cdots$) for installation 1 without any reference to the remaining parts of the multiple-echelon system. In other words, we solve the single-installation problem for the lowest echelon, assuming that delivery of any order, regardless of its size, will be effected in two periods, and using in our calculations a unit cost equal to the transportation cost from the higher echelon, without any reference to the original purchase cost. This would suggest that if at the beginning of the n^{th} period the stock on hand plus on order at installation 1 is less than $\bar{x}_n$, we order the difference; and if the stock is larger than $\bar{x}_n$, we do not order. The problem is, of course, that there may not be adequate stock at installation 2 to fill such an order. In the solution given in this paper, it is shown that we ship only that part of the order for which there is available stock at the next highest echelon. This describes the optimal policy at the lowest installation (Theorem 1, below).

The next question is that of the optimal quantity of stock to bring in at echelon 2. It will be shown in the next section that the optimal purchase quantities at echelon 2 are functions only of x_2 , the stock at the two installations plus the stock in transit. Moreover, the optimal policies for this echelon may be computed by the standard single-installation model using the ordering cost appropriate to this echelon, and the natural one-period costs described above ($\tilde{L}(x_2)$). The important idea is that we must in some fashion introduce a penalty at this echelon for keeping a quantity of stock on hand which is insufficient to meet the normal requests from the lower installation (Theorem 2, below). The procedure for doing this is quite simple: We merely introduce an additional one-period cost at the second echelon which is precisely equal to the expected incre-

ment in total cost at installation 1, because the stock at echelon 2 is inadequate to bring the lower level's stock up to the required point $\bar{x}_n$.

In the example that we are discussing, the specific form for this additional one-period cost may be found as follows: We recall the definition of the functions $C_n(x_1, w_1)$ to be the minimum expected discounted cost at echelon 1 if there are n periods remaining and if the stock on hand is x_1; and the stock on order, w_1. (This function is to be computed on the basis of an ordering cost equal to c_1, the transportation cost.) For $n = 1$, $C_1(x_1, w_1) = L(x_1)$, and also $C_2(x_1, w_1) = L(x_1) + \alpha \int_0^\infty L(x_1 + w_1 - t)\phi(t)\,dt$. In this expression for C_2, the first term represents the expected one-period costs in the immediate period, and the second term represents similar costs for the next period. Inasmuch as delivery of any order takes two periods, there is no modification that can be made in these costs. For $n > 2$, we use the decomposition described in Equation (3); that is,

$$C_n(x_1, w_1) = L(x_1) + \alpha \int_0^\infty L(x_1 + w_1 - t)\phi(t)\,dt + f_n(x_1 + w_1). \tag{6}$$

The first two terms on the left-hand side are as described above; the third term represents the optimal cost exclusive of those costs which it is impossible to modify by a request for a shipment.

As in Equation (4), the functions $f_n(u)$ satisfy

$$\begin{aligned} f_n(u) = \operatorname*{Min}_{y \geq u} \Big\{ & c_1(y - u) + \alpha^2 \iint L(y - t_1 - t_2)\phi(t_1)\phi(t_2)\,dt_1\,dt_2 \\ & + \alpha \int_0^\infty f_{n-1}(y - t)\phi(t)\,dt \Big\}, \end{aligned} \tag{7}$$

and the minimizing value is $\bar{x}_n$. In other words, if $x_1 + w_1 < \bar{x}_n$ so that ordering occurs, the minimum cost will be

$$\begin{aligned} & c_1(\bar{x}_n - u) + \alpha^2 \iint L(\bar{x}_n - t_1 - t_2)\phi(t_1)\phi(t_2)\,dt_1\,dt_2 \\ & + \alpha \int_0^\infty f_{n-1}(\bar{x}_n - t)\phi(t)\,dt. \end{aligned} \tag{8}$$

If, however, x_2, the stock at both installations, plus stock in transit, is less than $\bar{x}_n$, we will only be able to ship $x_2 - (x_1 + w_1)$ and therefore the minimum cost will be

$$\begin{aligned} & c_1(x_2 - u) + \alpha^2 \iint L(x_2 - t_1 - t_2)\phi(t_1)\phi(t_2)\,dt_1\,dt_2 \\ & + \alpha \int_0^\infty f_{n-1}(x_2 - t)\phi(t)\,dt. \end{aligned} \tag{9}$$

Expression (9) is of course larger than Expression (8), and the difference in cost is attributable exclusively to the insufficiency of stock at level 2. Therefore, the additional one-period loss to be charged to this echelon is given by Expression

(9) minus Expression (8), or

$$c_1(x_2 - \bar{x}_n) + \alpha^2 \iint [L(x_2 - t_1 - t_2) - L(\bar{x}_n - t_1 - t_2)]\phi(t_1)\phi(t_2)\, dt_1\, dt_2 \tag{10}$$
$$+ \alpha \int_0^\infty [f_{n-1}(x_2 - t) - f_{n-1}(\bar{x}_n - t)]\phi(t)\, dt,$$

if $x_2 < \bar{x}_n$ and zero if $x_2 > \bar{x}_n$. With this additional one-period loss to be charged to the second echelon, the optimal policy is then computed using standard techniques. Of course the specific values of Expression (10) involve the critical numbers $\bar{x}_n$ and the functions $f_n(u)$, but these will have been computed already for installation 1.

It is worth remarking that Expression (10) is a convex function of x_2, so that the optimal policy for the second echelon will be of the (S, s) type.

If there are more than two echelons, the same procedure is repeated, always augmenting the natural one-period loss at the echelon by the increment in total cost at the lower echelon due to the lack of available stock.

3. The Proof of Optimality

In this section we shall prove that the procedure suggested in the previous section is indeed optimal. Because of notational difficulties, we shall restrict our attention to the example described in the previous section although the ideas are quite general. In order to be specific we shall assume the time lag in delivery to installation 2 to be a single period.

Our approach will be to investigate the optimal solution for the entire system, and show that it reduces to the solution given by Clark. The first step is to write down a sequence of functional equations, analogous to Equation (3), but for the entire system rather than a single installation. We define $C_n(x_1, w_1, x_2)$ to be the minimum expected value of the discounted system costs if there are n periods remaining; if stock on hand at installation 1 is x_1; stock in transit, w_1; and system stock, x_2. At the beginning of the period two decisions are made: the first, a decision as to how much system stock to order for delivery next period; and the second, a decision as to the quantity of stock to be placed in transit to installation 1. The stock on hand plus in transit to installation 1 may be raised from $x_1 + w_1$ to y, where y is any number between $x_1 + w_1$ and x_2, at a cost of $c_1(y - x_1 - w_1)$; and if such a decision is taken, at the beginning of the next period stock on hand at installation 1 will be $x_1 + w_1 - t$ (t is the demand), and the stock in transit will be $y - x_1 - w_1$. The system stock is, of course, not modified by this decision; it can only be changed by a decision to introduce z units into the system (at a cost of $c(z)$), and will become $x_2 + z - t$. Therefore, if the two decisions described by y and z are taken, the inventories (x_1, w_1, x_2) become $(x_1 + w_1 - t, y - x_1 - w_1, x_2 + z - t)$, and the discounted value of expected future costs will be

$$\alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, y - x_1 - w_1, x_2 + z - t)\phi(t)\, dt. \tag{11}$$

In order to complete the accounting, we should consider the purchase (and transportation), holding and shortage costs. The purchase and transportation costs are given by

$$c(z) + c_1(y - x_1 - w_1). \tag{12}$$

The shortage and holding costs are given by

$$\tilde{L}(x_2) + L(x_1), \tag{13}$$

the terms of which apply, respectively, to echelon 2 and installation 1.

$C_n(x_1, w_1, x_2)$ is, of course, equal to the minimum of Expressions (11) + (12) + (13), when y and z are chosen optimally, and we therefore obtain the following functional equation:

$$\begin{aligned} C_n(x_1, w_1, x_2) = \operatorname*{Min}_{\substack{x_1+w_1 \leqq y \leqq x_2 \\ 0 \leqq z}} \Big\{ &c(z) + c_1(y - x_1 - w_1) + \tilde{L}(x_2) \\ &+ L(x_1) + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, y - x_1 - w_1, x_2 + z - t)\phi(t)\, dt \Big\}, \end{aligned} \tag{14}$$

with the condition $C_0 \equiv 0$.

Let us also introduce the functional equation which would be used to compute optimal policies for installation 1 in *isolation*. Let $C_n(x_1, w_1)$ be the minimum expected value of the discounted costs for an n period problem at installation 1, which begins with x_1 units on hand and w_1 units in transit. We are assuming that the unit purchase price is the transportation cost and that *all* orders are delivered in two periods. C_n satisfies

$$\begin{aligned} C_n(x_1, w_1) = \operatorname*{Min}_{y \geqq x_1+w_1} \Big\{ &c_1(y - x_1 - w_1) + L(x_1) \\ &+ \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, y - x_1 - w_1)\phi(t)\, dt \Big\}. \end{aligned} \tag{15}$$

Of course, the solution of Equation (15) is of no clear relevance to Equation (14) as yet.

Obviously $C_1(x_1, w_1) = L(x_1)$, and $C_1(x_1, w_1, x_2) = L(x_1) + \tilde{L}(x_2)$. In other words, $C_1(x_1, w_1, x_2) = C_1(x_1, w_1) + g_1(x_2)$. We shall show that $C_n(x_1, w_1, x_2)$ may always be written as $C_n(x_1, w_1)$ + a function of x_2 alone, and this is the important step in verifying that Clark's solution is optimal.

Theorem 1. There is a sequence of functions $g_n(x_2)$, with $g_1(x_2) = \tilde{L}(x_2)$, such that

$$C_n(x_1, w_1, x_2) = C_n(x_1, w_1) + g_n(x_2). \tag{16}$$

Moreover, it is optimal for installation 1 to provision without reference to installation 2, subject to the proviso that if insufficient stock is available at installation 2, then installation 1 will be content with getting as much as it can.

We shall demonstrate this theorem by induction. Let us suppose that Equation (16) is true for $(n - 1)$, and we shall then demonstrate its validity for n. Sub-

stituting in Equation (14), we obtain

$$
\begin{aligned}
C_n(x_1, w_1, x_2) = \min_{\substack{x_1+w_1 \leqq y \leqq x_2 \\ 0 \leqq z}} \Big\{ & c(z) + c_1(y - x_1 - w_1) + \bar{L}(x_2) \\
& + L(x_1) + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, y - x_1 - w_1)\phi(t)\, dt \\
& + \alpha \int_0^\infty g_{n-1}(x_2 + z - t)\phi(t)\, dt \Big\}.
\end{aligned} \tag{17}
$$

From Equation (17) we see that aside from the constraint that y be less than x_2, the optimal selection of y is such as to minimize

$$
c_1(y - x_1 - w_1) + L(x_1) + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, y - x_1 - w_1)\phi(t)\, dt,
$$

and this is, of course, the same as the single critical number $\bar{x}_n$ for the problem of installation 1 considered separately. If it turns out that $x_2 \geqq \bar{x}_n$, then the constraint $x_2 \geqq y$ is not operative, and we may therefore conclude that for $x_2 \geqq \bar{x}_n$,

$$
\begin{aligned}
C_n(x_1, w_1, x_2) &= C_n(x_1, w_1) \\
&\quad + \min_{z \geqq 0} \Big\{ c(z) + \bar{L}(x_2) + \alpha \int_0^\infty g_{n-1}(x_2 + z - t)\phi(t)\, dt \Big\}.
\end{aligned} \tag{18}
$$

On the other hand, if $x_2 < \bar{x}_n$ (and therefore $x_1 + w_1 < \bar{x}_n$), installation 1 will be thwarted in its attempt to bring its stock level up to $\bar{x}_n$. Because of the convexity of the one-period costs, it is optimal to bring the stock level up as high as possible and therefore $y = x_2$. As a consequence, we see that for $x_2 < \bar{x}_n$,

$$
\begin{aligned}
C_n(x_1, w_1, x_2) &= c_1(x_2 - x_1 - w_1) + L(x_1) \\
&\quad + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, x_2 - x_1 - w_1)\phi(t)\, dt \\
&\quad + \min_{z \geqq 0} \Big\{ c(z) + \bar{L}(x_2) + \alpha \int_0^\infty g_{n-1}(x_2 + z - t)\phi(t)\, dt \Big\}.
\end{aligned} \tag{19}
$$

Now we are interested in showing that $C_n(x_1, w_1, x_2) - C_n(x_1, w_1)$ is a function of x_2 alone. From Equations (18) and (19), we see that this difference is equal to

$$
\Lambda_n(x_1, w_1, x_2) + \min_{z \geqq 0} \Big\{ c(z) + \bar{L}(x_2) + \alpha \int_0^\infty g_{n-1}(x_2 + z - t)\phi(t)\, dt \Big\}, \tag{20}
$$

where

$$
\begin{aligned}
\Lambda_n(x_1, w_1, x_2) &= c_1(x_2 - x_1 - w_1) + L(x_1) \\
&\quad + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, x_2 - x_1 - w_1)\phi(t)\, dt - C_n(x_1, w_1),
\end{aligned} \tag{21}
$$

when $x_2 < \bar{x}_n$ and zero otherwise.

In order to demonstrate Theorem 1, it is therefore necessary to show that $\Lambda_n(x_1, w_1, x_2)$ is in reality a function of x_2 alone, and of course we need only consider the region $x_2 < \bar{x}_n$. In this region, however,

$$(22)\quad \begin{aligned} C_n(x_1, w_1) &= c_1(\bar{x}_n - x_1 - w_1) + L(x_1) \\ &\quad + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t, \bar{x}_n - x_1 - w_1)\phi(t)\, dt, \end{aligned}$$

and therefore Equation (21) may be written as

$$(23)\quad \begin{aligned} \Lambda_n(x_1, w_1, x_2) &= c_1(x_2 - \bar{x}_n) \\ &\quad + \alpha \int_0^\infty [C_{n-1}(x_1 + w_1 - t, x_2 - x_1 - w_1) \\ &\qquad - C_{n-1}(x_1 + w_1 - t, \bar{x}_n - x_1 - w_1)]\phi(t)\, dt. \end{aligned}$$

Our theorem will be demonstrated if we can show that the integrand in Equation (23) is independent of x_1 and w_1. But by Equation (6),

$$C_{n-1}(x_1, w_1) = L(x_1) + \alpha \int_0^\infty L(x_1 + w_1 - y)\phi(y)\, dy + f_n(x_1 + w_1),$$

and therefore the integrand in Equation (23) is given by

$$(24)\quad \begin{aligned} &\alpha \int_0^\infty L(x_2 - t - y)\phi(y)\, dy + f_{n-1}(x_2 - t) \\ &\qquad - \alpha \int_0^\infty L(\bar{x}_n - t - y)\phi(y)\, dy - f_{n-1}(\bar{x}_n - t), \end{aligned}$$

which is a function of x_2 alone. We have therefore demonstrated Theorem 1.

We have, however, demonstrated somewhat more. Λ_n is now known to be a function of x_2, which may be written as

$$(25)\quad \begin{aligned} \Lambda_n(x_2) &= c_1(x_2 - \bar{x}_n) \\ &\quad + \alpha^2 \int_0^\infty \int_0^\infty [L(x_2 - t - y) - L(\bar{x}_n - t - y)]\phi(t)\phi(y)\, dt\, dy \\ &\quad + \alpha \int_0^\infty [f_{n-1}(x_2 - t) - f_{n-1}(\bar{x}_n - t)]\phi(t)\, dt, \end{aligned}$$

for $x_2 < \bar{x}_n$ and zero for $x_2 > \bar{x}_n$. But Equation (20), which represents $g_n(x_2)$, may be written as

$$(26)\quad g_n(x_2) = \operatorname*{Min}_{z \geq 0} \left\{c(z) + \bar{L}(x_2) + \Lambda(x_2) + \alpha \int_0^\infty g_{n-1}(x_2 + z - t)\phi(t)\, dt\right\}.$$

The solution of this equation provides us with the optimal policy for the entire system. As we see, all that is required is to augment the natural costs at echelon 2 by $\Lambda(x_2)$.

Theorem 2. The functions $g_n(x_2)$ satisfy Equation (26), by means of which the optimal system stock may be obtained.

4. Several Installations with the Same Supplier

In this section we shall generalize the model considered in Section III so as to include the possibility of several installations with the same supplier. We shall, however, retain the restriction that no installation has two different suppliers. All of the assumptions that have previously been mentioned, such as backlogging, no setup cost for transportation, etc., will be retained also in this model. The only point in need of clarification is our assumption that the natural losses should be functions of echelon stock, rather than installation stock. The notion of an echelon in this model will be as follows: We begin by selecting a specific installation, say installation I. Associated with I will be a number of other installations which receive, directly or indirectly, stock from installation I. The total stock at I, plus the stock in transit or on hand at these other installations will comprise the echelon associated with installation I. With this definition our assumption will again be that the natural one-period costs are associated with echelons, rather than installations.

Let us consider the following example of such a complex:

$$\begin{tikzcd}
 & & & {[A_3]} \\
 & & {[B_2]} \arrow[ur] \arrow[dr] & \\
{} \arrow[r] & {[C_1]} \arrow[ur] \arrow[dr] & & {[A_2]} \\
 & & {[B_1]} \arrow[r] & {[A_1]}
\end{tikzcd}$$

The procedure which we have shown in the previous section to be optimal for a simpler problem suggests the following procedure in this complex:

(1) For installations A_1, A_2, and A_3 (which are terminal installations) compute the optimal sequences of single critical numbers, assuming the installations to be in isolation; also that all requests for shipment are supplied during the natural lead time, and that the purchase cost is given by the transportation cost from the higher echelon.

(2) Augment the natural costs at echelon B_1 by the increment in cost at A_1 because of the inability to satisfy requests for stock at A_1, and do the same for B_2. Then compute the optimal stock levels at B_1 and B_2 separately, assuming the availability of infinite stock from C_1.

(3) Modify the natural costs at C_1 by the increment in cost at B_1 and B_2 because of the inability to satisfy requests for stock, and then compute the optimal policy at C_1.

If the directions given above are examined closely, it may be seen that they are ambiguous on a number of points. The clarification of these points will show where the Clark procedure departs from optimality in this model, whereas it was optimal for the model considered in Section III. Even though the procedure is not optimal, it has considerable merit, both in its ease of application and in its approximate validity.

Point 1. Shall we permit an arbitrary pair of installations to exchange stock; and if so, at what cost, and with what lags?

As an example, we are posing the question as to whether A_2 shall be permitted

to ship excess stock to A_3. The desire to make this shipment might arise in two different ways. First of all, there may be insufficient stock at echelon B_2 to raise both A_2 and A_3 to their required critical levels, and the stocks left over at A_2 and A_3 may be out of balance by a sufficient amount so that it is wise to ship both B_2 and A_2 stock to A_3. Another possible cause of transshipment might be a substantial anticipated drop in demand at A_2 and an excess of carryover stock which might profitably be shipped to A_3.

In practice, however, transshipment of this sort would rarely take place. Moreover, if we permit this sort of transshipment to take place, the theoretical and computational aspects of the problem become quite complex. It would be meaningless for an installation to consider itself in isolation, inasmuch as its actual stock levels in the future would depend on the disposition of stock at all other installations. Since our primary aim is to be able to compute optimal policies at each installation separately, we shall assume that such transshipment is impossible. It is gratifying that such an assumption does not run contrary to what is done in practice.

Point 2. If all requests cannot be satisfied because of insufficient stock at a higher echelon, how is the available stock to be rationed among the requesting installations?

The answer to this question bears very heavily on the optimality of the procedure suggested above. We shall consider the following concrete case:

$$\begin{array}{ccc} & & [A_1] \\ & \nearrow & \\ \rightarrow [B] & & , \\ & \searrow & \\ & & [A_2] \end{array}$$

and assume, for definiteness, that all routes have a time lag of one period, and that the transportation cost c_1 is the same from B to A_1 as from B to A_2. Let L^1 and L^2 represent the one-period costs at installations 1 and 2, respectively, and $\tilde{L}$ the one-period cost at B. Let $C_n^1(x_1)$ and $C_n^2(x_2)$ represent the minimum costs for A_1 and A_2 computed separately, and let $\{\bar{x}_n^1\}$ and $\{\bar{x}_n^2\}$ be the sequence of single critical numbers for A_1 and A_2. The C functions satisfy the customary functional equations, for example:

$$C_n^1(x_1) = \operatorname*{Min}_{y_1 \geqq x_1} \left\{ c_1(y_1 - x_1) + L^1(x_1) + \alpha \int_0^\infty C'_{n-1}(y_1 - t)\phi_1(t)\,dt \right\}, \tag{27}$$

and similarly for C_n^2.

In the spirit of Section III, we define $C_n(x_1, x_2, x_3)$ to be the minimum *system* cost if A_1 has x_1 units, A_2 has x_2 units, and the B echelon has x_3 units. These functions satisfy a functional equation analogous to Equation (17), i.e.,

$$\begin{aligned} C_n(x_1, x_2, x_3) = \operatorname{Min} \Big\{ & c(z) + c_1(y_1 - x_1) \\ & + c_1(y_2 - x_2) + \tilde{L}(x_3) + L^1(x_1) + L^2(x_2) \\ & + \alpha \iint C_{n-1}(y_1 - t_1, y_2 - t_2, x_3 + z - t_1 - t_2)\phi_1(t_1)\phi_2(t_2)\,dt_1\,dt_2 \Big\}, \end{aligned} \tag{28}$$

where the minimization is over the region $y_1 \geqq x_1$, $y_2 \geqq x_2$, $x_3 \geqq y_1 + y_2$, $z \geqq 0$.

The crucial ideas behind the optimality results of the previous section were embodied in Theorems 1 and 2. The analogue of Theorem 1 for the model considered in this section would be that there exists a sequence of functions $g_n(x_3)$ with the property that

$$C_n(x_1, x_2, x_3) = C_n^1(x_1) + C_n^2(x_2) + g_n(x_3). \tag{29}$$

Does there exist such a sequence of functions? And if there does, what light is cast upon the question of allocation of stock (Point 2)?

Unfortunately the answer to the first of these questions is in the negative. The functions $C_n(x_1, x_2, x_3)$ *cannot* be broken down in the form of Equation (29). To see why this is so, let us assume that Equation (29) is valid for $n - 1$, and see what the consequences of Equation (28) and this assumption would be for $C_n(x_1, x_2, x_3)$. We would have

$$\begin{aligned} C_n(x_1, x_2, x_3) = \underset{\cdots}{\text{Min}} \Big\{ & c(z) + c_1(y_1 + y_2 - x_1 - x_2) \\ & + \tilde{L}(x_3) + L^1(x_1) + L^2(x_2) + \alpha \int_0^\infty C_{n-1}^1(y_1 - t_1)\phi_1(t_1)\,dt_1 \\ & + \alpha \int_0^\infty C_{n-1}^2(y_2 - t_2)\phi_2(t_2)\,dt_2 \\ & + \alpha \iint g_{n-1}(x_3 + z - t_1 - t_2)\phi_1(t_1)\phi_2(t_2)\,dt_1\,dt_2 . \Big\} \end{aligned} \tag{30}$$

Aside from the constraint that $y_1 + y_2$ be less than x_3, the optimal selection of y_1 would be $\bar{x}_n^1$ and the optimal selection of y_2 would be $\bar{x}_n^2$. If $x_3 > \bar{x}_n^1 + \bar{x}_n^2$, the constraint is not operative, and from Equation (30) we would have

$$\begin{aligned} C_n(x_1, x_2, x_3) &= C_n^1(x_1) + C_n^2(x_2) \\ &+ \underset{z \geqq 0}{\text{Min}} \left\{ c(z) + \tilde{L}(x_3) + \alpha \iint g_{n-1}(x_3 + z - t_1 - t_2)\phi_1(t_1)\phi_2(t_2)\,dt_1\,dt_2 \right\}. \end{aligned} \tag{31}$$

So far, so good. We run into a problem, however, when $x_3 < \bar{x}_n^1 + \bar{x}_n^2$. This is, of course, the problem raised by Point 2, and the answer is given by Equation (30). The numbers y_1 and y_2 should be selected according to the constraints

$$y_1 + y_2 = x_3,\ y_1 \geqq x_1,\ y_2 \geqq x_2 \tag{32}$$

and such as to minimize

$$\begin{aligned} & c_1(y_1 + y_2 - x_1 - x_2) + L^1(x_1) + L^2(x^2) \\ & + \alpha \int C_{n-1}^1(y_1 - t_1)\phi_1(t_1)\,dt_1 + \alpha \int C_{n-1}^2(y_2 - t_2)\phi_2(t_2)\,dt_2 . \end{aligned} \tag{33}$$

Therefore, in order to allocate properly we must solve the minimization problem (33) subject to the constraints (32). The problem is certainly solvable. The difficulty, however, is in the form of the answer. The answer *may* depend

not only on x_3, but also on x_1 and x_2 (the stock at A_1 and A_2). It *may* depend on x_1 and x_2; generally it will not unless the stock levels x_1 and x_2 are seriously out of balance. But if the solution to the minimization problem does depend on x_1 and x_2, it will depend on them jointly and a factorization of the type given by Equation (29) would not be obtained.

Let us assume that such a lack of balance does not occur. Then y_1 and y_2 would be selected to minimize (33) subject to

$$y_1 + y_2 = x_3, \quad \text{alone.} \tag{34}$$

Call these solutions $\bar{x}_n^1(x_3)$ and $\bar{x}_n^2(x_3)$. Then Equation (30) would read

$$\begin{aligned} C_n(x_1, x_2, x_3) &= C_n^1(x_1) + C_n^2(x_2) + \Lambda_n(x_1, x_2, x_3) \\ &+ \operatorname*{Min}_{z \geqq 0} \left\{ c(z) + \tilde{L}(x_3) + \alpha \iint g_{n-1}(x_3 + z - t_1 - t_2)\phi_1(t_1)\phi_2(t_2)\, dt_1\, dt_2 \right\}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \Lambda_n(x_1, x_2, x_3) &= c_1(\bar{x}_n^1(x_3) + \bar{x}_n^2(x_3) - x_1 - x_2) + L^1(x_1) + L^2(x_2) \\ &+ \alpha \int_0^\infty C_{n-1}^1(\bar{x}_n^1(x_3) - t_1)\phi_1(t_1)\, dt_1 - C_n^1(x_1) \\ &+ \alpha \int_0^\infty C_{n-1}^2(\bar{x}_n^2(x_3) - t_2)\phi_2(t_2)\, dt_2 - C_n^2(x_2), \end{aligned} \tag{36}$$

for $x_3 < \bar{x}_n^1 + \bar{x}_r^2$ and zero, otherwise. However, just as in Section III, if $x_1 < \bar{x}_n^1$ and $x_2 < \bar{x}_n^2$, this may be shown to be a function of x_3 alone, and this is the function that is to be taken to augment the natural costs at echelon B.

We repeat that Equations (35) and (36) are derivable only by means of the assumptions that the stock at installations A_1 and A_2 are not out of balance. Since this is expected to occur rather frequently, it suggests that Clark's approximation is an excellent one for this model.

5. Extensions

The discussion in Sections III and IV assumed that demand originates in the system at the lowest installation (echelon 1) and at no other point in the system. This, however, is not a necessary assumption and, in fact, the probability distributions used for the various echelons need have no relationship with each other. This may be demonstrated by considering the proof of optimality in Section III for the simple two-echelon example.

If $f_1(t_1)$ and $f_2(t_2)$ represent, respectively, the marginal demand distribution at echelon 1 and echelon 2, and $f(t_1, t_2)$ the joint distribution, then Equation (14) may be rewritten as follows:

$$\begin{aligned} C_n(x_1, w_1, x_2) = \operatorname*{Min}_{\substack{x_1 + w_1 \leqq y \leqq x_2 \\ 0 \leqq z}} &\left\{ c(z) + c_1(y - x_1 - w_1) + \tilde{L}(x_2) + L(x_1) \right. \\ &\left. + \alpha \int_0^\infty \int_0^\infty C_{n-1}(x_1 + w_1 - t_1, y - x_1 - w_1, x_2 + z - t_2) f(t_1, t_2)\, dt_1\, dt_2 \right\}. \end{aligned} \tag{37}$$

Following through the proof of Theorem 1 by substituting

$$C_n(x_1, w_1, x_2) = C_n(x_1, w_1) + g_n(x_2)$$

in Equation (37), we obtain

$$C_n(x_1, w_1, x_2) = \operatorname*{Min}_{\substack{x_1+w_1 \leq y \leq x_2 \\ 0 \leq z}} \Big\{ c(z) + c_1(y - x_1 - w_1) + \tilde{L}(x_2)$$

$$(38) \qquad + L(x_1) + \alpha \int_0^\infty C_{n-1}(x_1 + w_1 - t_1, y - x_1 - w_1) f_1(t_1)\, dt_1$$

$$+ \int_0^\infty g_{n-1}(x_2 + z - t_2) f_2(t_2)\, dt_2 \Big\},$$

which is the same form as Equation (17). The remainder of the proof is the same as in Section III.

The ability to assign different distributions to the various echelons has several interesting applications. For example, the N-installation problem of Section III may be interpreted as N stages of production, where the time required for production in each stage is analogous to the delivery times in the inventory problem. The final stage of production (analogous to installation 1 in the inventory problem) is faced with an exogenous demand while each production stage may incur random losses through spoilage. The probability distribution used for the final production stage is the exogenous demand distribution augmented by losses during the stage. This distribution is successively augmented by losses in the other production stages to obtain distributions for these stages. The per unit ordering cost for each stage is the fabrication cost in the immediately prior stage. This example represents the case when the mean demand is an increasing funcion of the echelon number, i.e., the higher the echelon, the higher the mean.

An example of the opposite case is encountered in the inventory problem where items are regenerated through repair. Considering the problem of Section III again, suppose that items issued from installation 1 are exchanged for damaged items (on a one for one basis) which then undergo repair cycles of different durations according to the degree of damage. Thus, if t items are issued, then t reparable items are generated, with different portions, $t_1, t_2, \cdots$ ($t = \sum_i t_i$) being successively more remote, timewise, from being available for reissue. Here, the net demand faced by echelon k is given by $t - \sum_{i=1}^{k} t_i$ which is a decreasing function of k. If, throughout the repair cycle, items are scrapped as being uneconomically reparable, then the mean demand as a function of echelon number may be more general than the monotonically increasing or decreasing functions considered above.

Problems of the type described in Sections III and IV, together with the interpretations mentioned above, may be combined to portray almost any inventory and/or production structure. Such combinations may be used to make supply repair, and production decisions in an integrated fashion. Of course, in each application, the assumptions of the method must be analyzed with respect to their validity or effect.

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IX–36

A REDISTRIBUTION MODEL WITH SET-UP CHARGE*[1]

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This paper considers the problem of redistributing stock among several user activities within the period between regular deliveries of new supplies to the system. The cost of redistribution is assumed to be proportional to the number of shipments among the activities. A procedure based on minimizing total redistribution and shortage costs within the period is given for determining the amounts (if any) to be shipped among activities.

Introduction

Suppose an item of material is carried in stock at each of several user activities. It is assumed that periodically the item is procured for the entire system of activities from an outside source of supply and delivered after a fixed time to individual activities to replenish their stock. Because of random usage at each activity, the stock at some of them may become insufficient to protect against shortages which may occur before the next delivery of new supplies to the system. Therefore at each regular review within a delivery cycle, it may be desirable to redistribute stock among activities. However it is assumed that this will involve a cost proportional to the number of shipments made.

The procedure developed in this paper provides criteria which enable a central inventory manager to determine whether any activity has excess or deficit stock for the period before the next scheduled delivery. It also permits him to determine amounts to be redistributed among activities to rectify the imbalance. These criteria are based on an objective of minimizing the cost of the redistribution and the shortage costs which may be incurred by activities within the period until the next delivery.

Neither the alternative allocations of new procurement to activities at the end of this period nor the redistributions possible at subsequent reviews within this period are considered. This simplification may nevertheless provide useful results if existing procurement and allocation policies can be relied upon to correct basic stock inadequacies over the long term and if any redistribution between scheduled deliveries will render subsequent redistributions within the same period unnecessary.

Background of the Problem

Reference 1 described a redistribution model which was essentially of the following form: The decision variables were the non-negative quantities x_{ij}

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to be currently and instantaneously distributed from the ith activity to the jth activity, $(i, j = 1, \cdots, N)$, where N was the total number of activities. The shortage penalty for a single activity to be associated with the redistributions x_{ij} was the expected number of shortages as of some future point of time (e.g., the time at which a future allocation was to be delivered into the system). Expected shortage at the ith activity was defined as

$$G_i = \int_{S_i^0+\sum_j x_{ji}-\sum_j x_{ij}}^{\infty} \left(y - S_i^0 - \sum_{j=1}^{N} x_{ji} + \sum_{j=1}^{N} x_{ij}\right) dF_i(y) \tag{1}$$

where

S_i^0 = stock on hand at i before the redistribution is effected;

F_i = the probability distribution of accumulated demand at i from the present to the future point of time in question.

The only cost parameter included in the model was c_{ij}, the ratio of the unit cost of transportation between the ith and jth activities to a common unit cost of shortage at these activities.

The total system cost to be minimized by the redistribution was then

$$G = \sum_{i=1}^{N} G_i + \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij} . \tag{2}$$

Theorem I of Ref. 1 completely characterized the solution; namely:

For $i, j = 1, \cdots, N$, either

$$\left.\begin{array}{l} \displaystyle\int_{S_j^0+\sum_k x_{kj}-\sum_k x_{jk}}^{\infty} dF_j - \int_{S_i^0+\sum_k x_{ki}-\sum_k x_{ik}}^{\infty} dF_i = c_{ij} \\ \text{or} \qquad x_{ij} = 0. \end{array}\right\} \tag{3}$$

In words, a positive redistribution between any two activities equates the difference between the shortage probability of the receiving activity and that of the shipping activity to the (relative) unit cost of transportation.

Of subsequent interest in this memorandum is the case where $c_{ij} = 0$ and the F_i denote normal distributions with mean μ_i and variance σ_i^2. Whenever $c_{ij} = 0$ for all i, j, redistribution equates the shortage probabilities of all activities to a common value. In addition, when the normal assumption is made, property (3) requires only that the total amount received or shipped by an activity is:

$$\sum_{j=1}^{N} x_{ij} = \sigma_i(t_i - \bar{t}), \tag{4}$$

in the case that activity i is a shipper, or

$$\sum_{i=1}^{N} x_{ij} = \sigma_j(\bar{t} - t_j) \tag{5}$$

in the case that activity j is a receiver, where

$$t_i = (S_i^0 - \mu_i)/\sigma_i , \qquad i = 1, \cdots, N, \tag{6}$$

and

$$\bar{t} = \left[\sum_{i=1}^{N} (S_i^0 - \mu_i)\right] \Big/ \sum_{i=1}^{N} \sigma_i \tag{7}$$

denote the normalized stock positions of activities before and after redistribution, respectively.

Unfortunately, after a redistribution satisfying these conditions has been made, the stock positions of activities would most certainly depart from the criteria stated. Thus, redistributions would be required among the same activities every time the criteria were applied. A possible remedy for this situation is the introduction of set-up charges into the model. In general, set-up charges would tend to deter frequent redistributions of small amounts among activities.

Assumptions of the Set-up Charge Model

Specifically, it will be assumed that a single set-up charge K is incurred for each shipment between a pair of activities. As in the case of unit transportation cost, K must actually be the ratio of the set-up charge to the unit shortage cost.

Because of inherent difficulties associated with a model including set-up charges, a redistribution with a set-up charge is considered here under the simplest of the transportation cost assumptions previously studied; namely, zero unit *transportation* cost. Thus, total system cost of redistribution becomes

$$G = \sum_{i=1}^{N} \left[G_i + K \sum_{j=1}^{N} \delta(x_{ij}) \right] \tag{8}$$

where

$$\delta(z) = \begin{cases} 0 & \text{if } \; z = 0 \\ 1 & \text{otherwise.} \end{cases}$$

A normal distribution of demand at each activity will also be assumed. Finally, to avoid trivial considerations in the discussion to follow, it will be assumed that for no pair (i, j) will $\sigma_i = \sigma_j$ and $t_i = t_j$.

The Class of Optimal Redistributions

Suppose $\{x_{ij}^*\}$ minimizes G in (8). Then clearly the x_{ij}^* *which are positive* must still satisfy the characterization given in (4) through (7), where the sums in the last expression include only those activities for which x_{ij}^* is positive. Otherwise, infinitesimal changes in the positive x_{ij}^* could reduce $\sum G_i$ in (8) with no increase in the amount of set-up charges. In effect, if the integers i and j for which $x_{ij}^* > 0$ were somehow known, a minimizing solution would be completely determined at this point in terms of the $\sum_j x_{ij}^*$ and $\sum_i x_{ij}^*$.

In any redistribution where there are exactly m integers $i_1, \cdots, i_m$ for which $\sum_j x_{ij} > 0$, and exactly n (different) integers $j_1, \cdots, j_n$ for which $\sum_i x_{ij} > 0$,

it will always be possible to accomplish this redistribution with no more than $m + n - 1$ of the x_{ij}'s positive. The smallest number possible is, of course, $\max(m, n)$.

To avoid computational difficulties, which appear to be of little practical significance, it will henceforth be assumed that *any* redistribution with $\sum_j x_{ij}$ and $\sum_i x_{ij}$ positive only for $i_1, \cdots, i_m$ and $j_1, \cdots, j_n$, respectively, requires exactly $m + n - 1$ values of x_{ij} that are positive. For, if less than this number occurred in redistributions satisfying expressions (4) through (7), as modified, then these expressions imply that there exist *proper* subsets I of the integers $i_1, \cdots, i_m$, and J of the integers $j_1, \cdots, j_n$, so that

$$\sum_{i \in I} (S_i^0 - \mu_i - \sigma_i \bar{t}) = \sum_{j \in J} (\sigma_j \bar{t} - S_j^0 + \mu_j) \tag{9}$$

where

$$\bar{t} = \frac{\sum_{k=1}^{m} (S_{i_k}^0 - \mu_{i_k}) + \sum_{k=1}^{n} S_{j_k}^0 - \mu_{j_k})}{\sum_{k=1}^{m} \sigma_{i_k} + \sum_{k=1}^{n} \sigma_{j_k}} \tag{10}$$

In other words, the initial stockpiles of the activities, *i.e.*, the S_i^0, must satisfy rather special linear dependencies. This is, perhaps, an unlikely state of affairs.

Consider the set of all redistributions $\{x_{ij}\}$ in which the x_{ij} satisfy property (4) for i belonging to a set of integers $\{i_1, \cdots, i_m\}$, which satisfy property (5) for j belonging to a (disjoint) set $\{j_1, \cdots, j_n\}$, and which are zero otherwise; where $m \geqq 1$, $n \geqq 1$, and $m + n \leqq N$; where $\bar{t}$ is given by (10); and where t_i is defined by (6) for all i. In view of the above assumption about the required number of set-ups, G may be written for such x_{ij} as

$$G = \sum_{i=1}^{N} G_i + K\left[\sum_{k=1}^{m} \delta\left(\sum_{j=1}^{N} x_{i_k j}\right) + \sum_{k=1}^{n} \delta\left(\sum_{i=1}^{N} x_{i j_k}\right)\right] - K. \tag{11}$$

G is now dependent only on the sums $\sum_j x_{ij}$ and $\sum_i x_{ij}$; *i.e.*, the total shipments from or to an activity.

Note also that in (11) the G_i terms are summed over all i. The relevant system cost of a redistribution which excludes a given activity as either shipper or receiver must, nevertheless, include the shortage penalty for that activity, in order that the cost of this redistribution may be compared with one that includes that activity.

In fact, since G_i simplifies to $\sigma_i g(z_i)$ for any i, where z_i denotes the normalized stock position of activity i *after* redistribution is effected, and where

$$g(z) = \int_z^{\infty} \frac{(w - z)}{\sqrt{2\pi}} e^{-(1/2)w^2}\, dw, \tag{12}$$

G may be written entirely in terms of the t_i variables and variables y_i defined by

$$y_i = \begin{cases} 1 & \text{if } i \in \{i_1, \cdots, i_m, j_1, \cdots, j_n\} \\ 0 & \text{otherwise;} \end{cases} \tag{13}$$

namely,

$$
\begin{aligned}
G = G(y) &= \sum_{i=1}^{N} (1 - y_i)\sigma_i\, g(t_i) + g[\bar{t}(y)] \sum_{i=1}^{N} y_i\, \sigma_i + K \sum_{i=1}^{N} y_i - K \\
&= \sum_{i=1}^{N} \sigma_i\, g(t_i) - K + \sum_{i=1}^{N} y_i\, \sigma_i \left\{ g[\bar{t}(y)] - \sum_{i=1}^{N} \frac{y_i\, \sigma_i}{\sum y_i\, \sigma_i} \right. \\
&\qquad \left. \cdot \left[g(t_i) - \frac{K}{\sigma_i} \right] \right\}
\end{aligned}
\tag{14}
$$

where

$$
\bar{t}(y) = \frac{\sum_{i=1}^{N} y_i\, \sigma_i\, t_i}{\sum_{i=1}^{N} y_i\, \sigma_i}\,. \tag{15}
$$

If a positive redistribution is optimal, the problem becomes that of minimizing G with respect to a set of integers i for which $y_i = 1$, *i.e.*, a set of activities which actually ship or receive material in redistribution. While the computation of G could be made very quickly for any specified combination of activities, the total number which must be considered, namely, $\sum_{n=2}^{N} \binom{N}{n}$, is obviously large for large N. In addition, note that *no redistribution* is always a potential candidate for the optimal redistribution. This particular redistribution, *i.e.*, $y_i = 0$ for all i, will subsequently be denoted by the symbol ω. The "combinations" involving only one activity in redistribution are ruled out of course.

A Related Problem

If the domain of definition of $G(y)$ in (14) is extended over the set

$$
Y = \{y: y = (y_1, \cdots, y_N), \qquad 0 \leqq y_i \leqq 1\},
$$

then a formally related (but not equivalent) minimizing problem to that of finding the optimal redistribution may be considered. The related problem can be subjected to a more complete analysis than the combinatorial one that is actually presented by redistribution. And furthermore, the minimization of the related problem turns up a solution that indicates a close approximation to a redistribution solution.

It is clear that G is differentiable at every point of Y except ω (where, nevertheless, it is continuous). Also it can be shown that

Theorem I. G is convex over Y.

Proof: In view of the form of G as written in (14), all that need be shown is that for $y \in Y$, $y^* \in Y$ and $y \neq y^*$, and for α such that $0 \leqq \alpha \leqq 1$,

$$
\begin{aligned}
& g\{\bar{t}[\alpha y + (1 - \alpha) y^*]\} [\alpha \sum y_i \sigma_i + (1 - \alpha) \sum y_i^* \sigma_i] \\
& \qquad \leqq \alpha g[\bar{t}(y)] \sum y_i \sigma_i + (1 - \alpha) g[\bar{t}(y^*)] \sum y_i^* \sigma_i
\end{aligned}
\tag{16}
$$

or, more explicitly,

$$
(17) \quad \begin{aligned} & g\left[\frac{\alpha\sum y_i\sigma_i t_i + (1-\alpha)\sum y_i^*\sigma_i t_i}{\alpha\sum y_i\sigma_i + (1-\alpha)\sum y_i^*\sigma_i}\right][\alpha\sum y_i\sigma_i + (1-\alpha)\sum y_i^*\sigma_i] \\ & \quad \leqq \alpha g\left(\frac{\sum y_i\sigma_i t_i}{\sum y_i\sigma_i}\right)\sum y_i\sigma_i + (1-\alpha)g\left(\frac{\sum y_i^*\sigma_i t_i}{\sum y_i^*\sigma_i}\right)\sum y_i^*\sigma_i. \end{aligned}
$$

Let $z = \sum t_i(y_i\sigma_i/\sum y_i\sigma_i)$, $z^* = \sum t_i(y_i^*\sigma_i/\sum y_i^*\sigma_i)$, and $\beta = \alpha(\sum y_i\sigma_i)/[\alpha\sum y_i\sigma_i + (1-\alpha)\sum y_i^*\sigma_i]$. Then (17) is equivalent to

$$
(18) \qquad g[\beta z + (1-\beta)z^*] \leqq \beta g(z) + (1-\beta)g(z^*)
$$

which, because $0 \leqq \beta \leqq 1$, is true from the convexity of g as defined in expression (12).

With these properties of $G(y)$ established, then, as a special case of Theorem I of reference 1, it follows that if a point $y \neq \omega$ minimizes G over Y, such a point is completely characterized by the property:

For every i, either

$$
(19) \quad \begin{cases} \dfrac{\partial G}{\partial y_i} = 0 \\ \text{or} \\ y_i = \begin{cases} 0 & \text{and} \quad \dfrac{\partial G}{\partial y_i} > 0 \\ 1 & \text{and} \quad \dfrac{\partial G}{\partial y_i} < 0, \end{cases} \end{cases}
$$

where

$$
(20) \qquad \frac{\partial G}{\partial y_i} = \sigma_i\{g[\bar{t}(y)] - g(t_i) + [t_i - \bar{t}(y)]g'[\bar{t}(y)]\} + K.
$$

The necessary and sufficient condition that $G(\omega) > \min G(y)$ is straightforward. The expression for G as given in (14) is composed of a component $\sum\sigma_i g(t_i) - K$, which is $G(\omega)$, and a component dependent on y, namely

$$
(21) \qquad (\sum y_i\sigma_i)\left\{g[\bar{t}(y)] - \sum\frac{y_i\sigma_i}{\sum y_i\sigma_i}\left[g(t_i) - \frac{K}{\sigma_i}\right]\right\}.
$$

Therefore $G(\omega) > \min G(y)$ if and only if there exists a $y \neq \omega$ such that expression (21) is negative.[2] The existence of such a point y is equivalent to the existence of $\alpha_1, \cdots, \alpha_N$ so that $0 \leqq \alpha_i \leqq 1$, $\sum \alpha_i = 1$, $0 < \alpha_i < 1$ for at least one i, and so that

$$
(22) \qquad g(\sum\alpha_i t_i) < \sum\alpha_i\left[g(t_i) - \frac{K}{\sigma_i}\right].
$$

[2] The necessity for this follows from the fact that if a y not equal to ω minimizes G, then expression (21) is identical to $\sum y_i(\partial G/\partial y_i)$, which by property (19) is non-positive. Sufficiency is obvious.

This in turn is equivalent to the existence of at least two integers, say i_1 and i_2, between 1 and N, and an α_{i_1}, where $0 < \alpha_{i_1} < 1$, such that (22) holds for $\alpha_{i_2} = 1 - \alpha_{i_1}$.

This is obviously a stronger requirement than the (strict) convexity of the function g and of course relates to the initial inventory positions of the several activities. Graphically it would be quite easy to see if the requirement is met: the curve described by the points $\{t, g(t)\}$ must intersect the convex hull of the points $\{t_i, g(t_i) - (K/\sigma_i)\}$.

These results might be summarized as

Theorem II. A point $y \neq \omega$ minimizes G over Y if and only if

(i) There exists a triple (i_1, i_2, α_{i_1}), where $0 < \alpha_{i_1} < 1$, so that (22) holds for $\alpha_{i_2} = 1 - \alpha_{i_1}$, and

(ii) For every i, (19) holds.

The above theorem effectively removes a possibly troublesome matter in the definition of G as given by (14). $G(y)$ assumes the value $\sum \sigma_i g(t_i) - K$ at $y = \omega$, whereas the original expression (8) assumes, appropriately, the value $\sum \sigma_i g(t_i)$. Theorem II demonstrates that if a point $y \neq \omega$ exists satisfying property (19) for each of its coordinates, then necessarily $G(y) < \sum \sigma_i g(t_i) - K$. Moreover, Theorem III below makes it possible to ignore how G is defined over the region $\{y: 0 < \sum y_i \leqq 1\}$.

This characterization given by Theorem II of the solution to the problem of minimizing G over Y would be a satisfactory resolution of the redistribution problem were it not for the possibility that the minimizing point may have a coordinate y_i such that $0 < y_i < 1$. As previously mentioned meaningful redistributions can only be associated with certain vertices of Y, in particular the set of points

$$\bar{Y} = [y: y \in Y, \quad y_i = 0 \quad \text{or} \quad 1, \quad \sum y_i \neq 1].$$

Clearly the $\min_Y G(y)$ need not be assumed at such points.

However, situations in which more than one of the y_i lie between zero and one in the optimal solution would appear to be unlikely. In the first place, the value $\bar{t}(y^*)$, where $G(y^*) = \min_Y G(y)$, is unique. Otherwise, if there exists a $y' \neq y^*$ such that $G(y') = G(y^*)$ and $\bar{t}(y^*) \neq \bar{t}(y')$, then Theorem I requires that

$$(23) \qquad G[\beta y^* + (1 - \beta)y'] = \beta G(y^*) + (1 - \beta)G(y') = \min_Y G(y)$$

for all β such that $0 \leqq \beta \leqq 1$. But note in the proof of Theorem I that if $z \neq z^*$ in expression (18), then that inequality and hence the inequality in expression (16) must be strict. Therefore, with $\bar{t}(y')$ and $\bar{t}(y^*)$ substituted for z and z^*, respectively, in these expressions, a contradiction is found to equation (23). Evidently $\bar{t}(y^*) = \bar{t}(y')$.

Therefore expression (20), which for each i is a function of $\bar{t}(y)$ alone, must have a zero at the unique value $\bar{t}(y^*)$ for more than one i. Again, here is a situation that would require rather special relationships among the initial stockpiles at activities.

Furthermore, one can always be assured that

Theorem III. There exists a y' such that $G(y') = \min_Y G(y)$ and $0 < y'_i < 1$ for at most one i.

Proof: Note that expression (20) for each i varies only with the value of $\bar{t}(y)$. Therefore if y^* minimizes $G(y)$ and if $P = \{i: 0 < y_i^* < 1\}$ has more than one element, then it is only necessary to exhibit a y' yielding $\bar{t}(y') = \bar{t}(y^*)$, where $y'_k = y_k^*$ for all $k \notin P$ and for at most one $k \in P$. Since expression (20) for each i must thereby have the same values for y' as for y^*, it follows that y' must also minimize $G(y)$.

Let $Q = \{i: y_i^* = 1\}$, $K_1 = \sum_{i \in Q} \sigma_i t_i$, $K_2 = \sum_{i \in Q} \sigma_i$, and $\bar{t}^A = (\sum_{i \in A} \sigma_i t_i)/(\sum_{i \in A} \sigma_i)$, when A is non-null. Suppose, for instance, $K_1 \leqq K_2 \bar{t}(y^*)$. (The case of equality is worth consideration only in the event Q is null, in which case K_1 and K_2 should be taken to be zero in the following; if equality holds and $K_2 > 0$, then the desired y' is already at hand.) The set $P^* = \{i: i \in P, t_i > \bar{t}(y^*)\}$ is evidently non-null, and, since $\bar{t}(y^*)$ is essentially a weighted average of the t_i, it follows that

$$\frac{K_1 + \sum_{i \in P^*} y_i^* \sigma_i t_i}{K_2 + \sum_{i \in P^*} y_i^* \sigma_i} \geqq \bar{t}(y^*)$$

or, equivalently,

$$K_1 + \sum_{i \in P^*} y_i^* \sigma_i [t_i - \bar{t}(y^*)] \geqq K_2 \bar{t}(y^*).$$

But

$$\sum_{i \in P^*} \sigma_i [t_i - \bar{t}(y^*)] > \sum_{i \in P^*} y_i^* \sigma_i [t_i - \bar{t}(y^*)],$$

and therefore

$$\bar{t}^{P^*+Q} > \bar{t}(y^*).$$

Because of this, there must exist a smallest set $P' \subset P^*$ such that $\bar{t}^{P'+Q} > \bar{t}(y^*)$; *i.e.*, a smallest set in terms of the number of its elements.

Now if just one element is excluded from P' (call the reduced set P''), then $\bar{t}^{P''+Q} \leqq \bar{t}(y^*)$. Hence, there exists a y_k such that $k \in P'$, $0 \leqq y_k \leqq 1$, and

$$\frac{K_1 + \sum_{i \in P''} \sigma_i t_i + y_k \sigma_k t_k}{K_2 + \sum_{i \in P''} \sigma_i + y_k \sigma_k} = \bar{t}(y^*).$$

The desired y' may then be defined by $y'_i = 1$ if $i \in P'' + Q$, $y'_k = y_k$, and $y'_i = 0$ otherwise.

A simple rephrasing of the argument handles the case where $K_1 > \bar{t}(y^*) K_2$. The above theorem[3] insures that if $G(y^*) = \min_Y G(y) < G(\omega)$, then $\sum y_i^* >$

[3] Geometrically it has been shown that a hyperplane known to have an intersection with the n-dimensional cube must have a point in common with one of its edges. The construction of the above proof gave a useful insight into the problem of finding y^*. It has been relatively easy to find the sets P' and P'' in numerical examples, although the procedures have not been reduced to an algorithm.

1. It also affirms the uniqueness of the minimizing solution in the case that expression (20), evaluated at $\bar{t}(y^*)$, is zero for at most one i.

A Computing Procedure

In this discussion it will be assumed that $\min_Y G(y) < G(\omega)$.

Expression (20) set equal to zero may be considered as a condition on $\bar{t} = \bar{t}(y)$ and rewritten as

$$\sigma_i[f(\bar{t}) + t_iF(\bar{t}) - f(t_i) - t_iF(t_i)] + K = 0 \tag{24}$$

where

$$f(t) = \frac{e^{-(1/2)t^2}}{\sqrt{2\pi}},$$

and

$$F(t) = \int_{-\infty}^{t} f(w)\, dw.$$

The left side of (24) is a strictly increasing function of $\bar{t}$ on the interval $(-\infty, t_i)$ and strictly decreasing on (t_i, ∞). Depending on its value at $\pm\infty$, it may have no roots, one root, or two roots. If y^* minimizing G over Y does have a coordinate between zero and one, then among the collection of roots for equation (24) for $i = 1, \cdots, N$, is one which is actually assumed by $\bar{t}(y)$ for y satisfying property (19) and with only one y_i between zero and one.

If this does not occur, *i.e.*, the minimum of G over Y occurs at a vertex of Y, then the simple behavior of the left-hand side of (24) and knowledge of the roots should be suggestive of the appropriate combination of activities to produce a $\bar{t}(y)$ satisfying property (19). (Incidentally, this vertex which minimizes G over Y must be unique; otherwise the convexity of G must permit a y with at least one coordinate between zero and one also to minimize G over Y.) Also available is a method like the simplex corrected gradient procedure described in Ref. 2. (See Section 6 of that article for the application to concave programming.) If the minimum of G occurs at a vertex, this procedure will indeed terminate.

Of course any procedure for extremizing convex or concave functions can be applied to the present minimizing problem. The advantage of making the analysis described in this paper lies in the possible avoidance either of examining a large number of activity combinations for redistribution purposes or of using an extensive iterative procedure which might be required in a more general purpose computing scheme.

It will be remembered that when $\min_Y G(y) < \min_{\bar{Y}} G(y)$, property (19) does not characterize the solution of the "real" redistribution problem. One might think that in this case the closest vertex $\bar{y}$ to the y^* minimizing G over Y would be a good approximation to the real solution; *i.e.*, choose $\bar{y}$ such that

$$\bar{y}_i = \begin{cases} 0 & \text{if} \quad y_i^* \leqq \frac{1}{2} \\ 1 & \text{if} \quad y_i^* > \frac{1}{2} \end{cases}$$

as a solution to the redistribution problem. In all numerical examples tried by the author, this has yielded a value of G very close to $\min_{\bar{Y}} G(y)$. But an even better "solution" has been a $\tilde{y} \in \bar{Y}$ which is closest to the hyperplane

$$\sum \sigma_i \, [t_i - \bar{t}(y^*)]y_i = 0.$$

In the following numerical example a value $K = .25$ was used in the calculations:

i	t_i	σ_i	y_i^*	$\bar{y}_i$	$\tilde{y}_i$	ω
1	−1.0	3.	1.	1.	1.	0
2	−0.5	3.	1.	1.	1.	0
3	0.0	4.	0	0	0	0
4	0.5	4.	0	0	0	0
5	1.0	1.	0	0	0	0
6	1.5	2.	0.2323	0	1.	0
7	2.0	1.	1.	1.	0	0
8	2.5	3.	1.	1.	1.	0

y	$G(y)$	$\bar{t}$
y^*	5.2531	0.5444
$\bar{y}$	5.2577	0.5000
$\tilde{y}$	5.2540	0.5455
ω	7.6371	—

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IX–37

DEVELOPMENT AND EVALUATION OF SURVEILLANCE SAMPLING PLANS*†

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The problem of maintaining quality of inventory in the presence of deterioration is studied. Repeated application of sampling inspections together with replacement policies are used to maintain quality. The effect of such repeated applications can be measured in terms of the proportion of poor product existing at any time. In this paper, the sampling plans studied are those commonly used in acceptance sampling, and the replacement policy consists in replacing lots judged defective by the sampling procedure. The theory of Markov processes is used to evaluate the effectiveness of the sampling plans and replacement policy. As illustrations, special cases are considered. Graphs are provided for these cases.

1. Introduction

Acceptance sampling plans have been traditionally used to evaluate the quality of manufactured items. They can also be used, by periodic application, to ascertain the quality of stored items. When an acceptance sampling plan is used for such a purpose, we shall refer to it as a surveillance sampling plan. We shall only consider manufactured items whose quality deteriorates over time. More specifically we desire to construct decision rules for surveillance inspectors such that a preassigned level of quality of product is available with high probability at all times. The operating characteristics of these rules will depend upon the deterioration rate of the quality of the item, the surveillance sampling plan employed, and the length of the surveillance period. In this paper we shall discuss surveillance sampling plans and their evaluation. Once evaluation is made possible questions of optimality can be studied.

Briefly we shall consider the following type of problem. Suppose we have k different lots, each of size N, which are stored. At regular intervals these lots are inspected. If a lot passes inspection, it is kept on hand until the next inspection; if it does not pass inspection, it is replaced by a new lot of acceptable quality. The only way that a lot can leave the system and be replaced is for it to be rejected in inspection. This situation will arise whenever the lots stored are to be used only in cases of emergency, as in the stockpiling of vital material. The quality of the lots can be expected to deteriorate with age. The aim of the surveillance plan is to maintain the quality of the k lots (Nk items) so that, for example, if an emergency arises, there will be on hand sufficient stock of good quality to meet the crisis. We shall show how to measure the effectiveness of a surveillance plan which accomplishes this purpose. Once we have an index

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of effectiveness of a plan, a catalogue of plans can be constructed. A plan can then be available for selection which will meet some desired cost or other relevant criteria.

Specifically, we shall assume that the surveillance procedures to be evaluated are given by standard acceptance sampling plans, for example, those used by government agencies in their procurement programs, where $L_j(p)(j = 1, \cdots)$ denotes the probability of accepting a lot when the proportion of defective items in the lot is given by p; j denotes the number of periods the lot has already been on hand. That is, if it is known that a lot has been on hand for j periods, sampling plan j which has operating characteristic (OC) curve $L_j(p)$ is used to accept or reject the lot for further storage. Thus, at least in our general formulation, we allow the possibility of adjusting the surveillance plan according to the age of the lot.

We shall assume the existence of a deterioration function $p(t)$ which denotes the proportion of defective items in a lot which has been on hand for t units of time. Thus $p(0)$ denotes the incoming quality of a lot. We are implicitly assuming (i) that the lot sizes are large enough so that $p(t)$ can be considered independent of the particular lot and (ii) that the manufacturer's process is either under control at level $p(0)$ or the initial inspection plan for new lots is such that the incoming level is $p(0)$.

One way to evaluate the effectiveness of a surveillance plan (i.e., the set of sampling plans characterized by $\{L_j\}$) is to determine the proportion of defective items on hand at any given time. This will be not only a function of $\{L_j\}$ but also of $p(t)$. We shall show how to evaluate this proportion under the assumption that the surveillance plan has been in operation for a long time, that is, assuming "steady-state" conditions. We shall also consider two special cases: one where $p(t)$ is a step-function with one jump, the other where $p(t)$ has an exponential form. In both cases we assume L_j is independent of j, that is, only one acceptance plan is used throughout surveillance.

A truncated model is also discussed and evaluated. Truncation occurs when a surveillance policy dictates that lots on hand for a preassigned length of time are automatically replaced by new lots.

This paper represents an initial attempt at the development and evaluation of sampling plans for the surveillance function in inventory management. Some questions are resolved but, of course, many are posed. It is hoped that several problems raised in this paper will receive the attention of other workers in the field.

2. Lot Age Viewed as a Stochastic Process

The age of the ith lot together with the deterioration function $p(t)$ and the application of the surveillance sampling plan generate a stochastic process $\{X_n(i)\}(n = 1, \cdots)$ where $X_n(i)$ denotes the number of periods the ith lot has been on hand *just after* the nth inspection. This stochastic process is, in

particular, a Markov chain[1] with stationary transition probabilities

$$(2.1)\qquad \begin{aligned} p_{j,j+1} &= L_{j+1}[p(j+1)] \\ p_{j,0} &= 1 - L_{j+1}[p(j+1)] \end{aligned} \qquad j = 0, 1, \cdots .$$

where $p_{j,j+1}$ is the probability that a lot on hand for j periods will survive $(j+1)$ periods, and $p_{j,0}$ is the probability that a lot on hand for j periods will be discarded.

If $X_n(i) = j$, this means that the proportion of defectives in the ith lot, τ units of time after the nth inspection, is $p(j+\tau)$. For the remainder of this paper we shall assume $0 \leqq \tau < 1$, i.e., the periods between inspection represent one unit of time. Let $Y_{n,\tau}(i) = p(j+\tau)$ when $X_n(i) = j$. Then the proportion of defective items in the entire population τ units of time after the nth inspection is

$$(2.2)\qquad P_{n,\tau} = \frac{1}{k}\sum_{i=1}^{k} Y_{n,\tau}(i).$$

We shall assume that k, the number of lots, is large so that by the law of large numbers

$$(2.3)\qquad P_{n,\tau} \sim E(P_{n,\tau}) = \frac{1}{k}\sum_{i=1}^{k}\sum_{j=0}^{\infty} p(j+\tau)P[X_n(i) = j].$$

3. Steady State Conditions

If the surveillance procedure has been in effect for a long time $E(P_{n,\tau})$ can be approximated by

$$(3.1)\qquad \pi_\tau = \lim_{n\to\infty} E(P_{n,\tau}).$$

Since the Y's are bounded random variables

$$(3.2)\qquad \pi_\tau = \frac{1}{k}\sum_{i=1}^{k}\sum_{j=0}^{\infty} p(j+\tau)\lim_{n\to\infty} P[X_n(i) = j].$$

Under reasonable conditions on $p(t)$ and L_j (for example, if for some T, $L_j[p(j)]$ is bounded away from 1 for all $j > T$) it is known from the theory of Markov chains that

$$(3.3)\qquad v_j = \lim_{n\to\infty} P[X_n(i) = j]$$

is independent of i and can be obtained by solving the system of equations

$$(3.4)\qquad \begin{aligned} v_{j+1} &= v_j p_{j,j+1} \qquad j = 0, 1, \cdots \\ \sum_{j=0}^{\infty} v_j &= 1 \end{aligned}$$

subject to the conditions that $v_j > 0$ for all j. The v_j's are called "steady-state" probabilities. If $P[X_0(i) = j] = v_j$, then $X_n(i)$ is a stationary stochastic process

[1] For an exposition on Markov chains, the reader is referred to *An Introduction to Probability Theory and its Application*, W. Feller, John Wiley & Sons, 2nd edition, 1957.

and $P[X_n(i) = j] = v_j$ for all n and $j = 0, 1, \cdots$. In our notation we have as solutions

$$v_j = v_0 \prod_{r=0}^{j-1} L_{r+1}[p(r+1)] \tag{3.5}$$

$$v_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{r=0}^{j-1} L_{r+1}[p(r+1)]}.$$

We then have

$$\pi_\tau = \sum_{j=0}^{\infty} p(j+\tau) v_j. \tag{3.6}$$

Thus, assuming "steady-state" conditions, that is, n large enough so that we can assume $P[X_n(i) = j] = v_j$, we have π_τ is the expected proportion of defectives existing in the Nk items τ units after an inspection.

4. Case 1: $p(t)$ a Step-function

In this case let us suppose that

$$\begin{aligned} p(t) &= p_0 && \text{for} \quad t < T \\ &= p_1 \qquad (p_1 > p_0) && \phantom{\text{for}} \quad t \geqq T \end{aligned} \tag{4.1}$$

and

$$L_j = L.$$

Study of this case can be justified on the basis that such a step function can be thought of as a first approximation to the more general case where the deterioration curve $p(t)$ is an ogive; T corresponds to the value of t where $p(t)$ has its maximum slope; p_0 and p_1 correspond to the average values of $p(t)$ for $t < T$ and $t \geqq T$, respectively.

$$\begin{aligned} v_j &= v_0 \prod_{r=0}^{j-1} L_{r+1}[p(r+1)] = v_0 L^j(p_0) && j < T \\ v_j &= v_0 L^{[T]}(p_0) L^{j-[T]}(p_1) && j \geqq T, \end{aligned} \tag{4.2}$$

and

$$v_0 = \frac{1}{\dfrac{1 - L^{[T]+1}(p_0)}{1 - L(p_0)} + \dfrac{L^{[T]}(p_0)L(p_1)}{1 - L(p_1)}}. \tag{4.3}$$

Here $[T]$ denotes the greatest integer less than T. Then we have for

$$0 \leqq \tau < (T - [T])$$

$$\begin{aligned} \pi_\tau &= p_0 \sum_{j=0}^{[T]} v_j + p_1 \sum_{j=[T]+1}^{\infty} v_j \\ &= \frac{v_0 L^{[T]}(p_0) L(p_1)}{1 - L(p_1)} (p_1 - p_0) + p_0 \end{aligned} \tag{4.4}$$

and for $0 < (T - [T]) \leqq \tau < 1$

$$\begin{aligned}\pi_\tau &= p_0 \sum_{j=0}^{[T]-1} v_j + p_1 \sum_{j=[T]}^{\infty} v_j \\ &= \frac{v_0 L^{[T]}(p_0)}{1 - L(p_1)} \cdot (p_1 - p_0) + p_0 .\end{aligned} \tag{4.5}$$

The expressions (4.4) and (4.5) can be used to evaluate a given surveillance plan. If p_0, p_1, and T are known, then (4.4) and (4.5) indicate the proportion of defectives in the entire population. However, it is in general unreasonable to expect p_1 and T to be known. If such is the case then a conservative evaluation (comparable to the AOQL in acceptance sampling) of the plan is given by the largest of the expressions (4.4) and (4.5) maximized over p_1 and T.

If we consider the special case $p_0 = 0$ and if L is continuous from the left at $p = 1$ then

$$\max_{p_1} \pi_\tau \geqq \frac{1}{[T] + 1} . \tag{4.6}$$

Now in order to find $\max_{p_1, T} \pi_\tau$ we note that expression (4.4), when T is any positive integer, is always less than or equal to expression (4.5) for any value of T' when $T < T' < T + 1$. Whenever T is not an integer (4.5) is greater than or equal to (4.4). Thus we need only maximize (4.5) over p_1 and T.

Since $v_0 L^{[T]}(p_0)$ can easily be shown to be non-increasing in $[T]$, it can easily be argued that the maximum of (4.5) over p_1 and T will always occur for $[T]$ as small as possible. If $T < 1$ the maximum is trivially equal to 1. To avoid this degenerate case we shall bound T away from 1, i.e., we assume $1 < T < 2$. We want to obtain

$$\begin{aligned}\max_{p_1, T > 1} \pi_\tau &= \max_{p_1} \frac{v_0 L(p_0)}{1 - L(p_1)} (p_1 - p_0) + p_0 \\ &= \max_{p_1} \frac{L(p_0)(p_1 - p_0)}{1 + L(p_0) - L(p_1)} + p_0 .\end{aligned} \tag{4.7}$$

Differentiating the appropriate function in (4.7) with respect to p_1 we get

$$\begin{aligned}\frac{d\left\{\dfrac{L(p_0)(p_1 - p_0)}{1 + L(p_0) - L(p_1)}\right\}}{dp_1} &= \frac{[1 + L(p_0) - L(p_1)]L(p_0) + L(p_0)(p_1 - p_0)L'(p_1)}{[1 + L(p_0) - L(p_1)]^2} \\ &= \frac{L(p_0)\{1 + L(p_0) - L(p_1) + (p_1 - p_0)L'(p_1)\}}{[1 + L(p_0) - L(p_1)]^2} .\end{aligned} \tag{4.8}$$

The expression (4.8) (assuming L monotone) is positive for $p_1 = p_0$; hence the maximizing value $\hat{p}_1$ of (4.7) is either at one of the roots

$$1 + L(p_0) - L(p_1) + (p_1 - p_0)L'(p_1) = 0 \tag{4.9}$$

or at $p_1 = 1$. The latter will always be the case if there are no roots to (4.9). If $\hat{p}_1 = 1$ then (4.7) becomes

$$\frac{L(p_0)(1 - p_0)}{1 + L(p_0)} + p_0 = \frac{p_0 + L(p_0)}{1 + L(p_0)}. \tag{4.10}$$

If $\hat{p} < 1$, then from (4.9) we have that

$$(\hat{p}_1 - p_0)L'(\hat{p}_1) = -[1 + L(p_0) - L(\hat{p}_1)]$$

and (4.7) becomes

$$\max_{p_1, T > 1} \pi_\tau = \frac{-L(p_0)}{L'(\hat{p}_1)} + p_0 .$$

Thus given a value p_0 and a sampling plan with OC curve $L(p)$ it is possible to compute a very conservative estimate of the proportion defective on hand. For the purpose of selecting plans it might now be profitable to tabulate this value of π_τ for some existing sampling plans and for various values of p_0.

5. Case 2: Exponential Deterioration

In this situation we suppose $p(t) = 1 - \theta e^{-\alpha t}(\alpha, \theta > 0)$. Then $1 - \theta = p(0)$, i.e., $p(t) = 1 - [1 - p(0)]e^{-\alpha t} = 1 - e^{-\alpha t} + p(0)e^{-\alpha t}$. Note that α is a function of the sampling period. We also assume that $L_j(p) = (1 - p)^k$; an OC curve arising from a single sampling plan with acceptance number (C) zero and k the size of the sample. This, to be sure, is a very special case. However, the numerical results obtained in this case can serve as an indication of the results in situations less amenable to calculation. At any rate we have

$$\begin{aligned} v_j &= v_0 \prod_{r=0}^{j=1} [1 - p(r + 1)]^k \\ &= v_0 \prod_{r=0}^{j-1} \theta^k e^{-\alpha(r+1)k} \\ &= v_0 \theta^{kj} \exp\left(-\alpha k \sum_{r=1}^{j} r\right) \\ &= v_0 \theta^{kj} e^{-(\alpha k/2) j(j+1)} \end{aligned} \tag{5.1}$$

$$v_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \theta^{kj} e^{-(\alpha k/2) j(j+1)}} = \frac{1}{\sum_{j=0}^{\infty} \theta^{kj} e^{-(\alpha k/2) j(j+1)}}.$$

We then have

$$\begin{aligned} \pi_\tau &= \sum_{j=0}^{\infty} v_j(1 - \theta e^{-\alpha(j+\tau)}) = 1 - \theta e^{-\alpha\tau} \sum_{j=0}^{\infty} v_j e^{-\alpha j} \\ &= 1 - \theta e^{-\alpha\tau} v_0 \sum_{j=0}^{\infty} \theta^{kj} e^{-(\alpha k/2) j(j+1)} e^{-\alpha j} \\ &= 1 - \theta v_0 e^{-\alpha\tau} \sum_{j=0}^{\infty} \theta^{kj} e^{-(\alpha k/2)(j^2 + (k+2/k) j)}. \end{aligned} \tag{5.2}$$

Since in any physical situation, α and t will be given, $p(t)$ will be determined in a specific time dimension (i.e., seconds, weeks, etc.). We can set $\alpha t = \alpha^* \tau$, determine α^* according to some reasonable criterion, and then obtain τ.

By numerical methods α can be found to minimize π_τ. Then the length of the period between inspections can be adjusted so that the deterioration function will have that particular value of α. The optimal interval between inspections is a function of the particular sampling plan used, i.e., we considered the plan with OC curve $(1 - p)^k$. It would be of interest to see how much the optimal α varies with a change in sampling plan.

6. The Truncated Model

In some situations, a surveillance policy might dictate that a lot, after having been on hand for a certain length of time, must be automatically replaced by a new lot. Thus if a lot is replaced after $M + 1$ periods we have in the Markov chain that $p_{M,0} = 1$. With this modification the "steady-state" probabilities are

$$(6.1) \qquad v_j = v_0 \prod_{r=0}^{j-1} p_{r,r+1} = v_0 \prod_{r=0}^{j-1} L_{r+1}\,[p(r+1)] \qquad (j = 1, \cdots, M)$$

$$v_0 = \frac{1}{1 + \sum_{j=1}^{M} \prod_{r=0}^{j-1} L_{r+1}\,[p(r+1)]}\,.$$

The analogue of π_τ is

$$(6.2) \qquad \pi_{\tau,M} = \sum_{j=0}^{M} v_j\, p(j + \tau)\,.$$

It is of interest to carry out computations to determine the effect of truncation on the level of quality maintained. This has been tried for several values of the parameters, where $p(t)$ is exponential and $\alpha = 1$, and some graphs have been drawn to depict the relationships. The graphs in Figures 1 through 3 are not expected to portray any realistic situation but merely to demonstrate the effect of truncation on several sampling plans. In Figure 1 each lot is replaced after two periods ($M = 1$), $\theta = .99$, and a sample size of 5 is considered with acceptance numbers C (number of defectives tolerated for acceptance) as depicted. Figure 2 considers exactly the same situation for $M = 2$, and Figure 3 for $M = 3$.

7. Selection of a Surveillance Plan

Discussion and analysis in the previous sections are not extensive enough to make possible the selection of an optimal surveillance sampling plan or set of sampling plans for some specific purpose. However, the expressions for v_0 and π_τ make it possible to choose the better of two offered plans or the best among a finite number of offered plans. For example, suppose we desire the proportion of defectives on hand to be never greater than π^*. Then among the possible

plans select those which maintain $\pi_\tau \leqq \pi^*$ for all τ. Now v_0 indicates the proportion of all the lots that are new. Since there is some expense incurred in obtaining a new lot, one would desire to select from among all plans where

$$\pi_\tau \leqq \pi^*$$

that plan having the smallest v_0.

Naturally much work remains before consideration can be given to a standard catalogue of surveillance plans but the thinking in this report should be helpful as a first step towards this goal. For example, in the case of a general deterioration function $p(t)$, the actual computation of π_τ and v_0 could be quite difficult. The use of the step-function model discussed in Section 4 only serves as a first approximation and the use of the specific exponential form discussed in Section 5 could be quite unrealistic. It would be interesting and useful to determine how well the step-function model serves as a first approximation for various deterioration functions.

[The original paper contained nine graphs illustrating the effects of parameter variations. Only three of those graphs are reproduced here. —The Editor.]

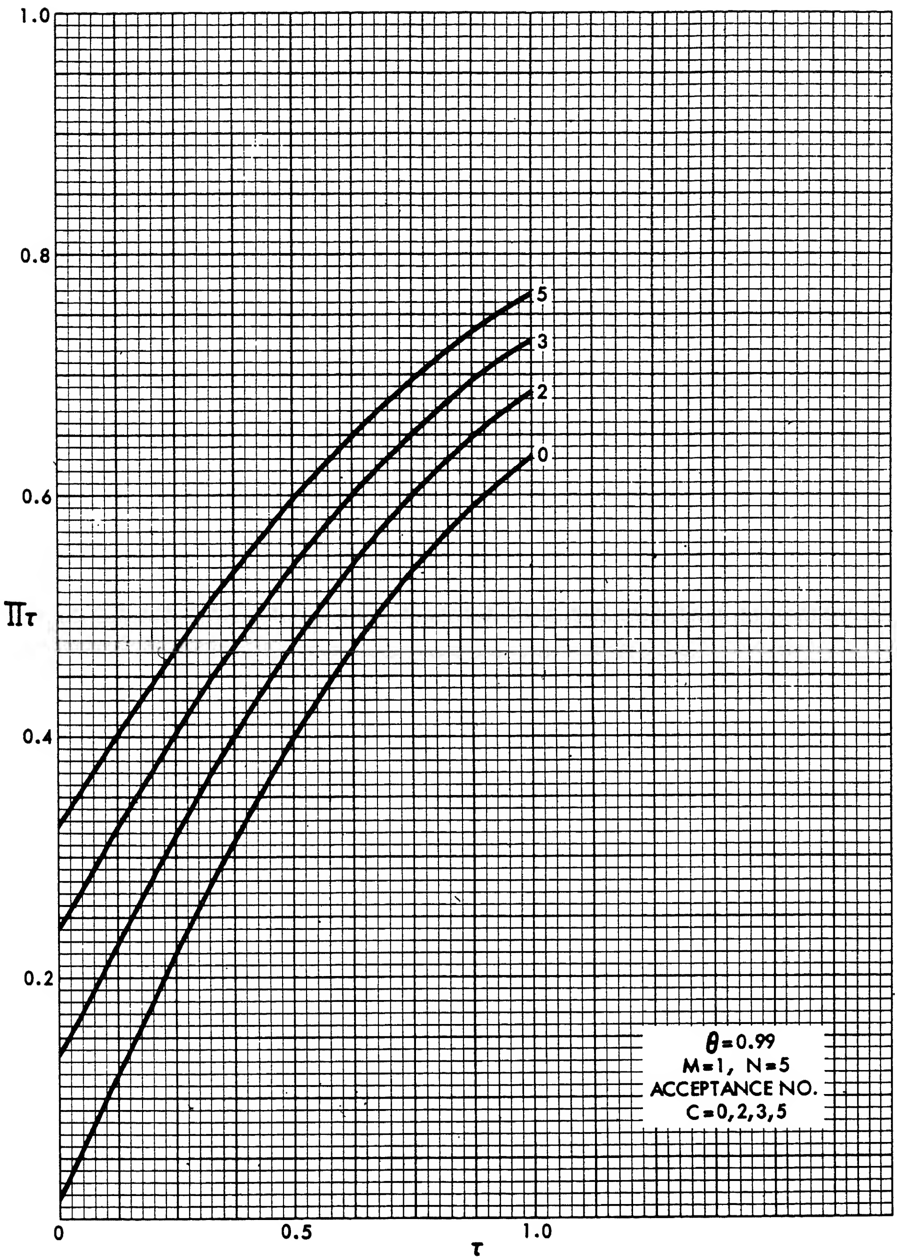
1.0
0.8
0.6
0.4
0.2
Π_τ
0
0.5
1.0
τ
5
3
2
0
θ=0.99
M=1, N=5
ACCEPTANCE NO.
C=0,2,3,5

Fig. 1

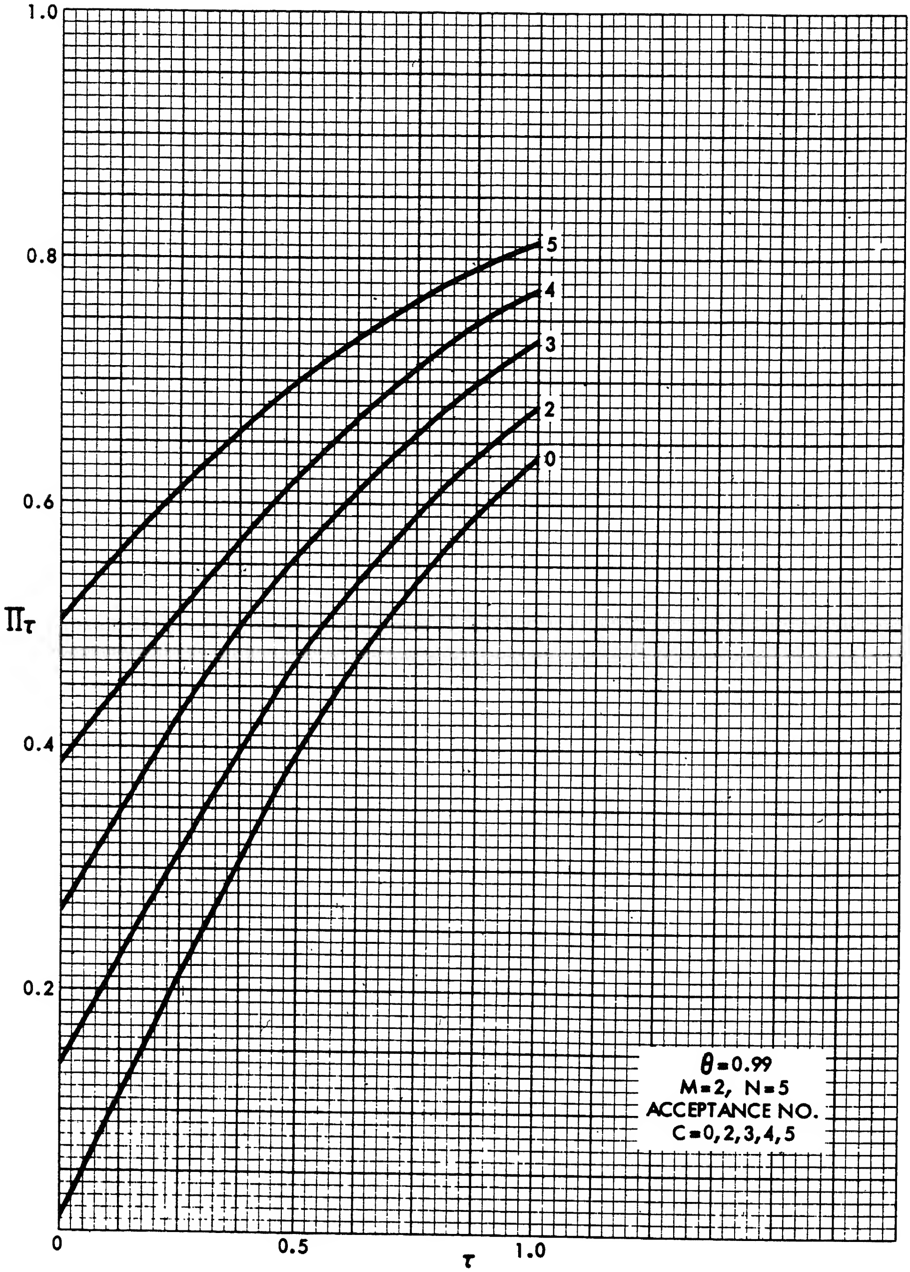
1.0
0.8
0.6
0.4
0.2
0
0.5
1.0
Π_τ
τ
5
4
3
2
0
θ=0.99
M=2, N=5
ACCEPTANCE NO.
C=0,2,3,4,5

FIG. 2

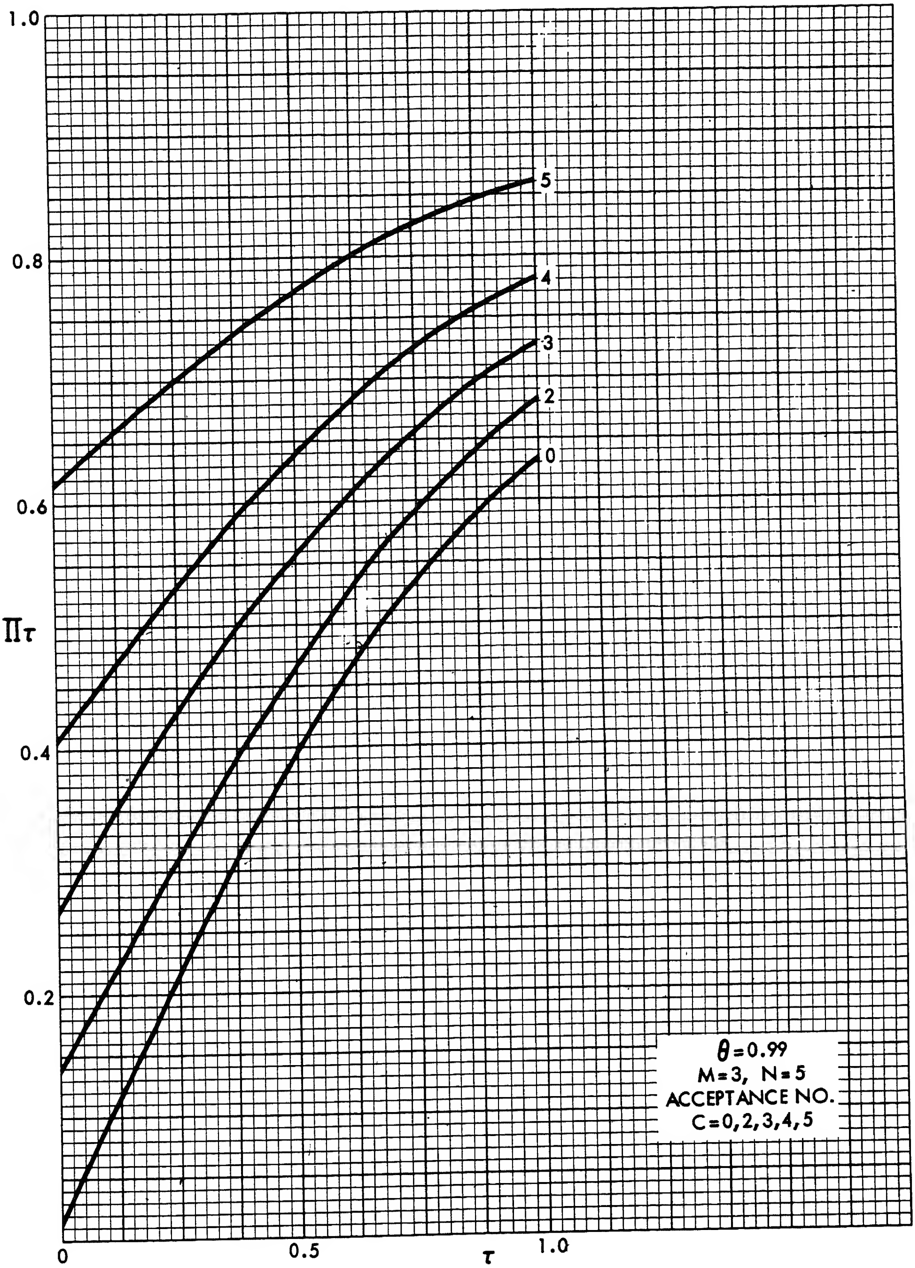

FIG. 3